ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2017, Vol. 11, No. 3, pp. 453–462. © Pleiades Publishing, Ltd., 2017.
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Solvability of a Nonstationary Problem of a Rigid Body Motion in the Flow of a Viscous Incompressible Fluid in a Pipe of Arbitrary Cross-Section

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Received April 4, 2016; in final form, January 21, 2017

Abstract—The existence of a generalized weak solution is proved for the nonstationary problem of motion of a rigid body in the flow of a viscous incompressible fluid filling a cylindrical pipe of arbitrary cross-section. The fluid flow conforms to the Navier—Stokes equations and tends to the Poiseuille flow at infinity. The body moves in accordance with the laws of classical mechanics under the influence of the surrounding fluid and the gravity force directed along the cylinder. Collisions of the body with the boundary of the flow domain are not admitted and, by this reason, the problem is considered until the body approaches the boundary.

DOI: 10.1134/S1990478917030164

Keywords: *viscous incompressible fluid, Navier—Stokes equations, rigid body, Poiseuille flow, straight pipe, noncompact boundary*

1. CLASSICAL STATEMENT OF THE PROBLEM

We study the problem of dynamics of a rigid body in the flow of a viscous incompressible fluid. The fluid fills a straight cylindrical pipe of arbitrary cross-section, its flow obeys the Navier-Stokes equations and tends to the Poiseuille flow at infinity. The body moves according to the laws of classical mechanics under the action of the ambient fluid and the force of gravity directed along the pipe.

Let Σ be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Sigma$ of class $C^{0,1}$. Suppose that the mechanical system under consideration occupies the cylindrical domain $\Omega = \Sigma \times \mathbb{R} \subset \mathbb{R}^3$. We denote the boundary of Ω by Γ; i.e., $\Gamma = \partial\Sigma \times \mathbb{R}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis for $\mathbb{R}^3$ such that Σ lies in the plane of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Let the coordinates of $\mathbf{y} \in \mathbb{R}^3$ be denoted by y_i, $i = 1, 2, 3$. We use the same notation for the vectors in $\mathbb{R}^3$.

Denote by $\widetilde{B}(t)$ the subdomain of Ω occupied by the body at time t and by $\widetilde{S}(t)$, its boundary of class $C^{0,1}$. The fluid occupies the domain $\widetilde{F}(t) = \Omega \setminus (\widetilde{B}(t) \cup \widetilde{S}(t))$. We assume that the fluid and body are homogeneous; i.e., their densities ρ_f and ρ_b are positive constants. In solving the problem, some difficulty arises connected with the fact that the body is carried away by the fluid flow and eventually leaves every finite segment of the pipe. We will try to eliminate this difficulty at the very beginning, already in the statement of the problem. Let $\tilde{\mathbf{v}}$ and $\tilde{p}$ be the velocity and pressure fields in $\widetilde{F}(t)$, and let $\hat{\mathbf{y}}(t)$ be the body center of gravity that coincides with the geometric center of the domain $\widetilde{B}(t)$ since the body is homogeneous. Let us introduce the function $\mathbf{h}(t) = (0, 0, \hat{y}_3(t))$ and perform the change of variables:

$$\mathbf{x} = \mathbf{y} - \mathbf{h}(t), \qquad \mathbf{v}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\mathbf{x} + \mathbf{h}(t), t), \qquad p(\mathbf{x}, t) = \tilde{p}(\mathbf{x} + \mathbf{h}(t), t),$$
$$B(t) = \widetilde{B}(t) - \mathbf{h}(t), \qquad S(t) = \widetilde{S}(t) - \mathbf{h}(t), \qquad F(t) = \Omega \setminus (B(t) \cup S(t)).$$

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Without loss of generality, we suppose that $\hat{y}_3(0) = 0$, i.e., at time $t = 0$ the geometric center $\hat{\mathbf{x}}(t)$ of $B(t)$ coincides with $\hat{\mathbf{y}}(0)$. The point $\hat{\mathbf{x}}$ moves in the plane of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$: $\hat{\mathbf{x}}(t) = \hat{\mathbf{y}}(t) - \mathbf{h}(t)$, $\hat{x}_1(t) = \hat{y}_1(t)$, $\hat{x}_2(t) = \hat{y}_2(t)$, and $\hat{x}_3(t) = 0$ for all t. Since Ω is a cylindrical domain, $\Omega = \Omega - \mathbf{h}(t)$ for all t, and $\mathbf{x} \in \Omega$ is equivalent to $\mathbf{y} \in \Omega$. Note that we replace the spatial variable $\mathbf{y}$ rather than the coordinate system: the velocity vector and pressure remain the same at the points $\mathbf{y}$ and $\mathbf{x}$ that correspond to each other.

Collisions of the body with the boundary of the flow region are not allowed. So we consider the problem on the time interval $[0, T]$, $T < \infty$, such that

$$\operatorname{dist}(B(t), \Gamma) = \operatorname{dist}(\widetilde{B}(t), \Gamma) \geq \delta_* > 0 \quad \text{for all} \quad t \in [0, T], \tag{1.1}$$

where δ_* is some fixed positive number.

The functions $\mathbf{v}$ and p obey the Navier–Stokes equations which are written in the new variables as follows:

$$\rho_f \big(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{h}' \cdot \nabla)\mathbf{v}\big) = \operatorname{div} P(\mathbf{v}, p) + \rho_f \mathbf{g} \qquad \text{in} \quad F(t), \tag{1.2}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in} \quad F(t), \tag{1.3}$$

$$P(\mathbf{v}, p) = -pI + 2\mu D(\mathbf{v}), \tag{1.4}$$

where $\mu = \text{const}$ is the fluid viscosity, $\mathbf{g} = (0, 0, -g)$, g is the value of the gravitational acceleration and $\mathbf{h}'$ means the derivative of $\mathbf{h}$ with respect to t, while I is the unit tensor, P is the stress tensor and $D(\mathbf{v})$ is the strain-rate tensor having the following components in the basis $\{\mathbf{e}_i\}$:

$$D_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Let us write the equations of dynamics of the body. The functions $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ are defined as follows:

$$\hat{\mathbf{y}}(t) = |\widetilde{B}(t)|^{-1} \int\limits_{\widetilde{B}(t)} \mathbf{y}\, d\mathbf{y}, \qquad \hat{\mathbf{x}}(t) = |B(t)|^{-1} \int\limits_{B(t)} \mathbf{x}\, d\mathbf{x},$$

where $|A|$ is the three-dimensional Lebesgue measure of $A \subset \mathbb{R}^3$. The body is rigid, and so we have the representation for its velocity field

$$\boldsymbol{\zeta}(t) + \boldsymbol{\omega}(t) \times (\mathbf{y} - \hat{\mathbf{y}}(t)) = \boldsymbol{\zeta}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \hat{\mathbf{x}}(t)) \quad \text{for} \quad \mathbf{y} \in \widetilde{B}(t), \quad \mathbf{x} \in B(t),$$

where $\boldsymbol{\zeta}(t) = \dfrac{d\hat{\mathbf{y}}(t)}{dt}$ and $\boldsymbol{\omega}(t)$ is the angular velocity vector.

According to the laws of classical mechanics, $\boldsymbol{\zeta}$ and $\boldsymbol{\omega}$ satisfy the equations

$$m\frac{d\boldsymbol{\zeta}}{dt} = \int\limits_{S(t)} P(\mathbf{v}, p)\mathbf{n}\, ds + \int\limits_{B(t)} \rho_b \mathbf{g}\, d\mathbf{x}, \tag{1.5}$$

$$\frac{dJ\boldsymbol{\omega}}{dt} = \int\limits_{S(t)} (\mathbf{x} - \hat{\mathbf{x}}) \times P(\mathbf{v}, p)\mathbf{n}\, ds, \tag{1.6}$$

where $m = \rho_b |B(t)|$ is the body mass, $\mathbf{n}$ is the normal vector to $S(t)$ directed towards the fluid,

$$J(t) = \rho_b \int\limits_{B(t)} \big(|\mathbf{x} - \hat{\mathbf{x}}(t)|^2 I - (\mathbf{x} - \hat{\mathbf{x}}(t)) \otimes (\mathbf{x} - \hat{\mathbf{x}}(t))\big)\, d\mathbf{x}$$

is the moment-of-inertia tensor of the body relative to the center of gravity. Note that the gravity torque, acting on the body, relative to $\hat{\mathbf{x}}$, is equal to zero.

We complement (1.2)–(1.6) with the boundary and initial conditions:

$$\mathbf{v}(\mathbf{x},t) = 0 \quad \text{for} \quad \mathbf{x} \in \Gamma, \tag{1.7}$$

$$\mathbf{v}(\mathbf{x},t) = \boldsymbol{\zeta}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \hat{\mathbf{x}}(t)) \quad \text{for} \quad \mathbf{x} \in S(t), \tag{1.8}$$

$$\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}^0, \qquad \boldsymbol{\omega}(0) = \boldsymbol{\omega}^0, \qquad B(0) = B_0, \tag{1.9}$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}^0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in F_0 = \Omega \setminus \overline{B}_0. \tag{1.10}$$

Finally, the flow tends to the Poiseuille flow at infinity:

$$(\mathbf{v}(\mathbf{x},t) - \mathbf{v}_p(\mathbf{x})) \to 0 \quad \text{as} \quad x_3 \to \pm\infty, \tag{1.11}$$

where $\mathbf{v}_p(\mathbf{x}) = (0,0,V_p(x_1,x_2))$ and the function $V_p : \Sigma \to \mathbb{R}$ is the solution of the problem

$$\partial_1^2 V_p + \partial_2^2 V_p = \lambda = \text{const}, \qquad (x_1,x_2) \in \Sigma, \qquad V_p(x_1,x_2) = 0, \qquad (x_1,x_2) \in \partial\Sigma.$$

To each value of λ there corresponds a particular Poiseuille flow. To select one of them, it is more convenient to use another parameter instead of λ, namely, the fluid flow rate across Σ:

$$Q = \int\limits_{\Sigma} V_p(x_1,x_2)\, dx_1 dx_2.$$

It is not difficult to see that

$$\lambda = -\frac{1}{Q} \int\limits_{\Sigma} |\nabla V_p|^2 \, dx_1 dx_2.$$

The problem of finding the functions $\mathbf{v}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\omega}$ that satisfy (1.2)–(1.11) will be called *Problem A*. We have not included $B(t)$ into the list of the sought-for functions because $B(t)$ can be uniquely determined, provided B_0, $\boldsymbol{\zeta}(t)$, and $\boldsymbol{\omega}(t)$ are given. Note that $\mathbf{h}$ in (1.2) can also be uniquely found from $\boldsymbol{\zeta}$ because $\mathbf{h}(0) = \mathbf{0}$ and $\mathbf{h}'(t) = (0,0,\zeta_3(t))$.

There are many papers considering the problems close to Problem A (see, for example, [1–9]). In most of them, the flow region Ω either coincides with the entire space or is bounded. In the first case, the velocity tends to zero at infinity, while in the second it vanishes at the boundary of the flow region. Nonstationary problems of this type with a noncompact boundary and nonzero conditions at infinity have not been studied by now. In [10, 11], the case of inhomogeneous conditions at infinity is considered for the problem of an unbounded flow of a viscous fluid satisfying the Navier–Stokes equations and streaming around a body. It seems that, for the first time, the problem in a domain with a noncompact boundary (namely, in a cylindrical pipe) was under study in [12]. In this article the fluid flow is described by the Stokes equations, whereas it tends to the Poiseuille flow at infinity. We prove the existence of a stationary solution of the problem.

We consider a nonstationary version of the problem in a cylindrical pipe. The main result is formulated in Theorem 1 which claims that a weak generalized solution of the problem exists at least until the first collision of the body with the boundary of the flow region. Generally speaking, it can be shown by the technique of [4] that the solution exists on an arbitrary span of time, while the collisions of the body with the boundary are allowed. But, such a study would be rather sophisticated and lengthy, and its significance would be questioned by the recent article [13] in which the impossibility of collisions is proved, provided that the fluid flow is described by the Navier–Stokes equations. It should be noted that, in that article, the body is a sphere and the boundary is a plane; thus this result cannot be applied to our case. Nevertheless, this result most likely can be generalized (we would even say, for sure) although the authors have not yet published any generalization. Since we are rather confident of the possibility of such a generalization, we decided to prove the solvability of our problem until the first collision.

2. THE GENERALIZED STATEMENT OF THE PROBLEM AND THE MAIN RESULT

We define the velocity field in the entire domain Ω as follows:

$$\mathbf{u}(\mathbf{x},t) = \begin{cases} \mathbf{v}(\mathbf{x},t), & \mathbf{x} \in F(t), \\ \boldsymbol{\zeta}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \hat{\mathbf{x}}(t)), & \mathbf{x} \in B(t). \end{cases}$$

By (1.8), the field $\mathbf{u}$ does not have any discontinuity on $S(t)$. Moreover, $\operatorname{div}\mathbf{u} = 0$ in Ω. Next, it is not difficult to check that $D(\mathbf{u}) = 0$ in $B(t)$.

The converse statement is well known (for example, see [14]); namely, if $D(\mathbf{u}) = 0$ in some domain of the space $\mathbb{R}^3$ then in this domain the field $\mathbf{u}$ is the velocity field of some rigid body motion. In accordance with these requirements, we define the function spaces that the velocity field solving Problem A must belong to.

Let the spaces of scalar and vector functions be denoted in the same way. For an arbitrary real number $\alpha > 0$, we define the domain $\Omega_\alpha = \Sigma \times (-\alpha, \alpha)$. Let G be a domain in $\mathbb{R}^3$ that coincides either with Ω or with one of the domains Ω_α. Along with the classical Lebesgue $L^p(G)$ spaces and the Sobolev spaces $H^1(G)$, we use the standard function spaces of mathematical hydrodynamics:

$$C^\infty_\sigma(G) = \{\mathbf{u} \in C^\infty(G) \mid \operatorname{div}\mathbf{u} = 0 \text{ in } G,\ \mathbf{u}|_{\Gamma\cap\overline{G}} = 0\},$$
$$L^2_\sigma(G) \text{ is the closure of } C^\infty_\sigma(G) \cap L^2(G) \text{ in } L^2(G),$$
$$H^1_\sigma(G) \text{ is the closure of } C^\infty_\sigma(G) \cap H^1(G) \text{ in } H^1(G).$$

Moreover, we define the function classes related to the problem of the motion of a rigid body in a fluid. For arbitrary domain $G_0 \subset \mathbb{R}^3$, we consider the spaces

$$L^2_{\mathcal{R}}(G, G_0) = \big\{\mathbf{u} \in L^2_\sigma(G) \mid D(\mathbf{u}) = 0 \text{ in } G_0 \cap G\big\},$$
$$H^1_{\mathcal{R}}(G, G_0) = \big\{\mathbf{u} \in H^1_\sigma(G) \mid D(\mathbf{u}) = 0 \text{ in } G_0 \cap G\big\}.$$

These space were introduced and studied in [1, 3] (also see [4, 15]). Let us use the same norms on $L^2_{\mathcal{R}}(G, G_0)$ and $H^1_{\mathcal{R}}(G, G_0)$ as on $L^2(G)$ and $H^1(G)$ respectively.

The norm on $L^2(\Omega)$ will be denoted by $\|\cdot\|$. As it follows from the Korn inequality, the norm on $H^1_\sigma(G)$ and $H^1_{\mathcal{R}}(G, G_0)$ can be equivalently given as $\big(\int_G |D(\mathbf{u})|^2\, d\mathbf{x}\big)^{1/2}$.

If G_0 depends on time then denote by $L^p\big(0, T; H^1_{\mathcal{R}}(G, G_0)\big)$ the set of functions from $L^p\big(0, T; H^1_\sigma(G)\big)$ which belong to $H^1_{\mathcal{R}}(G, G_0)$ for almost all $t \in [0, T]$. We use the same norms on these spaces.

Note that the velocity field of the Poiseuille flow belongs neither to $L^2_\sigma(\Omega)$ nor $H^1_\sigma(\Omega)$; however, it belongs to $H^1_\sigma(\Omega_\alpha)$ for all $\alpha > 0$. Another important fact for us is that if $\mathbf{w} \in H^1_\sigma(\Omega)$ then

$$\int\limits_\Sigma \mathbf{w} \cdot \mathbf{e}_3\, ds = 0, \tag{2.1}$$

where Σ is an arbitrary section of Ω by the plane orthogonal to $\mathbf{e}_3$.

The domain $B(t)$ moves without deformation; therefore, there exists a family of orientation preserving isometries $U_{t,s} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $U_{t,s}(B(s)) = B(t)$ for all $s, t \in [0, T]$. It is not difficult to see that $\mathbf{x}(t) = U_{t,s}(\boldsymbol{\xi})$ is a unique solution of the problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) - \mathbf{h}'(t), \qquad \mathbf{x}(s) = \boldsymbol{\xi} \in B(s). \tag{2.2}$$

In order to describe the motion of the body, we will use the characteristic function φ of $B(t)$; i.e., $\varphi(\mathbf{x}, t) = 1$ for $\mathbf{x} \in B(t)$ and $\varphi(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \Omega \setminus B(t)$. By (2.2), φ is the solution of the problem:

$$\partial_t\varphi + \operatorname{div}((\mathbf{u} - \mathbf{h}')\varphi) = 0, \qquad \varphi(\cdot, 0) = \varphi^0,$$

where φ^0 is the characteristic function of B_0. It is immediate from the definition of $U_{t,s}$ that $\varphi(\mathbf{x}, t) = \varphi^0(U_{0,t}(\mathbf{x})) = \varphi(U_{s,t}(\mathbf{x}), s)$.

We can also describe the density field in Ω in terms of φ:

$$\rho(\mathbf{x}, t) = \rho_b\varphi(\mathbf{x}, t) + \rho_f(1 - \varphi(\mathbf{x}, t)).$$

We are ready now to give the definition of generalized solution of Problem A:

Definition 1. Say that a pair of functions $\{\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^3,\ \varphi : \Omega \times [0, T] \to \mathbb{R}\}$ is a *generalized solution* of Problem A on $[0, T]$ if the following are fulfilled:

(1) $\varphi(\cdot,t)$ is the characteristic function of $B(t)\subset\Omega$ for all $t\in[0,T]$; $\varphi\in C(0,T;L^p(\Omega))$ for all $p\in[1,\infty)$; $\int_\Omega \varphi^0 x_3\,d\mathbf{x}=0$, where $\varphi^0=\varphi(\cdot,0)$ is the characteristic function of $B_0=B(0)$;

(2) for an arbitrary compactly-supported function $\eta\in C^1(\Omega\times[0,T])$ such that $\eta(\cdot,T)=0$, the function φ satisfies the integral identity

$$\int_0^T\int_\Omega \varphi\big(\partial_t\eta+(\mathbf{u}-\mathbf{h}')\cdot\nabla\eta\big)\,d\mathbf{x}dt=-\int_\Omega \varphi^0\eta^0\,d\mathbf{x}, \tag{2.3}$$

where

$$\eta^0=\eta(\cdot,0),\quad \mathbf{h}'(t)=\big(0,0,h_3'(t)\big),\quad h_3(0)=0,\quad h_3'(t)=|B(t)|^{-1}\int_\Omega \varphi(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{e}_3\,d\mathbf{x};$$

(3) there is a family of orientation-preserving isometries $U_{t,s}:\mathbb{R}^3\to\mathbb{R}^3$ such that $U_{t,s}(B(s))=B(t)$ for all $s,t\in[0,T]$;

(4) for all $\alpha>0$

$$\mathbf{u}\in L^\infty\big(0,T;L^2_\sigma(\Omega_\alpha)\big)\cap L^2\big(0,T;H^1_{\mathcal{R}}(\Omega_\alpha,B)\big),\qquad (\mathbf{u}-\mathbf{v}_p)\in L^2\big(0,T;H^1_\sigma(\Omega)\big), \tag{2.4}$$

and the integral identity

$$\int_0^T\int_\Omega (\rho\mathbf{u}\cdot(\partial_t\boldsymbol{\psi}+((\mathbf{u}-\mathbf{h}')\cdot\nabla)\boldsymbol{\psi})-2\mu D(\mathbf{u}):D(\boldsymbol{\psi})+\rho\mathbf{g}\cdot\boldsymbol{\psi})\,d\mathbf{x}dt=-\int_\Omega \rho^0\mathbf{u}^0\cdot\boldsymbol{\psi}^0\,d\mathbf{x} \tag{2.5}$$

holds for an arbitrary compactly-supported function $\boldsymbol{\psi}\in W^{1,2}(\Omega\times[0,T])\cap L^4\big(0,T;H^1_{\mathcal{R}}(\Omega,B)\big)$ such that $\boldsymbol{\psi}(\cdot,T)=0$. Here, ρ^0, $\mathbf{u}^0$, and $\boldsymbol{\psi}^0$ are the initial distributions of the functions ρ, $\mathbf{u}$, and ψ, respectively.

An analog of this definition in the case of a flow in bounded domain is a standard one by now. Its substantiation can be found in the papers on this topic, for example, in [1, 4]. If $\{\mathbf{u},\varphi\}$ is a generalized solution of Problem A then the functions $\mathbf{v}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\omega}$ appearing in the setting of the original problem can be reconstructed as follows: Firstly, by Condition 1, for each $t\in[0,T]$ the function $\varphi(\cdot,t)$ is the characteristic function of a certain set $B(t)$. Since $\mathbf{u}\in L^2\big(0,T;H^1_{\mathcal{R}}(\Omega_\alpha,B)\big)$, we have $D(\mathbf{u}(\cdot,t))=0$ for $\mathbf{x}\in B(t)$. Hence, there are functions $\boldsymbol{\zeta}:[0,T]\to\mathbb{R}^3$ and $\boldsymbol{\omega}:[0,T]\to\mathbb{R}^3$ such that $\mathbf{u}(\mathbf{x},t)=\boldsymbol{\zeta}(t)+\boldsymbol{\omega}(t)\times(\mathbf{x}-\hat{\mathbf{x}}(t))$ for $\mathbf{x}\in B(t)$, where $\hat{\mathbf{x}}(t)$ is the geometric center of $B(t)$, which, by Condition 2, moves without deformation. Moreover, it is not difficult to derive from Condition 2 that $\boldsymbol{\zeta}(t)=\dfrac{d\hat{\mathbf{x}}(t)}{dt}$ and $\boldsymbol{\omega}(t)$ is the angular velocity of $B(t)$. We find the function $\mathbf{v}(\cdot,t)$ as the restriction of $\mathbf{u}(\cdot,t)$ to the set $F(t)=\Omega\setminus\overline{B(t)}$. It is shown in [1, 4] that, if the so-defined $\mathbf{v}$ is differentiable with respect to time and twice differentiable with respect to spatial variables then there exists a function p such that for all t in the domain $F(t)$ the equations (1.2)–(1.4) are fulfilled. In this case, $\boldsymbol{\zeta}$ and $\boldsymbol{\omega}$ satisfy equations (1.5) and (1.6).

It may seem that Condition 3 in the definition is superfluous and follows from the fact that the velocity field $\mathbf{u}(\cdot,t)$ is a rigid body field on $B(t)$. However, Condition 3 is necessary since the smoothness required from $\mathbf{u}$ (Condition 4) does not allow us to uniquely determine the particle trajectories. Therefore, generally speaking, nothing forbids the body to "disperse to dust" at some time. We can only show that the body will not disintegrate into finitely many pieces. Condition 3 is just necessary in order for the body to remain "solid" during the time interval under consideration.

The purpose of this article is to prove the following

Theorem 1. *Suppose that* $(\mathbf{u}^0-\mathbf{v}_p)\in L^2_\sigma(\Omega)$, $B_0\subset\Omega$, *and* $\operatorname{dist}(B_0,\Gamma)>\delta_*$. *Then there is* $T>0$ *such that a generalized solution of Problem* A *exist on the interval* $[0,T]$. *Moreover,*

(1) *T can be taken arbitrarily large if* $\operatorname{dist}(B(t),\Gamma)>\delta_*$ *for all* $t\in[0,T]$;

(2) *the isometries* $U_{t,s}$ *are Lipschitz continuous in* t *and* s.

3. PROOF OF THEOREM 1

We fix an arbitrary $\alpha_* > 2\,\mathrm{diam}\,B_0$. In this case, $B(t) \subset \Omega_{\alpha_*/2}$ for all $t \geq 0$ since $\hat{\mathbf{x}}$ moves orthogonally to $\mathbf{e}_3$. Let $\tilde{\mathbf{v}}_p$ be a smooth stationary solenoidal vector field in Ω such that

(1) $\tilde{\mathbf{v}}_p(\mathbf{x}) = \mathbf{v}_p(\mathbf{x})$ in $\Omega \setminus \Omega_{\alpha_*}$;

(2) $\tilde{\mathbf{v}}_p(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma$;

(3) $\tilde{\mathbf{v}}_p(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega_{\alpha_*/2}$ such that $\mathrm{dist}(\mathbf{x}, \Gamma) > \delta_*/2$.

Obviously, such a vector field can be constructed.

Put $\mathbf{w} = \mathbf{u} - \tilde{\mathbf{v}}_p$. Then $\mathbf{w} \to 0$ as $x_3 \to \pm\infty$; and, more precisely $\mathbf{w} \in L^2\big(0, T; H^1_\sigma(\Omega)\big)$ as follows from (2.4).

Rewrite the equations from Definition 1 in terms of $\mathbf{w}$. Since $\varphi\tilde{\mathbf{v}}_p = 0$ in Ω, we simply replace $\mathbf{u}$ by $\mathbf{w}$ in (2.3):

$$\int_0^T \int_\Omega \varphi(\partial_t \eta + (\mathbf{w} - \mathbf{h}') \cdot \nabla\eta)\, d\mathbf{x}dt = -\int_\Omega \varphi^0 \eta^0\, d\mathbf{x}. \tag{3.1}$$

Then (2.5) looks as follows:

$$\int_0^T \int_\Omega \big(\rho\mathbf{w} \cdot (\partial_t \boldsymbol{\psi} + ((\mathbf{w} - \mathbf{h}') \cdot \nabla)\boldsymbol{\psi}) - 2\mu D(\mathbf{w}) : D(\boldsymbol{\psi})\big)\, d\mathbf{x}dt + \int_0^T \Phi(\mathbf{w}, \boldsymbol{\psi})\, dt + \int_0^T \Psi(\mathbf{h}, \boldsymbol{\psi})\, dt = -\int_\Omega \rho^0 \mathbf{w}^0 \cdot \boldsymbol{\psi}^0\, d\mathbf{x}, \tag{3.2}$$

where

$$\Phi(\mathbf{w}, \boldsymbol{\psi}) = \rho_f \int_\Omega (\mathbf{w} \otimes \tilde{\mathbf{v}}_p + \tilde{\mathbf{v}}_p \otimes \mathbf{w}) : D(\boldsymbol{\psi})\, d\mathbf{x},$$

$$\Psi(\mathbf{h}, \boldsymbol{\psi}) = \int_\Omega \boldsymbol{\psi} \cdot (\rho_f((\mathbf{h}' - \tilde{\mathbf{v}}_p) \cdot \nabla)\tilde{\mathbf{v}}_p + \mu\Delta\tilde{\mathbf{v}}_p + \rho\mathbf{g})\, d\mathbf{x},$$

$$\mathbf{h}'(t) = \big(0, 0, h_3'(t)\big), \qquad h_3(0) = 0,$$

$$h_3'(t) = |B(t)|^{-1} \int_\Omega \varphi(\mathbf{x}, t)\mathbf{w}(\mathbf{x}, t) \cdot \mathbf{e}_3\, d\mathbf{x}, \qquad \mathbf{w}^0 = \mathbf{u}^0 - \tilde{\mathbf{v}}_p.$$

Note that $|B(t)| = |B_0|$ for all $t > 0$. We do not include $\tilde{\mathbf{v}}_p$ into the list of arguments of Φ and Ψ since this function is fixed.

Let us consider Ψ in more detail. If $|x_3| > \alpha_*$ then $\tilde{\mathbf{v}}_p(\mathbf{x}) = \mathbf{v}_p(\mathbf{x})$ and, so, $((\mathbf{h}' - \tilde{\mathbf{v}}_p) \cdot \nabla)\tilde{\mathbf{v}}_p = 0$, whereas $\mu\Delta\tilde{\mathbf{v}}_p + \rho\mathbf{g} = \mathrm{const}\,\mathbf{e}_3$. Owing to (2.1), we obtain

$$\Psi(\mathbf{h}, \boldsymbol{\psi}) = \int_{\Omega_{\alpha_*}} \boldsymbol{\psi} \cdot (\rho_f((\mathbf{h}' - \tilde{\mathbf{v}}_p) \cdot \nabla)\tilde{\mathbf{v}}_p + \mu\Delta\tilde{\mathbf{v}}_p + \rho\mathbf{g})\, d\mathbf{x}. \tag{3.3}$$

Given a natural number $k > \alpha_*$, we consider Problem A_k in the domain Ω_k consisting in finding the functions $\mathbf{w}_k$ and φ_k vanishing in $\Omega \setminus \Omega_k$ and satisfying the integral identities:

$$\int_0^T \int_{\Omega_k} \varphi_k\big(\partial_t \eta + \big(\mathbf{w}_k - \mathbf{h}_k'\big) \cdot \nabla\eta\big)\, d\mathbf{x}dt = -\int_{\Omega_k} \varphi^0 \eta^0\, d\mathbf{x}, \tag{3.4}$$

$$\int_0^T \int_{\Omega_k} \big(\rho_k \mathbf{w}_k \cdot \big(\partial_t \boldsymbol{\psi} + \big((\mathbf{w}_k - \mathbf{h}_k') \cdot \nabla\big)\boldsymbol{\psi}\big) - 2\mu D(\mathbf{w}_k) : D(\boldsymbol{\psi})\big)\, d\mathbf{x}dt$$
$$+ \int_0^T \Phi(\mathbf{w}_k, \boldsymbol{\psi})\, dt + \int_0^T \Psi(\mathbf{h}_k, \boldsymbol{\psi})\, dt = -\int_\Omega \rho^0 \mathbf{w}_k^0 \cdot \boldsymbol{\psi}^0\, d\mathbf{x}, \quad (3.5)$$

$$\mathbf{h}_k'(t) = \big(0, 0, h_{k3}'(t)\big), \qquad h_{k3}(0) = 0, \qquad h_{k3}'(t) = |B(t)|^{-1} \int_\Omega \varphi_k(\mathbf{x}, t)\mathbf{w}_k(\mathbf{x}, t) \cdot \mathbf{e}_3\, d\mathbf{x},$$
$$\rho_k(\mathbf{x}, t) = \rho_b \varphi_k(\mathbf{x}, t) + \rho_f (1 - \varphi_k(\mathbf{x}, t)).$$

The function $\mathbf{w}_k^0$ is the initial value of $\mathbf{w}_k$. Also, $\mathbf{w}_k^0$ is zero in $\Omega \setminus \Omega_k$ and belongs to $L^2_{\mathcal{R}}(\Omega)$. In addition, we assume that $\mathbf{w}_k^0 \to \mathbf{w}^0$ in $L^2_{\mathcal{R}}(\Omega)$ as $k \to \infty$.

Given k, we have a problem in a bounded domain with zero boundary conditions for the function $\mathbf{w}_k$. Therefore, we can apply the well-known methods of [1, 4, 7] and prove that Problem A_k has a generalized solution on $[0, T_k]$ for $k > \alpha_*$, having the properties:

(1) $\varphi_k(\cdot, t)$ is the characteristic function of $B_k(t) \subset \Omega_{\alpha_*/2}$, and there exists some constant C such that $\|\varphi_k\|_{C(0,T_k;L^p(\Omega))} \le C$ for all k;

(2) there exist isometries $U^k_{t,s}$ Lipschitz continuous in t and s such that $B_k(t) = U^k_{t,s}(B_k(s))$ for all $t, s \in [0, T_k]$;

(3) $\mathbf{w}_k \in L^\infty\big(0, T_k; L^2_\sigma(\Omega)\big) \cap L^2\big(0, T_k; H^1_{\mathcal{R}}(\Omega, B_k)\big)$ and $\mathbf{w}_k(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \Omega \setminus \Omega_k$ for all k; (4) there exists a constant C such that for each k and almost all $t \in [0, T_k]$ we have

$$\int_\Omega |\mathbf{w}_k(\mathbf{x}, t)|^2\, d\mathbf{x} + \int_0^t \int_\Omega |D(\mathbf{w}_k)|^2\, d\mathbf{x}ds$$
$$\le \int_0^t |\Phi(\mathbf{w}_k, \mathbf{w}_k)|\, ds + \int_0^t |\Psi(\mathbf{h}_k, \mathbf{w}_k)|\, ds + C \int_\Omega \big|\mathbf{w}_k^0\big|^2\, d\mathbf{x}. \quad (3.6)$$

The only restriction on T_k is that $\operatorname{dist}(B_k(t), \Gamma) > \delta_*$ for all $t \in [0, T_k]$. After (3.8), we will estimate the value of T_k from below.

Since we will pass to the limit as $k \to \infty$, we need to deduce the estimates uniform in k from these properties. All constants independent of k will be denoted by C.

Let us start with the function $\mathbf{w}_k$ using (3.6). Taking it into account that the values of the modulus of $\mathbf{v}_p$ and all its derivatives up to the second order are bounded in Ω, it is not difficult to see that

$$|\Phi(\mathbf{w}_k(\cdot, t), \mathbf{w}_k(\cdot, t))| \le C\|\mathbf{w}_k(\cdot, t)\|^2 + \frac{1}{2}\|D(\mathbf{w}_k(\cdot, t))\|^2,$$

and, in view of (3.3), $|\Psi(\mathbf{h}_k(t), \mathbf{w}_k(\cdot, t))| \le C\|\mathbf{w}_k(\cdot, t)\|^2 + C$ for almost all $t \in [0, T_k]$. Inserting these estimates into (3.6) and using the Gronwall inequality, we have

$$\int_\Omega |\mathbf{w}_k(\cdot, t)|^2\, d\mathbf{x} + \int_0^t \int_\Omega |D(\mathbf{w}_k)|^2\, d\mathbf{x}ds \le C \quad (3.7)$$

for almost all $t \in [0, T_k]$.

For each k and almost all $t \in [0, T_k]$, the function $\mathbf{w}_k(\cdot, t)$ defines the velocity field of some rigid motion on $B_k(t)$, i.e., on the support of $\varphi_k(\cdot, t)$. Therefore, there exist $\boldsymbol{\zeta}_k(t)$ and $\boldsymbol{\omega}_k(t)$ such that

$\mathbf{w}_k(\mathbf{x},t) = \boldsymbol{\zeta}_k(t) + \boldsymbol{\omega}_k(t) \times (\mathbf{x} - \hat{\mathbf{x}}_k(t))$ for $\mathbf{x} \in B_k(t)$, where $\hat{\mathbf{x}}_k(t)$ is the geometric center of $B_k(t)$, whereas $\boldsymbol{\zeta}_k(t) = \dfrac{d\hat{\mathbf{x}}_k(t)}{dt}$. Using (3.7), we can easily obtain

$$\|\boldsymbol{\zeta}_k\|_{L^\infty(0,T_k)} + \|\boldsymbol{\omega}_k\|_{L^\infty(0,T_k)} \leq C. \tag{3.8}$$

For this purpose, the integral in (3.7) should be taken not over the entire domain Ω, but only over B_k. We note also that $h'_{k3}(t) = \zeta_{k3}(t)$, and so

$$\left\|\mathbf{h}'_k\right\|_{L^\infty(0,T_k)} \leq C. \tag{3.9}$$

It follows from (3.8) that the geometric center of $B_k(t)$ can move with the velocity not exceeding some constant C independent of k. The angular velocity of the body is also uniformly bounded. Therefore, there exists $T > 0$ such that

$$\operatorname{dist}(B_k(t), \Gamma) > \delta_* \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad k > \alpha_*.$$

Since $T \leq T_k$ for all k, estimates (3.7)–(3.9) are valid on $[0,T]$. It follows from (3.7) that there exist in $L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2\big(0,T; H^1_\sigma(\Omega)\big)$ a function $\mathbf{w}$ and a subsequence of the sequence $\{\mathbf{w}_k\}$, which we again denote by $\{\mathbf{w}_k\}$, such that

$$\begin{aligned} \mathbf{w}_k \to \mathbf{w} \quad &*\text{-weakly in } L^\infty\big(0,T; L^2_\sigma(\Omega_\alpha)\big), \\ \mathbf{w}_k \to \mathbf{w} \quad &\text{weakly in } L^2\big(0,T; H^1_\sigma(\Omega_\alpha)\big) \end{aligned} \tag{3.10}$$

for all $\alpha > \alpha_*$ as $k \to \infty$.

By analogy, (3.8) and (3.9) yield the existence of $\boldsymbol{\zeta}, \boldsymbol{\omega}, \mathbf{h}' \in L^\infty(0,T)$ such that, again up to a subsequence,

$$\boldsymbol{\zeta}_k \to \boldsymbol{\zeta}, \qquad \boldsymbol{\omega}_k \to \boldsymbol{\omega}, \qquad \mathbf{h}'_k \to \mathbf{h}' \quad *\text{-weakly in } L^\infty(0,T).$$

In addition, $\mathbf{h}' = \big(0,0,h'_3\big)$, $h'_3 = \zeta_3$, and $h_3(0) = 0$. It is not difficult to derive from these relations that $\hat{\mathbf{x}}_k \to \hat{\mathbf{x}}$ in $C^{0,\gamma}(0,T)$ for arbitrary $\gamma \in [0,1)$, where $\hat{\mathbf{x}}$ is some function acting from $(0,T)$ to Ω.

Since $U^k_{t,0}(\boldsymbol{\xi})$ is a solution of the problem

$$\frac{dU^k_{t,0}(\boldsymbol{\xi})}{dt} = \boldsymbol{\zeta}_k(t) - \mathbf{h}'_k(t) + \boldsymbol{\omega}_k(t) \times \big(U^k_{t,0}(\boldsymbol{\xi}) - \hat{\mathbf{x}}_k(t)\big), \qquad U^k_{0,0}(\boldsymbol{\xi}) = \boldsymbol{\xi},$$

the sequence $\big\{U^k_{t,0}\big\}$ converges in $C^{0,\gamma}(0,T)$, $\gamma \in [0,1)$, to the mapping $U_{t,0}$ solving the problem:

$$\frac{dU_{t,0}(\boldsymbol{\xi})}{dt} = \boldsymbol{\zeta}(t) - \mathbf{h}'(t) + \boldsymbol{\omega}(t) \times (U_{t,0}(\boldsymbol{\xi}) - \hat{\mathbf{x}}(t)), \qquad U_{0,0}(\boldsymbol{\xi}) = \boldsymbol{\xi}.$$

As follows from this equation, for each $t > 0$ the mapping $U_{t,0}$ is an orientation preserving isometry of $\mathbb{R}^3$; therefore, the inverse mapping $U_{0,t}$ is an isometry as well. Thus, for all t and s the mapping $U_{t,s}$ is an orientation-preserving isometry. From the same equation, we deduce the Lipschitz continuity of $U_{t,s}$ in t and s. Moreover, it is not difficult to see that $\hat{\mathbf{x}}(t)$ is the geometric center of $B(t) = U_{t,0}(B_0)$, $\boldsymbol{\zeta}(t) = \dfrac{d\hat{\mathbf{x}}(t)}{dt}$, while $\boldsymbol{\omega}(t)$ is the angular velocity of $B(t)$.

We introduce the function $\varphi(\mathbf{x},t) = \varphi^0(U_{0,t}(\mathbf{x}))$. It is clear that for each $t \in [0,T]$ this is the characteristic function of $B(t)$ and $\varphi_k \to \varphi$ as $k \to \infty$ in every space $L^p(\Omega \times [0,T])$, $1 \leq p < \infty$. Passing to the limit in (3.4), we find that φ satisfies (3.1). The convergence of $\{\varphi_k\}$ yields the convergence of the sequence $\{\rho_k\}$ to $\rho = \rho_b\varphi + \rho_f(1-\varphi)$ in every space $L^p(\Omega \times [0,T])$, $1 \leq p < \infty$.

The strong convergence of $\{\varphi_k\}$ in $L^p(\Omega \times [0,T])$ yields another important fact. Since

$$\int_0^T \int_\Omega \varphi_k D(\mathbf{w}_k)\, d\mathbf{x}dt = 0$$

for all k, this equality also remains valid in the limit as $k \to \infty$:

$$\int_0^T \int_\Omega \varphi D(\mathbf{w})\, d\mathbf{x}dt = 0.$$

Therefore, $D(\mathbf{w}(\cdot,t)) = 0$ in $B(t)$ and $\mathbf{w} \in L^2(0,T; H^1_{\mathcal{R}}(\Omega_\alpha, B))$ for all $\alpha > \alpha_*$.

We will prove now that the limit functions satisfy the integral identity (3.2) for arbitrary $\boldsymbol{\psi}$ satisfying the requirements of Condition 4 of Definition 1. To this end, we pass to the limit as $k \to \infty$ in (3.5). Generally speaking, there is an extra difficulty in carrying out this limit transition that is typical for the problems concerning the topic under consideration. The difficulty is that the test function $\boldsymbol{\psi}$ depends on the solution, to be more precise, on the domain B occupied by the body. On this set, $\boldsymbol{\psi}$ is subject to the condition $D(\boldsymbol{\psi}) = 0$. Thus, one more hidden nonlinearity is present in this integral identity.

We will overcome this difficulty as follows: First, we note that for arbitrary $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $B_k(t) \subset [B(t)]_\varepsilon$ for all $k > k_\varepsilon$ and all $t \in [0,T]$, where $[B(t)]_\varepsilon$ is the ε-neighborhood of $B(t)$ in $\mathbb{R}^3$. This follows from the uniform convergence of $U^k_{t,0}$ to $U_{t,0}$. We put

$$K^\alpha(\Omega_T) = \big\{\boldsymbol{\psi} \in L^4\big(0,T; H^1_{\mathcal{R}}(\Omega, B)\big) \mid \partial_t\boldsymbol{\psi} \in L^2(\Omega \times [0,T]),\ \boldsymbol{\psi} = 0 \text{ in } \Omega \setminus \Omega_\alpha\big\},$$
$$K^\alpha_\varepsilon(\Omega_T) = \big\{\boldsymbol{\psi} \in L^4\big(0,T; H^1_{\mathcal{R}}(\Omega, [B]_\varepsilon)\big) \mid \partial_t\boldsymbol{\psi} \in L^2(\Omega \times [0,T]), \boldsymbol{\psi} = 0 \text{ in } \Omega \setminus \Omega_\alpha\big\}.$$

Note that

$$K^\alpha_\varepsilon(\Omega_T) \subset K^\alpha(\Omega_T), \qquad K^\alpha(\Omega_T) = \lim_{\varepsilon \to 0} K^\alpha_\varepsilon(\Omega_T)$$

for all $\alpha > \alpha_*$ under the condition $\operatorname{dist}(B(t), \Gamma) > 0$, $t \in [0,T]$. This equality means that for each $\boldsymbol{\psi} \in K^\alpha(\Omega_T)$ we can find the functions $\boldsymbol{\psi}_\varepsilon \in K^\alpha_\varepsilon(\Omega_T)$ such that $\boldsymbol{\psi}_\varepsilon \to \boldsymbol{\psi}$ in $L^4(0,T; H^1(\Omega))$ and $\partial_t\boldsymbol{\psi}_\varepsilon \to \partial_t\boldsymbol{\psi}$ in $L^2(\Omega \times [0,T])$ as $\varepsilon \to 0$. The proof of these facts can be found in [1, 4, 7]. Fixing $\varepsilon > 0$ and $\alpha > \alpha_*$ arbitrarily, we may take in (3.5) $\boldsymbol{\psi} \in K^\alpha_\varepsilon(\Omega_T)$ independent of k, for $k > \max\{k_\varepsilon, \alpha\}$. In what follows, we assume that $\boldsymbol{\psi} \in K^\alpha_\varepsilon(\Omega_T)$.

In (3.5), only the convective terms depend nonlinearly on the solution, namely,

$$\int_0^T \int_{\Omega_k} \rho_k \mathbf{w}_k \cdot \big(\partial_t\boldsymbol{\psi} + \big((\mathbf{w}_k - \mathbf{h}'_k) \cdot \nabla\big)\boldsymbol{\psi}\big)\, d\mathbf{x}dt.$$

Passage to the limit in other terms presents no difficulty. Since the sequence $\{\rho_k\}$ converges strongly in $L^p(\Omega \times [0,T])$, $p \geq 1$, it follows from (3.10) that

$$\lim_{k\to\infty} \int_0^T \int_{\Omega_k} \rho_k \mathbf{w}_k \cdot \partial_t\boldsymbol{\psi}\, d\mathbf{x}dt = \int_0^T \int_\Omega \rho\mathbf{w} \cdot \partial_t\boldsymbol{\psi}\, d\mathbf{x}dt.$$

To perform passage to the limit in the remaining convective terms, we use the technique of [7]. We need to apply for $s \in (0, 1/2)$ the results that are proved for $H^s_{\mathcal{R}}(\Omega_\alpha, B)$ in Sec. 7 of that paper. Recall that $s \in (0,1)$ there. A useful property of the functions from these spaces is that they have no trace for $x_3 = \pm\alpha$. In other words, $C^\infty_0(\Omega_\alpha)$ is dense in $H^s_{\mathcal{R}}(\Omega_\alpha, B)$ for $s \in (0, 1/2)$.

As it follows from (3.5) and (3.7), there exists some constant C such that

$$\left| \int_0^T \int_\Omega \rho_k \mathbf{w}_k \cdot \partial_t\boldsymbol{\psi}\, d\mathbf{x}dt \right| \leq C\|\boldsymbol{\psi}\|_{L^4(0,T;H^1(\Omega))}$$

for all $k > \max\{k_\varepsilon, \alpha\}$ and $\boldsymbol{\psi} \in K^\alpha_\varepsilon(\Omega_T)$, satisfying the condition $\boldsymbol{\psi}|_{t=0} = \boldsymbol{\psi}|_{t=T} = 0$. Since ε can be taken however small in the last estimate, the results of [7, Sec. 7] allows us to prove that $\mathbf{w}_k \to \mathbf{w}$ in $L^2(\Omega_\alpha \times [0,T])$ as $k \to \infty$ for an arbitrary $\alpha > \alpha_*$. Using this fact, it is not difficult to pass to the limit in the convective term.

Thus, (3.2) is valid for an arbitrary ψ in $K_\varepsilon^\alpha(\Omega_T)$. Since ε is arbitrary and

$$\lim_{\varepsilon\to 0} K_\varepsilon^\alpha(\Omega_T) = K^\alpha(\Omega_T),$$

this equation also holds for an arbitrary ψ in $K^\alpha(\Omega_T)$. Owing to the arbitrariness of α, we conclude that (3.2) is true for an arbitrary function ψ satisfying the requirements of Condition 4 of Definition 1. Thus, Theorem 1 is proved.

ACKNOWLEDGMENTS

The authors were supported by the Russian Science Foundation (project no. 15–11–20019).

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