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Creep and Stress Relaxation in the Material of a Cylindrical Layer in Its Linear Motion

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Abstract—Within the framework of the theory of large deformations, we consider deformation of some material with nonlinear elastic and viscous properties that is located in the gap between two rigid coaxial cylindrical surfaces when the inner surface moves rectilinearly. We study the uniformly accelerated motion of the inner cylinder, its subsequent motion with a constant speed, and further deceleration till the full stop. We calculate stresses, reversible and irreversible deformations, displacements and study the stress relaxation after the full stop of the cylinder.

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Cold flow under creep conditions is the preferable method for considerably changing the form of some materials [1]. This caused interest in the formulation of the problems of the theory of large deformations, when, during the kinematic impact on the deformable materials, the irreversible deformations accumulate under conditions of creep at low temperatures. The problem under study is one of these examples.

1. THE BASIC MODEL RELATIONS

Various aspects of the simulation of large deformations of materials were considered in [2–11]. In this article, for solving the problem we use some mathematical model of large deformations in which the reversible and irreversible components of total strains are determined by the differential equations of change [6, 9, 11]. The main advantage of this model is the conformance with the requirements of the classical theory of elastoplasticity when the plastic deformations change during unloading and in the areas of elastic deformation in the same way as in the rigid displacement of the medium. This fact, together with the simplifying hypothesis of the independence of the thermodynamic potential from the plastic deformations, allows us to obtain solutions of a number of boundary value problems concerning elastoplastic and elastoviscoplastic [12–16] deformation of materials with large deformations, including some exact solutions. In the orthogonal system of Euler's Cartesian spatial coordinates x_i, the kinematics of the medium is defined by the relations

$$\begin{gathered} d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) = e_{ij} + p_{ij} - \frac{1}{2}e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{km}e_{mj}, \\ \frac{De_{ij}}{Dt} = \varepsilon_{ij} - \gamma_{ij} - \frac{1}{2}((\varepsilon_{ik} - \gamma_{ik} + z_{ik})e_{kj} + e_{ik}(\varepsilon_{kj} - \gamma_{kj} - z_{kj})), \\ \frac{Dp_{ij}}{Dt} = \gamma_{ij} - p_{ik}\gamma_{kj} - \gamma_{ik}p_{kj}, \qquad \frac{Dn_{ij}}{Dt} = \frac{dn_{ij}}{dt} - r_{ik}n_{kj} + n_{ik}r_{kj}, \end{gathered} \tag{1}$$

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$$\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \qquad v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_{i,j}v_j, \qquad u_{i,j} = \frac{\partial u_i}{\partial x_j},$$

$$r_{ij} = w_{ij} + z_{ij}(\varepsilon_{sk}, e_{sk}), \qquad w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}),$$

$$z_{ij}=A^{-1}[(\varepsilon_{ik}e_{kj} - e_{ik}\varepsilon_{kj})B^2 + (\varepsilon_{ik}e_{ks}e_{sj} - e_{ik}e_{ks}\varepsilon_{sj})B + e_{ik}\varepsilon_{ks}e_{st}e_{tj} - e_{ik}e_{ks}\varepsilon_{st}e_{tj}],$$

$$A = 8 - 8E_1 + 3E_1^2 - E_2 - \frac{1}{3}E_1^3 + \frac{1}{3}E_3, \qquad B = 2 - E_1,$$

$$E_1 = e_{kk}, \qquad E_2 = e_{ij}e_{ji}, \qquad E_3 = e_{ij}e_{jk}e_{ki},$$

where u_i and v_i are the components of displacements and velocities of the medium points; d_{ij} are the components of the Almansi strain tensor; e_{ij} and p_{ij} are their reversible and irreversible components; $\dfrac{D}{Dt}$ is the operator of the objective derivative of the tensors with respect to time which is written for an arbitrary tensor n_{ij}; and r_{ij} are the components of the tensor of rotations. The sources γ_{ij} and $\varepsilon^e_{ij} = \varepsilon_{ij} - \gamma_{ij}$ in the equations of change of irreversible and reversible deformations are the rates of their accumulation. When $\gamma_{ij} = 0$ then the components of the tensor of irreversible deformations change in the same way as in rotation of the coordinate system or, equivalently, in the deformation-free motion of the medium; i.e., when $\dfrac{Dp_{ij}}{Dt} = 0$. The tensor of rotations r_{ij} used here differs from the classical vorticity tensor w_{ij} by the nonlinear term z_{ij}. In addition, the tensor of irreversible deformations remains unchanged too, but the rotation of the coordinate system for different points of the body occurs differently (depending on the level of reversible deformations and the velocity of their changes). Note that at the zero nonlinear component z_{ij} of the tensor of rotations r_{ij}, the derivative in (1) is transformed into the Yauman derivative.

Following [6, 9, 11], we assume that the thermodynamic potential (the density distribution ψ of free energy) is an isotropic function of only reversible deformations. Then, by the law of energy conservation, the stresses in the medium are completely determined by the reversible deformations and related to them by a formula analogous to the Murnaghan's formula in the nonlinear theory of elasticity [17, 18]. Here we write this relation for the case of an incompressible medium:

$$\sigma_{ij} = -p\delta_{ij} + \frac{\partial W}{\partial e_{ik}}(\delta_{kj} - e_{kj}), \tag{2}$$

where σ_{ij} are the components of the Euler–Cauchy stress tensor and p is an unknown function of the additional hydrostatic pressure. For the elastic potential $W = \rho_0\psi$ (ρ_0 is the density), we consider its expansion into the Maclaurin series relative to the free state as follows:

$$\begin{gathered} W = -2\mu I_1 - \mu I_2 + bI_1^2 + (b-\mu)I_1I_2 - \chi I_1^3 + \ldots, \\ I_1 = e_{kk} - \frac{1}{2}\, e_{ks}e_{sk}, \qquad I_2 = e_{ks}e_{sk} - e_{ks}e_{st}e_{tk} + \frac{1}{4}\, e_{ks}e_{st}e_{tn}e_{nk}. \end{gathered} \tag{3}$$

Here μ is the shear modulus, while b and χ are the constant parameters of the material.

The dissipative mechanism of deformation which determines the accumulation of irreversible deformations is connected with rheological and plastic properties of the material. In what follows, we assume that the irreversible deformations are accumulated since the beginning of the deformation process and are connected with the creep of the material.

To define the corresponding dissipative mechanism of deformation, we introduce the thermodynamic potential $V(\sigma_{ij})$ in the form of the Norton creep power law [19]:

$$V(\sigma_{ij}) = B\Sigma^n(\sigma_1, \sigma_2, \sigma_3), \qquad \Sigma = \max|\sigma_i - \sigma_j|, \qquad \gamma_{ij} = \varepsilon^v_{ij} = \frac{\partial V(\Sigma)}{\partial \sigma_{ij}}. \tag{4}$$

Here σ_1, σ_2, and σ_3 are the principal values of the stress tensor, B and n are creep parameters, while ε^ν_{ij} are the components of the creep strain rate tensor.

Taking it into account that the plastic flow in the material does not occur, the stress state should not reach the yield surface. Using the Tresca yield criterion as such a surface, we have $\max|\sigma_i - \sigma_j| < 2k$ throughout the deformation process, where k is the yield strength of the material.

2. STATEMENT AND SOLUTION OF THE PROBLEM

Consider the deformation of an incompressible material located in the gap between two coaxial cylindrical surfaces with rigid walls under rectilinear motion of the inner surface of radius $r = r_0$, while the outer surface of radius $r = R$ remains fixed. The no-slip conditions are fulfilled on the rigid walls. Then the boundary conditions in the cylindrical coordinate system r, φ, and z are written as

$$v|_{r=R} = u|_{r=R} = 0, \qquad v|_{r=r_0} = v_0, \qquad u|_{r=r_0} = u_0 = \int_0^t v_0\,dt, \qquad \sigma_{rr}|_{r=R} = a_0. \tag{5}$$

Here $v = v_z(r,t)$ and $u = u_z(r,t)$ are the only nonzero components of the vectors of velocity and displacement respectively, while v_0 and a_0 are given functions. By (1), the kinematics of the medium in this case is given by the dependences

$$d_{rr} = -\frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2, \qquad d_{rz} = \frac{1}{2}\frac{\partial u}{\partial r}, \qquad \varepsilon_{rz} = -w_{rz} = \frac{1}{2}\frac{\partial v}{\partial r}, \qquad r_{rz} = \frac{2\varepsilon_{rz}(1-e_{zz})}{e_{rr}+e_{zz}-2}.$$

Granted that for this class of problems the diagonal components of the strain tensor are small quantities of a higher order as compared with the off-diagonal components [12–16], we will further restrict ourselves to the first order terms in the diagonal and the second order terms, off the diagonal. This restriction is not essential for solving the problem, but can significantly simplify the calculations. From (2) and (3) we find the stresses in the medium in the case under study:

$$\begin{gathered}
\sigma_{rr} = -(p+2\mu) + 2(b+\mu)e_{rr} + 2be_{zz} + \mu e_{rz}^2 = -P + 2\mu e_{rr},\\
\sigma_{\varphi\varphi} = -(p+2\mu) + 2b(e_{rr}+e_{zz}) - 2\mu e_{rz}^2 = -P - 3\mu e_{rz}^2,\\
\sigma_{zz} = -(p+2\mu) + 2(b+\mu)e_{zz} + 2be_{rr} + \mu e_{rz}^2 = -P + 2\mu e_{zz},\\
\sigma_{rz} = 2\mu e_{rz}, \qquad \frac{\sigma_{rr}-\sigma_{zz}}{\sigma_{rz}} = \frac{e_{rr}-e_{zz}}{e_{rz}}.
\end{gathered} \tag{6}$$

Neglecting the inertia forces, i.e., staying in the framework of quasistatic approximation, we write the equilibrium equations as

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r} = 0, \qquad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \tag{7}$$

Assuming the stresses finite, i.e. $\dfrac{\partial P}{\partial z} = 0$, and integrating (7), we find

$$\sigma_{rz} = \frac{c(t)}{r}, \qquad e_{rz} = \frac{c(t)}{2\mu r}, \qquad P = f(r,t). \tag{8}$$

Taking into account the remark before formula (6), we restrict ourselves to the terms up to the nth order with respect to stresses in the potential $V(\sigma_{ij})$, which, in the considered case, assumes the following form in the cylindrical coordinate system:

$$V(\sigma_{ij}) = B\big(4\sigma_{rz}^2 + (\sigma_{rr}-\sigma_{zz})^2\big)^{n/2}.$$

Then, by (4) and (6), we obtain the relations for the creep deformation rates:

$$\varepsilon_{rz}^v = (-1)^n 2^{n-1} Bn\sigma_{rz}^{n-1}, \qquad \varepsilon_{rr}^v = -\varepsilon_{zz}^v = \frac{\varepsilon_{rz}^v}{2}\frac{e_{rr}-e_{zz}}{e_{rz}}. \tag{9}$$

Using (8), (9), and $\varepsilon_{rz} = \varepsilon_{rz}^e + \varepsilon_{rz}^v$ together with the nonslip conditions (5) on the wall $r = R$, we obtain the following dependences for the velocity and displacement

$$\begin{gathered}
v = \frac{\dot c}{\mu}\ln\frac{r}{R} - \frac{(-1)^n 2^n Bnc^{n-1}}{2-n}\left(\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right),\\
u = \frac{c}{\mu}\ln\frac{r}{R} - \frac{(-1)^n 2^n Bnc_1}{2-n}\left(\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right), \qquad c_1(t) = \int_0^t c^{n-1}(t)\,dt.
\end{gathered} \tag{10}$$

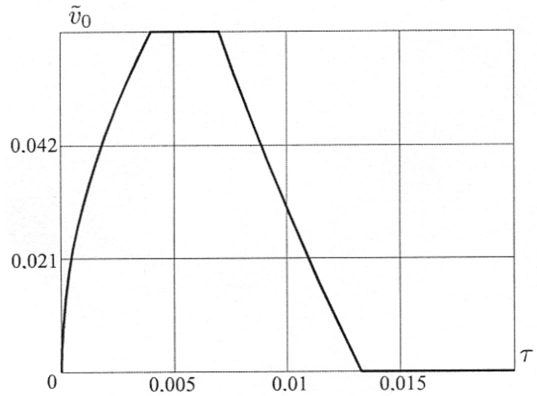

Fig. 1. Dimensionless velocity of the inner cylindrical surface $\tilde{v}_0 = v_0/\sqrt{\alpha r_0}$.

Taking into account the boundary conditions (5) on the inner surface $r = r_0$, we deduce from (10) the differential equation for the unknown function $c_1(t)$:

$$\dot{c}_1 = \left(\mu \ln^{-1}\frac{r_0}{R}\right)^{n-1}\left(u_0 - \frac{(-1)^n 2^n Bnc_1}{2-n}\left(\frac{1}{R^{n-2}} - \frac{1}{r_0^{n-2}}\right)\right)^{n-1}, \qquad c_1(0) = 0.$$

Using the equations of change of the deformation components from (1), we obtain the system of equations for finding the components of the tensors of elastic deformations e_{rr}, e_{zz} and creep strain p_{rr}, p_{rz}, and p_{zz} which in the considered approximation has the form

$$\frac{\partial p_{rz}}{\partial t} = \varepsilon_{rz}^v, \qquad \frac{\partial p_{zz}}{\partial t} = -\varepsilon_{rz}^v \frac{p_{zz} - e_{rz}^2}{e_{rz}} + \frac{4\varepsilon_{rz} p_{rz}}{2 + e_{rz}^2}\left(1 + p_{zz} - \frac{1}{2}e_{rz}^2 - 2e_{rz}p_{rz}\right),$$
$$e_{rr} = p_{zz} - \frac{3}{2}e_{rz}^2 - 2e_{rz}p_{rz}, \qquad p_{rr} + p_{zz} = -2p_{rz}^2, \qquad e_{rr} + e_{zz} = -e_{rz}^2. \tag{11}$$

System (11) is integrated numerically with the help of Wolfram Mathematica. The stress component σ_{rr} is obtained from the first equilibrium equation using the boundary condition from (5). Then, the hydrostatic pressure P, σ_{zz}, and $\sigma_{\varphi\varphi}$ are determined from (6).

We assume that the movement velocity of the inner cylindrical surface first increases ($0 \le t \le t_1$), then becomes constant ($t_1 \le t \le t_2$), decreases to zero ($t \ge t_3$), and next becomes equal to zero ($t \ge t_3$). Then the value of v_0 is chosen as (Fig. 1)

$$v_0 = \begin{cases} \alpha t, & 0 \le t \le t_1, \\ \alpha t_1, & t_1 \le t \le t_2, \\ \alpha t_1 - \beta(t - t_2), & t_2 \le t \le t_3, \\ 0, & t \ge t_3. \end{cases} \tag{12}$$

The calculations were carried out with dimensionless variables $\tilde{r} = r/R$ and $\tau = \alpha t^2/r_0$ for the values of the constants

$$k/\mu = 0.003, \qquad r_0/R = 0.2, \qquad n = 3, \qquad B\mu^2\sqrt{r_0/\alpha} = 3, \qquad \alpha/\beta = 0.5.$$

In Fig. 2, the displacement $\tilde{u} = u/R$ is shown at times $\tau_1 = 0.004$, $\tau_2 = 0.007$, $\tau_3 = 0.013$, and $\tau_4 > \tau_3$. The change of irreversible deformations at the points of the inner surface $r = r_0$ is shown in Fig. 3. The relaxation of the stress components σ_{rz} and σ_{zz} (the largest among the diagonal components) after the inner surface stops is given in Fig. 4 ($\tau_5 \gg \tau_3$).

In the article we solve the boundary value problem connected with the study of rectilinear motion of an elastoviscoplastic material in the gap between the two rigid coaxial cylindrical surfaces under conditions of the cold creep. The solution is obtained in the framework of the theory of large deformations. We consider the simplest case when the elastic deformations are assumed so small that, in calculating the stresses by them, the terms containing the third powers of these quantities can be neglected. The obtained solution simulates such a technological method as cold forming.

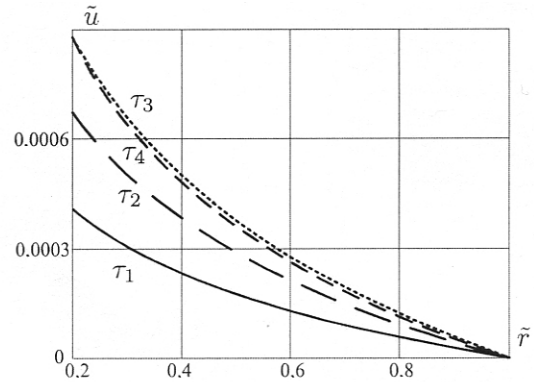

Fig. 2. Distribution of displacements over the layer at various time.

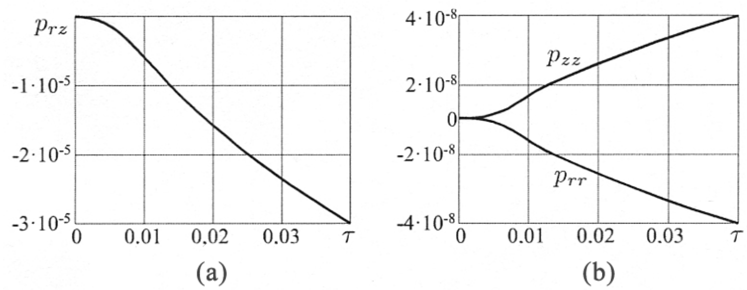

Fig. 3. Irreversible deformations p_{rz} (a) and p_{rr}, p_{zz} (b) for $r = r_0$.

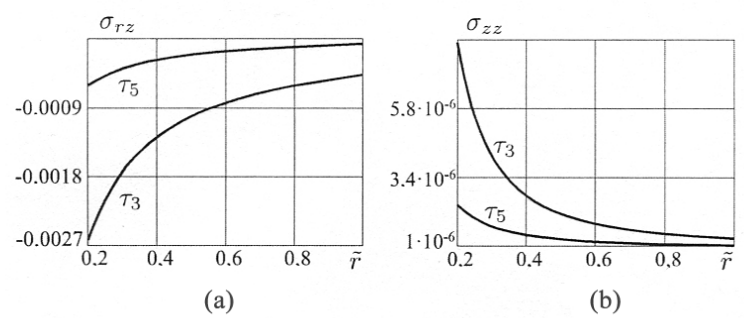

Fig. 4. Stress components σ_{rz} (a) and σ_{zz} (b).

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