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Network Flow Assignment as a Fixed Point Problem

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Abstract—This paper deals with the user equilibrium problem (flow assignment with equal journey time by alternative routes) and system optimum (flow assignment with minimal average journey time) in a network consisting of parallel routes with a single origin-destination pair. The travel time is simulated by arbitrary smooth nondecreasing functions. We prove that the equilibrium and optimal assignment problems for such a network can be reduced to the fixed point problem expressed explicitly. A simple iterative method of finding equilibrium and optimal flow assignment is developed. The method is proved to converge geometrically; under some fairly natural conditions the method is proved to converge quadratically.

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INTRODUCTION

In 1952, Wardrop formulated the two principles of flow assignment in a network in [24]. The first principle states that the routes in the network are chosen so that the journey times in all routes used are equal and less than those in the unused routes for every fixed origin-destination pair. In other words, the Wardrop first principle describes an equilibrium assignment of flows, which is called a *user equilibrium*. The second principle states that the routes in the network are chosen so that the average journey times for all used routes is minimal. In other words, the Wardrop's second principle describes an optimal assignment of flows, which is called *system optimum*.

Today, the problems of finding a user equilibrium and system optimum are extremely relevant research problems in terms of both theory and practice. The growing scales of networks with high degree of interconnectedness, such as transportation and telecommunication networks, require the development of new methodological tools for analysis and design. As a result, many researchers all around the world are studying the equilibrium and optimal traffic assignment problems [8, 20, 25, 30].

The most frequent application of the models and assessments of flow assignment is to modernize networks (network design problem) [15]. Modernization is often understood to be the addition of new links to an existing network or removal of old ones. In the process of solving such a problem, the Braess' paradox may occur, which is directly connected to main principles of network flow assignment [7]. The development of algorithms of directed transfer of information, as a rule, requires the optimal route assignment approaches [10]. The signal traffic control problem also can seldom be solved without using assignment principles [13]. The development of assessment methods of the dynamic flow assignment has complicated simulation of the network processes [14].

There are some game-theoretic formulation of the flow assignment problem. In [2], we can find the statement of the flow assignment problem as a two-level problem with two competitive groups of users at the lower level [29]. Flow assignment in competitive systems with an arbitrary number of user groups is under study in [3, 5, 9, 10, 26]. An extensive survey of foreign literature on the game theory applications in modeling the network flow assignment is provided in [18]. The article [27] is devoted to the problem of flow assignment when there are some dedicated lanes for environment-friendly transport. The more extensive analysis of this problem can be found in [28].

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In this article, we focus on the flow assignment problem for the network of parallel routes. In the previous papers dealing with networks of parallel routes, we used the linear BPR functions to simulate the travel time from origin to destination [4, 6]. It was demonstrated that an equilibrium flow assignment for those networks can be obtained explicitly. The method of finding nonzero routes in the network of parallel routes with an arbitrary BPR function was developed in [19]. Now, the journey time from origin to destination is simulated by an arbitrary smooth nondecreasing function. We show that, in this case, the flow assignment problem can be reduced to a fixed point problem. Moreover, we prove that implementation of a simple iteration procedure that arises from the transformation of the initial problem to the fixed point problem yields geometric convergence. In addition, under some natural conditions, we achieve quadratic convergence.

Let us note that the problem of the equilibrium flow assignment in a network was formulated in the form of a fixed point problem in [16, 17]. However, that form is provided in general terms and it is only mentioned that this form is possible in general and that there is a possibility of using the methods of finding a fixed point to solve the flow assignment problem. However, no specific methods for solving the flow assignment problem in the form of a fixed point problem were provided, and this approach has not been developed [22]. In this article, we are able to reduce the flow assignment problem on the network of parallel routes to the problem of finding a fixed point of an explicitly expressed mapping without using the methods for solving the fixed point problem. Moveover, it is shown that the resulting simple iterative procedures converge quadratically under some natural conditions.

Thus, in this paper we present a new flow assignment algorithm for networks with a single origin-destination pair and parallel routes that converges quadratically. This algorithm contributes to the development of parallel decomposition methods (or algorithms) [22]. This class of methods can solve the flow assignment problem in an arbitrary network with multiple origin-destination pairs (multi-commodity) by breaking down the network into subnetworks with a single origin-destination pair and solving the flow assignment problem for each subnetwork in parallel [12, 21, 22].

The paper is organized as follows: Section 1 deals with a user equilibrium in a network with parallel routes. The optimized user equilibrium problem is reduced to the fixed point problem, the obtained iterative process is proved to converge to a user equilibrium. In Section 2, we study system optimum in the network of parallel routes. Section 3 describes an algorithm for solving a nonlinear optimization problem.

1. A USER EQUILIBRIUM IN THE NETWORK OF PARALLEL ROUTES

Consider a network represented as an oriented graph G that consists of two nodes (*origin* and *destination*) and n nonintersecting links (*routes*). We call this network the *network of parallel routes*. Denote the total flow between the origin and the destination by F, and the assignment F to the links, by $f_i \geq 0$, $i = \overline{1,n}$, so that

$$\sum_{i=1}^{n} f_i = F.$$

Let us describe the travel time from origin to destination along a link by a smooth nondecreasing function on the set of real nonnegative numbers: $t_i \in C^1(\mathbb{R}^+)$, $t_i(x) - t_i(y) \geq 0$ for $x - y \geq 0$, $x, y \in \mathbb{R}^+$, and $i = \overline{1,n}$, where $\mathbb{R}^+$ is a set of real nonnegative numbers. Moreover, we suppose that $t_i(x) \geq 0$, $x \geq 0$, and $\partial t_i(x)/\partial x > 0$ for $x > 0$ and $i = \overline{1,n}$. This function is called the *delay* or the *cost* of moving along the link. Define also $f = (f_1, \ldots, f_n)$, $t(f) = (t_1(f_1), \ldots, t_n(f_n))$, and $\mathbf{f} = f^{\mathrm{T}}$.

A *user equilibrium* in the network of parallel routes is a flow assignment F to available links $f^* = (f_1^*, \ldots, f_n^*)$ such that the travel times from origin to destination are equal for each used link and less than those for every unused link [11, 24]:

$$t_i(f_i^*) \begin{cases} = t^* > 0, & \text{for } f_i^* > 0, \\ > t^* & \text{for } f_i^* = 0, \end{cases} \qquad i = \overline{1,n}.$$

It is proved that the user equilibrium problem can be formulated as an optimization problem [23]. In case of a network of parallel routes, the following optimization problem can be used to find an user equilibrium:

$$\min_f z(f) = \min_f \sum_{i=1}^{n} \int_0^{f_i} t_i(u)\, du \tag{1}$$

under the constraints

$$\sum_{i=1}^{n} f_i = F, \tag{2}$$

$$f_i \geq 0. \tag{3}$$

Let us introduce the additional notations:

$$\varphi_i(x) = t_i(x) - \frac{dt_i(x)}{dx}x, \qquad \psi_i(x) = \left(\frac{dt_i(x)}{dx}\right)^{-1}, \qquad i = \overline{1,n}.$$

Theorem 1. *Optimization problem* (1)–(3) *is equivalent to the fixed point problem*

$$\mathbf{f} = \Phi(\mathbf{f}), \tag{4}$$

where the components $\mathbf{f}$ *and* $t(f)$ *are indexed so that*

$$\varphi_1(f_1) \leq \cdots \leq \varphi_n(f_n), \tag{5}$$

and the components $\Phi(\mathbf{f}) = (\Phi_1(f), \ldots, \Phi_n(f))^{\mathrm{T}}$ *look like*

$$\Phi_i(f) = \begin{cases} \psi_i(f_i)\Big(F + \sum\limits_{s=1}^{m} \varphi_s(f_s)\psi_s(f_s)\Big) \Big/ \Big(\sum\limits_{s=1}^{m} \psi_s(f_s)\Big) - \varphi_i(f_i)\psi_i(f_i), & \textit{for } i \leq m, \\ 0, & \textit{for } i > m, \end{cases} \tag{6}$$

where m *is defined from the condition*

$$\sum_{i=1}^{m} \psi_i(f_i)[\varphi_m(f_m) - \varphi_i(f_i)] \leq F < \sum_{i=1}^{m+1} \psi_i(f_i)[\varphi_{m+1}(f_{m+1}) - \varphi_i(f_i)]. \tag{7}$$

Proof. For ease of explanation, we split the proof into the three parts.

(I) Let us approximate the following smooth nondecreasing functions $t_i(f_i) \in C^1(\mathbb{R}^+)$ for $f_i \geq 0$ by piecewise linear functions:

$$t_i(f_i) \approx \widehat{t}_i(f_i) = \sum_{j=1}^{q_i} \big[a_i^j + b_i^j f_i\big]\delta I_i^j, \qquad i = \overline{1,n},$$

where

$$\delta I_i^j = \begin{cases} 1, & \text{for } f_i \in I_j, \\ 0, & \text{otherwise,} \end{cases}$$

and $I_i = \big(I_i^1, \ldots, I_i^{q_i}\big)$ is the set of intervals corresponding to the linear segments of some piecewise linear functions $\widehat{t}_i(f_i)$ with precision ε:

$$\big|t_i(f_i) - \widehat{t}_i(f_i)\big| < \varepsilon, \qquad f_i \geq 0, \qquad i = \overline{1,n}.$$

We now formulate the user equilibrium problem for the network of parallel routes (1)–(3) when the delay functions are piecewise linear functions

$$\min_f \widehat{z}(f) = \min_f \sum_{i=1}^{n} \int_0^{f_i} \sum_{j=1}^{q_i} \big[a_i^j + b_i^j u\big]\delta I_i^j\, du \tag{8}$$

under the constraints

$$\sum_{i=1}^{n} f_i = F, \tag{9}$$

$$f_i \geq 0. \tag{10}$$

Apply the Khun–Tucker conditions to optimization problem (8)–(10) by differentiating the Lagrangian

$$L = \sum_{i=1}^{n} \int_0^{f_i} \sum_{j=1}^{q_i} \left[a_i^j + b_i^j u\right] \delta I_i^j \, du + \omega\left(F - \sum_{i=1}^{n} f_i\right) + \sum_{i=1}^{n} (-\eta_i) f_i$$

with respect to f_i and equating the result to zero:

$$\frac{\partial L}{\partial f_i} = \sum_{j=1}^{q_i} \left[a_i^j + b_i^j f_i\right] \delta I_i^j - \omega - \eta_i = 0,$$

where ω and η_i, $i = \overline{1, n}$, are Lagrange multipliers. Thus, we have

$$\widehat{t}_i(f_i) = \sum_{j=1}^{q_i} \left[a_i^j + b_i^j f_i\right] \delta I_i^j = \omega + \eta_i, \quad i = \overline{1, n}. \tag{11}$$

The complementary slackness condition requires that the equalities $\eta_i f_i = 0$ for all $i = \overline{1, n}$ be true. In this case, if $f_i > 0$ then $\eta_i = 0$ and, by (11), we have

$$\widehat{t}_i(f_i) = \sum_{j=1}^{q_i} \left[a_i^j + b_i^j f_i\right] \delta I_i^j = \omega.$$

On the other hand, if $f_i = 0$ then $\eta_i \geq 0$ and, by (11),

$$\widehat{t}_i(f_i) = \sum_{j=1}^{q_i} \left[a_i^j + b_i^j f_i\right] \delta I_i^j \geq \omega.$$

These correlations can be rewritten as the constraint

$$\widehat{t}_i(f_i) = \sum_{j=1}^{q_i} \left[a_i^j + b_i^j f_i\right] \delta I_i^j \begin{cases} = \omega, & \text{if } f_i > 0, \\ \geq \omega, & \text{if } f_i = 0, \end{cases} \qquad i = \overline{1, n}. \tag{12}$$

Express f_i in terms of ω using (12):

$$f_i = \begin{cases} \sum\limits_{j=1}^{q_i} \left[\frac{\omega - a_i^j}{b_i^j}\right] \delta I_i^j, & \text{if } \sum\limits_{j=1}^{q_i} a_i^j \delta I_i^j \leq \omega, \\ 0, & \text{if } \sum\limits_{j=1}^{q_i} a_i^j \delta I_i^j > \omega, \end{cases} \qquad i = \overline{1, n}. \tag{13}$$

The functions $\widehat{t}_i(f_i)$ for all $i = \overline{1, n}$ are convex, and so, the Khun–Tucker conditions are both sufficient and necessary. Therefore, we can say that the assignment f^* creates a user equilibrium in the network of parallel routes if and only if there is ω^* such that f^* and ω^* satisfy (12) and, therefore, (13).

(II) Assume that f^* and ω^* are defined. Then each f_i^*, $i = \overline{1, n}$, corresponds to a fixed interval $I_i^{j_i{}^*}$ and the coefficients $a_i^{j_i{}^*}$ and $b_i^{j_i{}^*}$ correspondingly. Without loss of generality, let us reindex the edges so that

$$a_1^{j_1{}^*} \leq \cdots \leq a_n^{j_n{}^*}. \tag{14}$$

Let m be the index of the edge such that $a_m^{j_m*} \le \omega^* < a_{m+1}^{j_{m+1}*}$. Then, inserting (13) into (9), we have

$$\sum_{i=1}^{n} f_i = \sum_{i=1}^{m} \left[\frac{\omega^* - a_i^{j_i*}}{b_i^{j_i*}}\right] = F. \tag{15}$$

Therefore, ω^* can be expressed as

$$\omega^* = \left(F + \sum_{i=1}^{m} a_i^{j_i*}\right) \Big/ \left(\sum_{i=1}^{m} \frac{1}{b_i^{j_i*}}\right). \tag{16}$$

Put (16) into (13) to obtain f^* explicitly:

$$f_i^* = \begin{cases} \frac{1}{b_i^{j*}}\left(F + \sum\limits_{i=1}^{m} a_i^{j_i*}\right) \Big/ \left(\sum\limits_{i=1}^{m} \frac{1}{b_i^{j_i*}}\right) - \frac{a_i^{j*}}{b_i^{j*}}, & \text{if } i \le m, \\ 0, & \text{if } i > m, \end{cases} \qquad i = \overline{1,n}. \tag{17}$$

Find the value of m using (15) and the following equation: $a_m^{j_m*} \le \omega^* < a_{m+1}^{j_{m+1}*}$:

$$\sum_{i=1}^{m} \left[\frac{a_m^{j_m*} - a_i^{j_i*}}{b_i^{j_i*}}\right] \le F < \sum_{i=1}^{m+1} \left[\frac{a_{m+1}^{j_{m+1}*} - a_i^{j_i*}}{b_i^{j_i*}}\right]. \tag{18}$$

Therefore, the user equilibrium in the network of parallel routes with piecewise linear delay functions on the edges can be expressed as (17) under condition that (14) and (18) hold.

(III) The piecewise linear functions $\widehat{t}_i(f_i)$, $i = \overline{1,n}$, are defined so that

$$\left|\widehat{t}_i(f_i) - t_i(f_i)\right| < \varepsilon, \qquad f_i \ge 0,$$

therefore, $\left|a_i^j + b_i^j f_i - t_i(f_i)\right| < \varepsilon$ for $f_i \in I_i^j$, $j = \overline{1,q_i}$, and $i = \overline{1,n}$. Moveover, for all $i = \overline{1,n}$ and $j = \overline{1,q_i}$ at the endpoints of $I_i^j = \left[\dot{f}_i^j,\ \ddot{f}_i^j\right]$ the value of the piecewise linear function $\widehat{t}_i(f_i)$ equals the value of the function $t_i(f_i)$ (by the definition of piecewise linear function):

$$\begin{cases} a_i^j + b_i^j \dot{f}_i^j = t_i(\dot{f}_i^j), \\ a_i^j + b_i^j \ddot{f}_i^j = t_i(\ddot{f}_i^j), \end{cases} \qquad j = \overline{1,q_i}, \qquad i = \overline{1,n}.$$

Therefore,

$$b_i^j = \frac{t_i(\ddot{f}_i^j) - t_i(\dot{f}_i^j)}{\ddot{f}_i^j - \dot{f}_i^j}, \qquad j = \overline{1,q_i}, \qquad i = \overline{1,n}, \tag{19}$$

$$a_i^j = t_i(\ddot{f}_i^j) - \frac{t_i(\ddot{f}_i^j) - t_i(\dot{f}_i^j)}{\ddot{f}_i^j - \dot{f}_i^j}\ddot{f}_i^j, \qquad j = \overline{1,q_i}, \qquad i = \overline{1,n}. \tag{20}$$

As $\varepsilon \to 0$, the intervals I_i^j for all $j = \overline{1,q_i}$ and $i = \overline{1,n}$ "collapse" to the point $\widehat{f}_i^j$ and we have the following limits for (19) and (20):

$$\lim_{\varepsilon \to 0} b_i^j = \frac{dt_i(\widehat{f}_i^j)}{df_i}, \qquad j = \overline{1,q_i}, \qquad i = \overline{1,n}, \tag{21}$$

$$\lim_{\varepsilon \to 0} a_i^j = t_i(\widehat{f}_i^j) - \frac{dt_i(\widehat{f}_i^j)}{df_i}\widehat{f}_i^j, \qquad j = \overline{1,q_i}, \qquad i = \overline{1,n}. \tag{22}$$

In item (II), we proved that a user equilibrium in the network of parallel routes with piecewise linear delay functions on the edges can be expressed as (17) under condition that (14) and (18) hold. However, this statement holds for piecewise linear functions that approximate the initial functions to every degree of accuracy. Therefore, by smoothness of the delay function as $\varepsilon \to 0$ and taking into account (21) and (22), the correlations (17), (14), and (18) will become (6), (5), and (7) correspondingly.

The proof of Theorem 1 is complete. □

Theorem 1 states that the user equilibrium problem formulated as the optimization problem (1)–(3) is equivalent to fixed point problem (4) for the network of parallel routes. Representation of the user equilibrium problem as (4) allows us to reformulate the problem of finding the equilibrium assignment in the network of parallel routes in the form of the simple iterative process

$$\mathbf{f}^{k+1} = \Phi(\mathbf{f}^k), \tag{23}$$

where $\mathbf{f}^k$ and $t(f)$ at the $(k+1)$th step are indexed so that $\varphi_1(f_1^k) \le \dots \le \varphi_n(f_n^k)$, while the components $\Phi(\mathbf{f}^k) = (\Phi_1(f^k), \dots, \Phi_n(f^k))^{\mathrm{T}}$ are as follows:

$$\Phi_i(f^k) = \begin{cases} \psi_i(f_i^k)\Big(F + \sum\limits_{s=1}^{m^k} \varphi_s(f_s^k)\psi_s(f_s^k)\Big)\Big(\sum\limits_{s=1}^{m^k} \psi_s(f_s^k)\Big) - \varphi_i(f_i^k)\psi_i(f_i^k), & \text{for } i \le m^k, \\ 0, & \text{for } i > m^k, \end{cases}$$

where m^k is defined by

$$\sum_{i=1}^{m^k} \psi_i(f_i^k)\left[\varphi_{m^k}(f_{m^k}^k) - \varphi_i(f_i^k)\right] \le F < \sum_{i=1}^{m^k+1} \psi_i(f_i^k)\left[\varphi_{m^k+1}(f_{m^k+1}^k) - \varphi_i(f_i^k)\right].$$

Theorem 2. *There is a δ-neighborhood of $\mathbf{f}^*$ such that, for every choice of the initial approximation $\mathbf{f}^0$ from this neighborhood, the series $\{\mathbf{f}^k\}$ of (23) remains within the neighborhood and converges geometrically to $\mathbf{f}^*$.*

Proof. For ease of presentation, we will use the following general notation of the derivative: $\frac{dt_i(f_i)}{df_i} = t_i'(f_i)$. In this case, the product of $\varphi_i(f_i)$ and $\psi_i(f_i)$ for all $i = \overline{1,n}$ will take the form

$$\varphi_i(f_i)\psi_i(f_i) = \frac{t_i(f_i) - t_i'(f_i)f_i}{t_i'(f_i)} = \frac{t_i(f_i)}{t_i'(f_i)} - f_i. \tag{24}$$

Consider Step $(k+1)$ of the iterative process (23). In view of (24) and for $i \le m^k$, we see that $\Phi_i(f^k)$ at this step is as follows:

$$\Phi_i(f^k) = f_i^k - \frac{t_i(f_i^k)}{t_i'(f_i^k)} + \left(F - \sum_{s=1}^{m^k}\left[f_s^k - \frac{t_s(f_s^k)}{t_s'(f_s^k)}\right]\right) \Big/ \left(\sum_{s=1}^{m^k} \frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right).$$

Therefore, the iterative process (23) at Step $(k+1)$ for $i \le m^k$ can be written as

$$f_i^{k+1} = f_i^k - \frac{t_i(f_i^k)}{t_i'(f_i^k)} + \left(F - \sum_{s=1}^{m^k}\left[f_s^k - \frac{t_s(f_s^k)}{t_s'(f_s^k)}\right]\right) \Big/ \left(\sum_{s=1}^{m^k} \frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right). \tag{25}$$

Let f^* be a user equilibrium assignment of flows in the network. Consider $(f_i^{k+1} - f_i^*)$ for $i \le m^k$. Let us assume that f^k belongs to the δ-neighborhood of f^* such that $m^k = m^*$, Then,

$$F = \sum_{s=1}^{m^*} f_s^* = \sum_{s=1}^{m^k} f_s^*, \qquad t_i(f_i^*) = t^* > 0$$

for all $i = \overline{1, m^k}$, and so,

$$f_i^{k+1} - f_i^* = (f_i^k - f_i^*) - \frac{t_i(f_i^k)}{t_i'(f_i^k)} + \left(F - \sum_{s=1}^{m^k}\left[f_s^k - \frac{t_s(f_s^k)}{t_s'(f_s^k)}\right]\right) \Big/ \left(\sum_{s=1}^{m^k} \frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right)$$

$$= (f_i^k - f_i^*) - \frac{(t_i(f_i^k) - t_i(f_i^*)) + t^*}{t_i'(f_i^k)} - \left[\sum_{s=1}^{m^k}\left[(f_s^k - f_s^*) - \frac{(t_s(f_s^k) - t_s(f_s^*)) + t^*}{t_s'(f_s^k)}\right]\right] \Big/ \sum_{s=1}^{m^k} \frac{t_i'(f_i^k)}{t_s'(f_s^k)}$$

$$= (f_i^k - f_i^*) - \frac{\big(t_i(f_i^k) - t_i(f_i^*)\big)}{t_i'(f_i^k)} - \frac{t^*}{t_i'(f_i^k)} + \frac{t^*}{t_i'(f_i^k)}\left(\sum_{s=1}^{m^k}\frac{1}{t_s'(f_s^k)}\right)\Big/\left(\sum_{s=1}^{m^k}\frac{1}{t_s'(f_s^k)}\right)$$

$$- \left(\sum_{s=1}^{m^k}\left[(f_s^k - f_s^*) - \frac{(t_s(f_s^k) - t_s(f_s^*))}{t_s'(f_s^k)}\right]\right)\Big/\left(\sum_{s=1}^{m^k}\frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right)$$

$$= (f_i^k - f_i^*) - \frac{\big(t_i(f_i^k) - t_i(f_i^*)\big)}{t_i'(f_i^k)} - \left(\sum_{s=1}^{m^k}\left[(f_s^k - f_s^*) - \frac{\big(t_s(f_s^k) - t_s(f_s^*)\big)}{t_s'(f_s^k)}\right]\right)\Big/\left(\sum_{s=1}^{m^k}\frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right),$$

Thus,

$$f_i^{k+1} - f_i^* = (f_i^k - f_i^*) - \frac{\big(t_i(f_i^k) - t_i(f_i^*)\big)}{t_i'(f_i^k)} - \frac{\sum\limits_{s=1}^{m^k}\left[(f_s^k - f_s^*) - \frac{(t_s(f_s^k) - t_s(f_s^*))}{t_s'(f_s^k)}\right]}{\sum\limits_{s=1}^{m^k}\frac{t_i'(f_i^k)}{t_s'(f_s^k)}}. \tag{26}$$

For ease of explanation, let us introduce the notations

$$\xi_i(x) = (x - f_i^*) - \frac{\big(t_i(x) - t_i(f_i^*)\big)}{t_i'(x)}, \qquad \varsigma_i(x) = \sum_{s=1}^{m^k}\frac{t_i'(x)}{t_s'(f_s^k)}, \quad i = \overline{1, m^k},$$

in this case, (26) becomes

$$f_i^{k+1} - f_i^* = \xi_i(f_i^k) - \frac{\sum\limits_{s=1}^{m^k}\xi_s(f_s^k)}{\varsigma(f_i^k)}. \tag{27}$$

Note that, by the Lagrange mean value theorem, for all $i = \overline{1, m^k}$ we have $t_i(f_i^k) - t_i(f_i^*) = t_i'(\theta_i^k)(f_i^k - f_i^*)$, where θ_i^k belongs to the interval between f_i^k and f_i^* so that we have

$$\xi_i(f_i^k) = \left[1 - \frac{t_i'(\theta_i^k)}{t_i'(f_i^k)}\right](f_i^k - f_i^*), \qquad i = \overline{1, m^k}. \tag{28}$$

Define

$$U_i(f_i^k) = \big[1 - \frac{t_i'(\theta_i^k)}{t_i'(f_i^k)}\big], \quad i = \overline{1, m^k}, \qquad U(f^k) = \big(U_1(f_1^k), \ldots, U_{m^k}(f_{m^k}^k)\big)$$

Putting (28) into (27)

$$f_i^{k+1} - f_i^* = U_i(f_i^k)(f_i^k - f_i^*) - \left(\sum_{s=1}^{m^k}U_s(f_s^k)(f_s^k - f_s^*)\right)\Big/\varsigma(f_i^k),$$

we have

$$|f_i^{k+1} - f_i^*| \le |U_i(f_i^k)| \cdot |f_i^k - f_i^*| + \frac{1}{\varsigma(f_i^k)} \cdot \sum_{s=1}^{m^k}|U_s(f_s^k)| \cdot |f_s^k - f_s^*|. \tag{29}$$

Sum up the left parts of (29) over $i = \overline{1, m^k}$ and note that

$$\sum_{i=1}^{m^k}\frac{1}{\varsigma(f_i^k)} = \sum_{i=1}^{m^k}\frac{1}{t_i'(f_i^k)} \cdot 1\Big/\left(\sum_{s=1}^{m^k}\frac{1}{t_s'(f_s^k)}\right) = 1.$$

Then

$$\sum_{i=1}^{m^k} \left|f_i^{k+1} - f_i^*\right| \le \sum_{i=1}^{m^k} \left|U_i\left(f_i^k\right)\right| \cdot \left|f_i^k - f_i^*\right| + \sum_{s=1}^{m^k} \left|U_s\left(f_s^k\right)\right| \cdot \left|f_s^k - f_s^*\right|,$$

and therefore,

$$\sum_{i=1}^{m^k} \left|f_i^{k+1} - f_i^*\right| \le \sum_{i=1}^{m^k} \left|2\,U_i\left(f_i^k\right)\right| \cdot \left|f_i^k - f_i^*\right|. \tag{30}$$

We have $\left\|U\left(f^k\right)\right\| \to 0$ as $f^k \to f^*$; i.e.,

$$\left(\forall \varepsilon,\ 0 < \varepsilon < 1\right)\ \left(\exists \rho\right) \left\|2U\left(f^k\right)\right\| \le \varepsilon < 1 \quad \text{for } f^k \in S_\rho(f^*),$$

where

$$S_\rho(f^*) = \{f : \|f - f^*\| \le \rho\}.$$

Therefore, since $f^k \in S_\rho(f^*)$ and due to (30), we have

$$\sum_{i=1}^{m^k} \left|f_i^{k+1} - f_i^*\right| \le \varepsilon \sum_{i=1}^{m^k} \left|f_i^k - f_i^*\right| \le \varepsilon\rho < \rho,$$

and so, $f^{k+1} \in S_\rho(f^*)$ Thus, if $f^0 \in S_\rho(f^*)$ then $\{f^k\} \in S_\rho(f^*)$ and

$$\sum_{i=1}^{m^k} \left|f_i^k - f_i^*\right| \le \varepsilon^k \sum_{i=1}^{m^k} \left|f_i^0 - f_i^*\right|,$$

whence it follows that $\{f^k\}$ converges geometrically.

The proof of Theorem 2 is complete. □

Note that Theorem 2 holds for the delay functions that are smooth on the edges of the network of parallel routes. However, if the delay functions are twice continuously differentiable then we have

Theorem 3. *Let in some neighborhood of* $\mathbf{f}^*$ *the functions* $t_i(\cdot) \in C^2$, $i = \overline{1,n}$ *be such that*

$$\left|t_i'(x)\right| \ge \alpha_i > 0, \qquad \left|t_i''(x)\right| \le \beta_i < \infty. \tag{31}$$

Then there is a δ*-neighborhood of* $\mathbf{f}^*$ *such that, for arbitrary choice of the initial approximation* $\mathbf{f}^0$ *from this neighborhood, the series* $\{\mathbf{f}^k\}$ *of the iterative process* (23) *remains within the neighborhood and converges quadratically to* $\mathbf{f}^*$.

Proof. If $t_i(\cdot) \in C^2(\mathbb{R}^+)$ for all $i = \overline{1,n}$ then, by the Taylor expansion formula, we have

$$t_i\left(f_i^*\right) = t_i\left(f_i^k\right) + t_i'\left(f_i^k\right)\left(f_i^* - f_i^k\right) + \frac{1}{2}t_i''\left(\Theta_i^k\right)\left(f_i^* - f_i^k\right)^2, \qquad i = \overline{1, m^k}, \tag{32}$$

where Θ_i^k is some point between f_i^k and f_i^*. Since $t_i\left(f_i^*\right) = t^*$, $i = \overline{1, m^k}$, let us rewrite (32) as

$$t^* = t_i\left(f_i^k\right) + t_i'\left(f_i^k\right)\left(f_i^* - f_i^k\right) + \frac{1}{2}t_i''\left(\Theta_i^k\right)\left(f_i^* - f_i^k\right)^2, \qquad i = \overline{1, m^k}. \tag{33}$$

Rewrite (25) as follows:

$$t_i\left(f_i^k\right) + t_i'\left(f_i^k\right)\left(f_i^{k+1} - f_i^k\right) = \left(F - \sum_{s=1}^{m^k}\left[f_s^k - \frac{t_s(f_s^k)}{t_s'(f_s^k)}\right]\right) \Big/ \left(\sum_{s=1}^{m^k} \frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right)$$

and add this expression to (33):

$$t^* + t_i'\left(f_i^k\right)\left(f_i^{k+1} - f_i^*\right) = \frac{1}{2}t_i''\left(\Theta_i^k\right)\left(f_i^* - f_i^k\right)^2 + \Omega, \qquad i = \overline{1, m^k}, \tag{34}$$

where

$$\Omega = \Bigl(F - \sum_{s=1}^{m^k} \bigl[f_s^k - \frac{t_s(f_s^k)}{t'_s(f_s^k)}\bigr]\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{1}{t'_s(f_s^k)}\Bigr).$$

Since $F = \sum_{i=1}^{m^k} f_i^*$, we have

$$\Omega = \Bigl(\sum_{s=1}^{m^k} \bigl[(f_s^* - f_s^k) + \frac{t_s(f_s^k)}{t'_s(f_s^k)}\bigr]\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{1}{t'_s(f_s^k)}\Bigr). \tag{35}$$

Find $t_i(f_i^k)$, $i = \overline{1, m^k}$, from (33) and insert it into (35):

$$\Omega = \Bigl(\sum_{s=1}^{m^k} \Bigl[(f_s^* - f_s^k) + \frac{t^*}{t'_s(f_s^k)} - (f_s^* - f_s^k) - \frac{1}{2}\frac{t''_s(\Theta_s^k)}{t'_s(f_s^k)}(f_s^* - f_s^k)^2\Bigr]\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{1}{t'_s(f_s^k)}\Bigr),$$

whence it follows that

$$\Omega = t^* - \frac{1}{2}\Bigl(\sum_{s=1}^{m^k} \frac{t''_s(\Theta_s^k)}{t'_s(f_s^k)}(f_s^* - f_s^k)^2\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{1}{t'_s(f_s^k)}\Bigr). \tag{36}$$

Inserting (36) into (34), we have for each $i = \overline{1, m^k}$

$$t'_i(f_i^k)(f_i^{k+1} - f_i^*) = \frac{1}{2}\, t''_i(\Theta_i^k)(f_i^* - f_i^k)^2 - \frac{1}{2}\Bigl(\sum_{s=1}^{m^k} \frac{t''_s(\Theta_s^k)}{t'_s(f_s^k)}(f_s^* - f_s^k)^2\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{1}{t'_s(f_s^k)}\Bigr), \tag{37}$$

then,

$$f_i^{k+1} - f_i^* = \frac{1}{2}\frac{t''_i(\Theta_i^k)}{t'_i(f_i^k)}(f_i^* - f_i^k)^2 - \frac{1}{2}\Bigl(\sum_{s=1}^{m^k} \frac{t''_s(\Theta_s^k)}{t'_s(f_s^k)}(f_s^* - f_s^k)^2\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{t'_i(f_i^k)}{t'_s(f_s^k)}\Bigr);$$

and so,

$$\begin{aligned}|f_i^{k+1} - f_i^*| \le \frac{1}{2}\Bigl|\frac{t''_i(\Theta_i^k)}{t'_i(f_i^k)}\Bigr| \cdot |f_i^* - f_i^k|^2 \\ + \frac{1}{2} \cdot \frac{1}{|t'_i(f_i^k)|} \cdot \Bigl(\sum_{s=1}^{m^k} \bigl|\frac{t''_s(\Theta_s^k)}{t'_s(f_s^k)}\bigr| \cdot |f_s^* - f_s^k|^2\Bigr) \Big/ \Bigl(\sum_{s=1}^{m^k} \frac{1}{|t'_s(f_s^k)|}\Bigr),\end{aligned}$$

which, after summation over $i = \overline{1, m^k}$, looks like

$$\sum_{i=1}^{m^k} |f_i^{k+1} - f_i^*| \le \sum_{i=1}^{m^k} \Bigl|\frac{t''_i(\Theta_i^k)}{t'_i(f_i^k)}\Bigr| \cdot |f_i^* - f_i^k|^2.$$

Since (31) holds in some neighborhood of $\mathbf{f}^*$, we have

$$\sum_{i=1}^{m^k} |f_i^{k+1} - f_i^*| \le \sum_{i=1}^{m^k} \frac{\beta_i}{\alpha_i} \cdot |f_i^* - f_i^k|^2. \tag{38}$$

Therefore, if follows from the definition in [1] and (38) that the series $\{\mathbf{f}^k\}$ of the iterative process (23) quadratically converges for an arbitrary choice of $\mathbf{f}^0$ from some δ-neighborhood of $\mathbf{f}^*$.

The proof of Theorem 3 is complete. □

Remark. Expressed explicitly, the iterative process (4) looks like

$$f_i^{k+1} = f_i^k - \frac{t_i(f_i^k)}{t_i'(f_i^k)} + \left(F - \sum_{s=1}^{m^k}\left[f_s^k - \frac{t_s(f_s^k)}{t_s'(f_s^k)}\right]\right) \Big/ \left(\sum_{s=1}^{m^k} \frac{t_i'(f_i^k)}{t_s'(f_s^k)}\right).$$

Note that at each new iterative step optimization problem (1)–(3) is solved for the case of linear delay functions that correspond to the tangents to the initial delay functions at the points obtained at the previous step at which an optimal assignment of flows is achieved.

Theorems 2 and 3 suggest that the iterative process (23) for $\mathbf{f}^0$ from some δ-neighborhood of $\mathbf{f}^*$ converges to a user equilibrium point for the flow assignment F in the network of parallel routes; moreover, when the delay functions meet some reasonable conditions on the links, we have quadratic convergence. As may be seen from the proofs of both theorems, one of the key requirements to the neighborhood is the equality $m^k = m^*$. In other words, as soon as the iterative process (23) finds an optimal number of routes, the quadratic convergence is achieved. On the other hand, we can to give a hint to the algorithm about which routes are optimal, for example, by having implemented the optimal route search procedure described in [19]. Moreover, it should be noted that in a loaded network all possible routes are used; and so, we can put $\mathbf{f}^0$ to be $\mathbf{f}^0 = (F/n, \dots, F/n)_n$.

2. A SYSTEM OPTIMUM IN THE NETWORK OF PARALLEL ROUTES

A *system optimum* in the network of parallel routes is a flow assignment F to the available edges $f^\dagger = (f_1^\dagger, \dots, f_n^\dagger)$ such that the average travel time of the total flow of F from origin to destination is minimized [11, 24].

By the definitions of user equilibrium and system optimum, it is clear that these two principles imply two different flow assignments. However, a user equilibrium is used when we need to simulate the behavior of independent users in a network who try to minimize their travel time. A system optimum, in turn, is useful to simulate a situation of some external influence on all users in a network with the purpose of minimizing the average travel time of the total flow.

It is proved that the system optimum problem can be stated in the form of an optimization problem [23]. In case of a network of parallel routes, the following optimization problem can be used to find the system optimum:

$$\min_f \mathbf{z}(f) = \min_f \sum_{i=1}^n t_i(f_i) f_i \tag{39}$$

under the constraints

$$\sum_{i=1}^n f_i = F, \tag{40}$$

$$f_i \ge 0. \tag{41}$$

In studying the system optimum problem for a network of parallel routes we assume that $t_i(\cdot) \in C^2(\mathbb{R}^+)$, $i = \overline{1,n}$, and that all other properties of the delay functions defined above remain true. Let us introduce the additional notations:

$$\varphi_i(x) = t_i(x) - \frac{dt_i(x)}{dx}x - \frac{d^2t_i(x)}{dx^2}x^2, \qquad \gamma_i(x) = \left(2\frac{dt_i(x)}{dx} + \frac{d^2t_i(x)}{dx^2}x\right)^{-1}, \qquad i = \overline{1,n}.$$

Theorem 4. *Optimization problem* (39)–(41) *is equivalent to the fixed point problem*

$$\mathbf{f} = \Psi(\mathbf{f}), \tag{42}$$

where the components $\mathbf{f}$ *and* $t(f)$ *are indexed so that*

$$\varphi_1(f_1) \le \dots \le \varphi_n(f_n), \tag{43}$$

and the components $\Psi(\mathbf{f}) = (\Psi_1(f), \dots, \Psi_n(f))^{\mathrm{T}}$ *are of the form*

$$\Psi_i(f) = \begin{cases} \gamma_i(f_i)\Big(F + \sum\limits_{s=1}^{m} \varphi_s(f_s)\gamma_s(f_s)\Big) \Big/ \Big(\sum\limits_{s=1}^{m} \gamma_s(f_s)\Big) - \varphi_i(f_i)\gamma_i(f_i), & \text{for } i \le m, \\ 0, & \text{for } i > m, \end{cases} \tag{44}$$

where m *is defined by the condition*

$$\sum_{i=1}^{m} \gamma_i(f_i)[\varphi_m(f_m) - \varphi_i(f_i)] \le F < \sum_{i=1}^{m+1} \gamma_i(f_i)[\varphi_{m+1}(f_{m+1}) - \varphi_i(f_i)]. \tag{45}$$

Proof. Let $g_i(x) = \frac{d(t_i(x)x)}{dx}$ for $i = \overline{1,n}$. Then (39)–(41) can be reformulated as

$$\min_f \mathbf{z}(f) = \min_f \sum_{i=1}^{n} \int\limits_0^{f_i} g_i(u)\,du \tag{46}$$

under the constraints (40) and (41). By Theorem 1, problem (46) under the constraints (40) and (41) is equivalent to the fixed point problem (42) under the conditions (43)–(45).

The proof of Theorem 4 is complete. □

Theorem 4 allows us to reduce the process of finding the optimal flow assignment in the network of parallel routes to a simple iterative procedure:

$$\mathbf{f}^{k+1} = \Psi(\mathbf{f}^k), \tag{47}$$

where $\mathbf{f}^k$ satisfies (43)–(45).

Theorem 5. *There is a* δ*-neighborhood of* $\mathbf{f}^*$ *such that, for every choice of the initial approximation* $\mathbf{f}^0$ *from this neighborhood, the series* $\{\mathbf{f}^k\}$ *of the iterative process* (42) *remains within the neighborhood and converges geometrically to* $\mathbf{f}^*$*.*

Moreover, if in some neighborhood of $\mathbf{f}^*$ *the functions* $t_i(\cdot) \in C^3$*,* $i = \overline{1,n}$*, satisfy*

$$\left|2t_i'(x) + t_i''(x)x\right| \ge \alpha_i > 0, \qquad \left|3t_i''(x) + t_i'''(x)x\right| \le \beta_i < \infty$$

then there is a δ*-neighborhood of* $\mathbf{f}^*$ *such that for an arbitrary choice of the initial approximation* $\mathbf{f}^0$ *from this neighborhood the series* $\{\mathbf{f}^k\}$ *of the iterative process* (42) *remains within the neighborhood and converges quadratically to* $\mathbf{f}^*$*.*

The proof of Theorem 5 follows from the representation of the optimization problem (39)–(41) in the form of the problem (46) with the constraints (40) and (41) and Theorems 2 and 3.

Thus, as in the case of solution to a user equilibrium, the system optimum problem in the network of parallel routes was reduced to the fixed point problem, which allows us to implement a simple iterative procedure. Moreover, Theorem 5 allows us to state that the iterative process converges to the system optimum with high enough rate.

3. A PROBLEM OF NONLINEAR PROGRAMMING

The methods used to solve the user equilibrium problem in the network of parallel routes with a single origin-destination pair (1)–(3) by reducing it to the fixed point problem and using a simple iterative procedure, can be used to solve a more general optimization problem.

Indeed, consider the following constrained optimization problem:

$$Z(x^*) = \min_x \sum_{i=1}^{n} z_i(x_i) \tag{48}$$

under the constraints

$$\sum_{i=1}^{n} a_i x_i = A, \tag{49}$$

$$x_i \geq 0, \tag{50}$$

where

$$x = (x_1, \dots, x_n), \qquad \mathbf{x} = x^{\mathrm{T}}, \qquad a_i > 0 \quad \text{for all } i = \overline{1, n}, \qquad A > 0,$$

and the first derivatives of $z_i(x) \in C^2(\mathbb{R}^+)$, $i = \overline{1, n}$, are nondecreasing:

$$\frac{dz_i}{dx_i} \in C^1(\mathbb{R}^+), \qquad \frac{dz_i(x)}{dx_i} - \frac{dz_i(y)}{dx_i} \geq 0 \qquad \text{for } x - y \geq 0, \quad x, y \in \mathbb{R}^+, \quad i = \overline{1, n},$$

$$\frac{dz_i(x)}{dx_i} \geq 0, \qquad x \geq 0, \quad i = \overline{1, n}.$$

Introduce the additional notations:

$$\mu_i(\nu) = \frac{1}{a_i}\left(\frac{dz_i(\nu)}{dx_i} - \frac{d^2 z_i(\nu)}{dx_i^2}\nu\right),$$

$$\varrho_i(\nu) = a_i\left(\frac{d^2 z_i(\nu)}{dx_i^2}\right)^{-1}, \qquad i = \overline{1, n},$$

$$\mathfrak{z}(x) = (z_1(x_1), \dots, z_n(x_n)).$$

Theorem 6. *Optimization problem* (48)—(50) *is equivalent to the fixed point problem*

$$\mathbf{x} = \Upsilon(\mathbf{x}), \tag{51}$$

where the components $\mathbf{f}$ *and* $\mathfrak{z}(x)$ *are indexed in so that*

$$\mu_1(x_1) \leq \dots \leq \mu_n(x_n), \tag{52}$$

the components $\Upsilon(\mathbf{x}) = (\Upsilon_1(x), \dots, \Upsilon_n(x))^{\mathrm{T}}$ *look like*

$$\Upsilon_i(x) = \begin{cases} \varrho_i(x_i)\Big(A + \sum\limits_{s=1}^{m} \mu_s(x_s)\varrho_s(x_s)\Big) \Big/ \Big(\sum\limits_{s=1}^{m} \varrho_s(x_s)\Big) - \mu_i(x_i)\varrho_i(x_i), & \text{for } i \leq m, \\ 0, & \text{for } i > m, \end{cases} \tag{53}$$

and m *is defined by the constraint*

$$\sum_{i=1}^{m} \varrho_i(x_i)[\mu_m(x_m) - \mu_i(x_i)] \leq A < \sum_{i=1}^{m+1} \varrho_i(x_i)[\mu_{m+1}(x_{m+1}) - \mu_i(x_i)]. \tag{54}$$

The proof of Theorem 6 is similar to that of Theorem 1.

Owing to Theorem 6, the process of seeking a solution of the optimization problem (48)–(50) can be turned into a simple iterative process:

$$\mathbf{x}^{k+1} = \Upsilon(\mathbf{x}^k), \tag{55}$$

where $\mathbf{x}^k$ satisfies (52)–(54).

Theorem 7. *There is a* δ*-neighborhood of the point* $\mathbf{x}^*$ *such that for every choice of the initial approximation* $\mathbf{x}^0$ *from this neighborhood the series* $\{\mathbf{x}^k\}$ *of the iterative process* (55) *remains within the neighborhood and geometrically converges to* $\mathbf{x}^*$.

Moreover, if in some neighborhood of $\mathbf{x}^*$ *the functions* $z_i(\cdot) \in C^3$, $i = \overline{1, n}$, *are such that*

$$|z_i''(\nu)| \geq \alpha_i > 0, \qquad |z_i'''(\nu)| \leq \beta_i < \infty$$

then there is a δ-neighborhood of $\mathbf{x}^*$ such that for every choice of the initial approximation $\mathbf{x}^0$ from this neighborhood the series $\{\mathbf{x}^k\}$ of the iterative process (55) *remains within the neighborhood and quadratically converges to* $\mathbf{x}^*$.

The proof is immediate from Theorems 2 and 3.

Therefore, the nonlinear programming optimization problem (48)–(50) can be solved by the iterative process (55) which has a high convergence rate under some natural conditions.

CONCLUSION

The key result of this paper is a new method of finding a user equilibrium and a system optimum in the network of parallel routes with a single origin-destination pair. The method consists in representing the problems as fixed point problems which, by their nature, allow for an iterative solution. It is proved that the so-obtained iterative process has high rate of convergence; and, so this approach is very efficient in practice.

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