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Polytopes of Special Classes of the Balanced Transferable Utility Games

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Abstract—Under study are the polytopes of $(0,1)$-normalized convex and 1-convex (dual simplex) n-person TU-games and monotonic big boss games. We solve the characterization problems of the extreme points of the polytopes of 1-convex games, symmetric convex games, and big boss games symmetric with respect to the coalition of powerless agents. For the remaining polytopes, some subsets of extreme points are described.

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INTRODUCTION

The study of the structure of the polyhedral games consisting of imputations satisfying some "fairness" principles and the investigation of the auxiliary polyhedra is an actual direction of the development of the theory of cooperative games. For example, the extreme points were obtained of the core of convex games (see [11]) and some other special classes. The barycenters of the cores of these games coincide with the available singleton solutions—the Shapley values in a convex game [11], the nucleolus in a 1-convex game [7], the τ-value in a big boss game [8], etc. The description of the extreme points of the Weber polytope (see [2]) made it possible to deduce some additional properties of the core, the Weber set, and the Shapley value and obtain easy proofs of the already known results.

A game with transferable utility (a TU-*game*) (N,ν), where $N=\{1,\dots,n\}$, $\nu:2^N\to\mathbb{R}$, and $\nu(\varnothing)=0$, can be identified with a vector in the space $\mathbb{R}^{2^n-1}$ whose components are equal to $\nu(S)$, $S\in 2^N\setminus\{\varnothing\}$. A parametric description of subsets of games forming cones or polytopes in this space is useful for studying the behavior of their solutions. Moreover, knowing the extreme elements of a polyhedral set, we can obtain any number of nonrepeating games of the class under consideration. The game generators are necessary for checking hypotheses, estimating the efficiency of algorithms in the mean, testing programs, and in developing learning tasks.

A game (N,ν) is called *balanced* if it has nonempty core. There is an alternative definition that uses the Bondareva–Shapley condition [1, 10]. In this article, we consider the polytopes of $(0,1)$-normalized TU-games satisfying some simpler conditions for the nonemptiness of the core than the Bondareva–Shapley condition and study the relationship between these sets. We solve the characterization problem of the extreme points of the polytopes of 1-convex games and symmetric convex n-person games and also big boss games symmetric with respect to the coalition of powerless agents. For the remaining polytopes, we describe subsets of extreme points.

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1. THE MAIN NOTIONS

A TU-game (N, ν) is called *nonnegative* if $\nu(S) \geq 0$, $S \in 2^N$; N*-essential* if $\nu(N) > \sum_{i \in N} \nu(i)$; *symmetric* if $S, T \in 2^N$ and $|S| = |T|$ imply $\nu(S) = \nu(T)$; $(0,1)$*-normalized* if

$$\nu(N) = 1, \qquad \nu(i) = 0, \qquad i \in N, \tag{1}$$

simple if $\nu(N) = 1$ and $\nu(S) \in \{0, 1\}$, $S \in 2^N$; *convex* if

$$\nu(S) + \nu(T) \leq \nu(S \cap T) + \nu(S \cup T), \qquad S, T \in 2^N;$$

and the *unanimity game* u_T of a coalition T if $T \neq \varnothing$, $u_T(S) = 1$ for $S \supseteq T$, and $u_T(S) = 0$ otherwise.

Let us identify a game (N, ν) with its characteristic function ν, assume that $n \geq 3$, and use the abbreviations: $\nu(i)$ instead of $\nu(\{i\})$, $K \cup i$ instead of $K \cup \{i\}$, etc. For a coalition $S \in 2^N \setminus \{\varnothing\}$ and a vector $x \in \mathbb{R}^n$, we have

$$x(S) = \sum_{i \in S} x_i, \qquad x(\varnothing) = 0.$$

Agents $i, j \in N$ are *symmetric* in a game ν if $\nu(S \cup i) = \nu(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. An agent $i \in N$ is called a *veto-player* if $\nu(S) = 0$ for $S \not\ni i$, $\mathrm{Veto}(\nu)$ is the set of the veto-players of a game ν, $m^\nu = \left(m_i^\nu\right)_{i \in N}$ is the vector of the contributions $m_i^\nu = \nu(N) - \nu(N \setminus i)$ of the players to the grand coalition. Put

$$\Omega = 2^N \setminus \{N, \varnothing\}, \qquad \Omega_{[i,j]} = \{S \in 2^N \mid i \leq |S| \leq j\}, \qquad i, j \in N, \quad i \leq j,$$

$$X(\nu) = \{x \in \mathbb{R}^n \mid x(N) = \nu(N)\}.$$

The sets of *imputations* and *dual imputations* and also the *core* of a game ν are defined as follows:

$$I(\nu) = \{x \in X(\nu) \mid x_i \geq \nu(i),\ i \in N\}, \qquad I^*(\nu) = \{x \in X(\nu) \mid x_i \leq m_i^\nu,\ i \in N\},$$

$$C(\nu) = \{x \in X(\nu) \mid x(S) \geq \nu(S),\ |S| \in \Omega\}.$$

A balanced game ν is called *exact* if, for each coalition $S \in \Omega$, there exists an imputation $x \in C(\nu)$ such that $x(S) = \nu(S)$.

The main solutions of a game ν are invariant under strategic equivalence, and every N-essential game has a unique $(0,1)$-form; therefore, we consider the normalized TU-games.

2. CONVEX GAMES

Denote the polytope of $(0,1)$-normalized convex n-person games by CO^n. Every game $\nu \in \mathrm{CO}^n$ is nonnegative. The properties of solutions of convex games are well studied. Various characterizations of a convex game are proposed. All extreme rays of the cone of convex 4-person games are listed in [11, p. 14]. They correspond to 36 extreme points of the polytope CO^4. It is observed in [11] that the extreme elements of the cones of games of five or more persons are unknown. Show that the set of integer extreme points $\mathrm{Ex}_I\mathrm{CO}^n$ of the polytope CO^n consists of $2^n - n - 1$ unanimity games.

Theorem 1. $\mathrm{Ex}_I\mathrm{CO}^n = \{u_T\}_{T \in \Omega_{[2,n]}}$.

Proof. Given $\nu \in \mathrm{CO}^n$, we have $0 \leq \nu(S) \leq 1$, $S \in 2^N$; i.e., CO^n is contained in the unit hypercube. Hence, only simple games can be integer extreme points. All unanimity games are simple and convex. If $|T| \geq 2$ then $u_T \in \mathrm{CO}^n$. Consequently,

$$\{u_T\}_{T \in \Omega_{[2,n]}} \subseteq \mathrm{Ex}_I\mathrm{CO}^n.$$

Take $\nu \in \mathrm{Ex}_I\mathrm{CO}^n$. In the expansion

$$\nu = \sum_{T \in 2^N \setminus \{\varnothing\}} \delta_T u_T$$

of a convex game in a basis of unanimity games, the coefficients δ_T are nonnegative [12]. From $\nu(i) = 0$, $i \in N$, and $\nu(N) = u_T(N) = 1$, $T \in 2^N \setminus \{\varnothing\}$, it follows that

$$\sum_{T \in \Omega_{[2,n]}} \delta_T = 1;$$

i.e., $\nu \in \operatorname{conv}\{u_T\}_{T \in \Omega_{[2,n]}}$.

Assuming that $\nu \in \mathrm{Ex}_I \mathrm{CO}^n$, we infer that ν coincides with one of the games u_T. We thus proved that $\{u_T\}_{T \in \Omega_{[2,n]}} \supseteq \mathrm{Ex}_I \mathrm{CO}^n$. Theorem 1 is proved. □

The extreme points of the polytope $S\mathrm{CO}^n \subset \mathrm{CO}^n$ of symmetric, convex, and $(0,1)$-normalized games are characterized by

Theorem 2. $\mathrm{Ex}S\mathrm{CO}^n = \{\nu^m\}_{m=1}^{n-1}$, *where*

$$\nu^m(S) = \begin{cases} f^m_{|S|}, & S \in \Omega_{[2,n-1]}, \\ 1, & |S| = n, \\ 0, & |S| = 1, \end{cases} \qquad f^m_{|S|} = \begin{cases} 0, & S \in \Omega_{[2,m]}, \\ \frac{|S|-m}{n-m}, & S \in \Omega_{[m+1,n-1]}. \end{cases}$$

Proof. For $\nu \in S\mathrm{CO}^n$ and $S \in \Omega_{[2,n-1]}$, put $f_{|S|} = \nu(S)$. Consider the polytope F^n, $n \geq 4$, defined by the system

$$\begin{cases} 2f_2 - f_3 \leq 0, \\ -f_{|S|} + 2f_{|S|+1} - f_{|S|+2} \leq 0, \quad S \in \Omega_{[2,n-3]}, \\ -f_{n-2} + 2f_{n-1} \leq 1, \\ f \in \mathbb{R}^{n-2}_+, \end{cases} \tag{2}$$

which is obtained from the convexity condition (see [12])

$$\nu(S \cup i) + \nu(S \cup j) \leq \nu(S \cup \{i, j\}) + \nu(S),$$

where $S \subseteq N \setminus \{i, j\}$, $i, j \in N$, $i \neq j$. The normalization condition (1) takes into account the assumption of the game symmetry. The matrix $A \in \mathbb{R}^{(n-2)\times(n-2)}$ of (2) and the inverse matrix are defined by the formulas

$$[A]_{i,j} = \begin{cases} 2, & i = j, \\ -1, & (i = j+1) \vee (i = j-1), \\ 0, & \text{otherwise}, \end{cases} \qquad [A]^{-1}_{i,j} = \begin{cases} \frac{j(n-1-i)}{n-1}, & i > j, \\ \frac{i(n-1-j)}{n-1}, & \text{otherwise}, \end{cases}$$

which imply that the system obtained from (2) by replacing the main constraints by equalities has the unique solution coinciding with f^1.

Define the matrices $P, R \in \mathbb{R}^{(n-2)\times(n-2)}$, where

$$[P]_{i,j} = \begin{cases} \frac{n-1-j}{n-1-i}, & i \leq j, \\ 0, & \text{otherwise}, \end{cases} \qquad [R]_{i,j} = \begin{cases} \frac{j}{i}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Transform (2) by using left multiplication by P. We obtain

$$\begin{cases} f_2 \leq \frac{1}{n-1}, \\ -f_{|S|-1} + \frac{n-|S|+1}{n-|S|} f_{|S|} \leq \frac{1}{n-|S|}, \quad S \in \Omega_{[3,n-1]}, \\ f \in \mathbb{R}^{n-2}_+. \end{cases} \tag{3}$$

Hence, for each $m = \overline{2, n-1}$, the vector f^m is a solution to the system of $n-2$ linearly independent equations

$$\begin{cases} f_{|S|} = 0, & S \in \Omega_{[2,m]}, \\ -f_{|S|-1} + \frac{n-|S|+1}{n-|S|} f_{|S|} = \frac{1}{n-|S|}, & S \in \Omega_{[m+1,n-1]}, \end{cases}$$

and satisfies (3). Thus, $\{f^m\}_{m=1}^{n-1} \subseteq \mathrm{Ex}F^n$.

Take $f^* \in \mathrm{Ex}F^n$ and suppose that $f^* \neq f^m$ for all $m = \overline{1, n-1}$. Using the left multiplication by R, represent (2) as

$$\frac{|S|}{|S|-1} f_{|S|} - f_{|S|+1} \le 0, \quad |S| \in \Omega_{[2,n-2]}, \qquad f_{n-1} \le \frac{n-2}{n-1}, \quad f \in \mathbb{R}_+^{n-2}.$$

If $f^*_{|S|} = 0$ for all $S \in \Omega_{[2,n-1]}$ then $f^* = f^{n-1}$; a contradiction. Otherwise (as is seen from the last system), there exists a coalition $K \in \Omega_{[2,n-1]}$ such that $f^*_{|S|} = 0$ for $S \in \Omega_{[2,|K|-1]}$ and $f^*_{|S|} > 0$ for $S \in \Omega_{[|K|,n-1]}$. If f^* satisfies the constraints of (3) corresponding to the coalitions $S \in \Omega_{[|K|,n-1]}$ as equalities then $f^* = f^{|K|-1}$; a contradiction. Hence, f^* satisfies at least one of these inequalities as a strict one; i.e.,

$$-f^*_{|T|-1} + \frac{n-|T|+1}{n-|T|} f^*_{|T|} < \frac{1}{n-|T|}$$

for some coalition $T \in \Omega_{[|K|,n-1]}$. Put

$$\theta = \begin{cases} \frac{n-|T|}{n-|T|+1}\left(\frac{1}{n-|T|} + f^*_{|T|-1} - \frac{n-|T|+1}{n-|T|} f^*_{|T|}\right), & T \in \Omega_{[3,n-1]}, \\ \frac{1}{n-1} - f^*_2, & |T| = 2, \end{cases}$$

$$\beta = \min\left\{\theta, \min_{0 \le k \le n-|T|-1} \frac{n-|T|}{n-|T|-k} f^*_{|T|+k}\right\}$$

and consider $\dot{f}, \ddot{f} \in \mathbb{R}_+^{n-2}$, where

$$\dot{f}_{|S|} = \ddot{f}_{|S|} = f^*_{|S|}, \qquad S \in \Omega_{[2,|T|-1]},$$

$$\dot{f}_{|T|+r} = f^*_{|T|+r} + \frac{n-|T|-r}{n-|T|}\beta, \qquad \ddot{f}_{|T|+r} = f^*_{|T|+r} - \frac{n-|T|-r}{n-|T|}\beta,$$

$r = \overline{0, n-|T|-1}$. The definition of β implies that $\dot{f}$ and $\ddot{f}$ are nonnegative and satisfy the inequalities of (3) corresponding to $S \in \Omega_{[2,|T|]}$. For all $S \in \Omega_{[|T|+r,n-1]}$ and $r = \overline{1, n-|T|-1}$, we have

$$-\left(f^*_{|T|+r-1} \pm \frac{n-|T|-r+1}{n-|T|}\beta\right) + \frac{n-|T|-r+1}{n-|T|-r}\left(f^*_{|T|+r} \pm \frac{n-|T|-r}{n-|T|}\beta\right) = -f^*_{|T|+r-1} + \frac{n-|T|-r+1}{n-|T|-r} f^*_{|T|+r}.$$

Consequently, $\dot{f}$ and $\ddot{f}$ satisfy the remaining inequalities of (3). We have $\dot{f}, \ddot{f} \in F^n$ and $f^* = (\dot{f} + \ddot{f})/2$, which contradicts the relation $f^* \in \mathrm{Ex}F^n$. So, $\mathrm{Ex}F^n \subseteq \{f^m\}_{m=1}^{n-1}$. Finally, $\mathrm{Ex}F^n = \{f^m\}_{m=1}^{n-1}$ for $n \ge 4$.

For the polytope F^3 defined by the condition $2f_2 \le 1$, $f_2 \in \mathbb{R}_+$, the equality $\mathrm{Ex}F^3 = \{f^m\}_{m=1}^{2}$ is obvious. The operator $\Psi : \mathbb{R}^{n-2} \to \mathbb{R}^{2^n-1}$ assigning to each vector $f \in F^n$, $n \ge 3$, the game ν, where

$$\nu(S) = \Psi_{|S|}(f) = \begin{cases} f_{|S|}, & S \in \Omega_{[2,n-1]}, \\ 1, & |S| = n, \\ 0, & \text{in the other cases}, \end{cases}$$

Table 1. Extreme points of the polytopes F^n, $n \in \{3,4,5\}$

	$f^m = (f_2^m, f_3^m, \ldots, f_{n-1}^m)$			
$n=3$	$f^2=(0)$,	$f^1=(1/2)$,		
$n=4$	$f^3=(0,0)$,	$f^2=(0,1/2)$,	$f^1=(1/3,2/3)$,	
$n=5$	$f^4=(0,0,0)$,	$f^3=(0,0,1/2)$,	$f^2=(0,1/3,2/3)$,	$f^1=(1/4,2/4,3/4)$

maps F^n onto $S\mathrm{CO}^n$ preserving the adjacency of faces. Consequently,

$$\mathrm{Ex}S\mathrm{CO}^n = \{\Psi(f^m)\}_{m=1}^{n-1} = \{\nu^m\}_{m=1}^{n-1}.$$

Theorem 2 is proved. □

Note that the game ν^1 of Theorem 2 satisfies the sufficient condition for coincidence of the core with the set of dual imputations obtained in [13], and the game ν^{n-1} satisfies the sufficient condition for coincidence of the core with the set of imputations.

Table 1 contains the vectors that are the preimages of the extreme points of the polytopes of symmetric, convex, and $(0,1)$-normalized three-, four-, and five-person games.

3. BIG BOSS GAMES

In some socioeconomic situations modelled by cooperative games, one of the participants (the *powerful agent*) has more opportunities than the others (*powerless agents*). For example, a market with one seller and several buyers, a parliament with one big party and several small parties, similar situations of bankruptcy and investment, and a holding. A coalition of powerless agents is called a *union*. The corresponding cooperative games are often *big boss games*. Several types of those games are known: nonmonotonic, monotonic, total, generalized, and strong boss games. They all are balanced. In big boss games, many concepts of solution (the bargaining set, the τ-value, the nucleolus, and the Shapley value) have special properties [8], and the core is a supercore. A monotonic big boss game (see [8]) with player k as the boss is defined by the following conditions:

$$\begin{cases} \nu(S) \le \nu(T), \quad S \subset T \subseteq N & \text{(monotonicity)}, \\ \nu(S) = 0, \quad k \notin S \subset N & \text{(the boss property)}, \\ \nu(N) - \nu(S) \ge \sum\limits_{i \in N \setminus S} m_i^\nu, \ \ k \in S \subset N & \text{(the union property)}. \end{cases}$$

If we replace monotonicity by nonnegativity of the game ν and the vector m^ν then we obtain a nonmonotonic big boss game that is a particular case of a clan game (see [9]).

The extreme rays of the cones of clan games and nonmonotonic big boss games are simple games [9]. If monotonicity is added then this property no longer holds. Denote the polytope of (0,1)-normalized big boss games with player k as the boss by MB_k^n. The monotonicity implies the nonnegativity of the game $\nu \in \mathrm{MB}_k^n$. A nonredundant system for MB_k^n was obtained in [14]. It was proved that MB_k^3 and MB_k^4 are integer polytopes, and the binary relation $\nu \equiv \omega \ \leftrightarrow \ C(\nu) = C(\omega)$ on MB_k^n partitions all integer extreme points of this polytope into n equivalence classes. In [14], we described some types of integer extreme points, their Shapley values, and consensus values. We also considered the polytope $S\mathrm{MB}_k^n \subset \mathrm{MB}_k^n$ of the games symmetric with respect to the coalition $N \setminus k$. It was showed that its integer extreme points $\nu \in \mathrm{Ex}_{NI}S\mathrm{MB}_k^n$ satisfy

$$\nu(S) = \frac{n-2}{n-1}, \qquad S \ni k, \qquad |S| = n-1,$$

and there exists a bijection between $\mathrm{Ex}_{NI}S\mathrm{MB}_k^n$, $n \ge 4$, and the set

$$Y = \left\{y \in \mathbb{R}^{n-3} : y_i \in \{0,1\},\ i = \overline{2, n-2}\right\}$$

Table 2. Binary codes and the vectors g^y

(y_2, y_3, y_4)	$(0,0,0)$	$(0,\ 1,\ 0)$	$(1\ ,0,\ 1)$	$(1,\ 1,\ 1)$
$(g_2^y, g_3^y, g_4^y, g_5^y)$	$(0,0,0,4/5)$	$(0,2/5,2/5,4/5)$	$(1/5,1/5,3/5,4/5)$	$(1/5,2/5,3/5,4/5)$

of binary codes of length $n-3$. In [14], we proved the following characterization

Theorem 3. *We have*:

$$\mathrm{Ex}S\mathrm{MB}_k^n = \Big(\bigcup_{l=2}^{n-1} \nu^l\Big) \cup \Big(\bigcup_{y\in Y} \nu^y\Big),$$

where

$$\nu^l(S) = \begin{cases} 1, & (|S| \ge l) \wedge (S \ni k), \\ 0, & \textit{otherwise}, \end{cases} \qquad \nu^y(S) = \begin{cases} 1, & |S| = n, \\ 0, & (S \not\ni k) \vee (|S| = 1), \\ g_{|S|}^y, & \textit{otherwise}, \end{cases}$$

$$g_{|S|}^y \begin{cases} 0, & (y_2 = 0) \wedge (|S| = 2), \\ (|S|-1)/(n-1), & (y_{|S|} = 1) \vee (|S| = n-1), \\ g_{|S|-1}^y, & (y_{|S|} = 0) \wedge (S \in \Omega_{[3,n-2]}). \end{cases}$$

$\left|\mathrm{Ex}S\mathrm{MB}_k^n\right| = 2^{n-3} + n - 2.$

Table 2 contains some of the binary codes and the corresponding vectors g^y defining the extreme points of the polytope $S\mathrm{MB}_1^6$.

4. DUAL SIMPLEX (1-CONVEX) GAMES

The dual simplex (1-*convex*) games (see [7]) appear, for example, in collective insurance [6]. Let $1\mathrm{CO}^n$ denote the polytope of 1-convex, nonnegative, and $(0,1)$-normalized n-person games. It is defined by (1) and the following conditions: $\nu \in \mathbb{R}_+^{2^n-1}$,

$$\sum_{i\in N} \nu(N \setminus i) \le n-1, \tag{4}$$

$$\sum_{i\in N\setminus j} \nu(N \setminus i) \ge n-2, \quad j \in N, \tag{5}$$

$$-\nu(S) + \sum_{i\in N\setminus S} \nu(N \setminus i) \ge n - |S| - 1, \;\; S \in \Omega_{[2,n-2]}. \tag{6}$$

The core of a 1-convex game coincides with the nonempty dual imputations set, i.e., it is a simplex. The extreme points of the core are easily calculated. Some simple method (see [7]) is known for finding the τ-value, which is additive in the class of 1-convex games and also coincides with the nucleolus and the barycenter of the core. In [15], we described the Neumann—Morgenstern solutions to some 1-convex games. Each of them consists of the core and an complementary polyhedral set.

The polytope $1\mathrm{CO}^n$ has dimension $2^n - 2 - n$ since the values $\nu(S)$ for the empty, maximal, and one-element coalitions are fixed, and the symmetric game $\widehat{\nu}$ whose nonfixed components are defined by the formula

$$\widehat{\nu}(S) = \begin{cases} (n^2-n-1)/(n^2-1), & |S| = n-1, \\ 1/(n+1), & S \in \Omega_{[2,n-2]}, \end{cases}$$

belongs to $1\mathrm{CO}^n$ and satisfies all inequalities in (4)–(6) as strict ones. As far as the author knows, the structure of $1\mathrm{CO}^n$ has not been studied.

The following two theorems contain description of the sets of the integer extreme points $\mathrm{Ex}_I 1\mathrm{CO}^n$ and the noninteger extreme points $\mathrm{Ex}_{NI} 1\mathrm{CO}^n$ of this polytope:

Theorem 4. $\mathrm{Ex}_I 1\mathrm{CO}^n = \bigcup_{k\in N} E^k$, *where* E^k *is the set of the games defined by the formula*

$$\nu(S) = \begin{cases} 0, & (S \subseteq N \setminus k) \vee (S = \{k\}), \\ 1, & (|S| = n-1) \wedge (S \ni k) \vee (S = N), \\ 0 \ or \ 1, & otherwise. \end{cases} \tag{7}$$

$|\mathrm{Ex}_I 1\mathrm{CO}^n| = n \cdot 2^{(2^{n-1}-n-1)}$.

Proof. It follows from (4) and (5) that $\nu(N \setminus i) \le 1$ for all $i \in N$. Reckoning with (6), we have $\nu(S) \le 1$, $S \in \Omega_{[2,n-2]}$; i.e., $\mathrm{Ex}_I 1\mathrm{CO}^n$ consists of simple games $\nu \in 1\mathrm{CO}^n$. The games defined by (7) are simple and belong to $1\mathrm{CO}^n$; hence,

$$\bigcup_{k\in N} E^k \subseteq \mathrm{Ex}_I 1\mathrm{CO}^n.$$

Let $\overline{\nu} \in \mathrm{Ex}_I 1\mathrm{CO}^n$. Then $\overline{\nu}$ is a simple game. If $\overline{\nu}(N \setminus i) = 1$ for all $i \in N$ then $\overline{\nu}$ does not satisfy (4); a contradiction. Hence, there exists $p \in N$ such that $\overline{\nu}(N \setminus p) = 0$. Inequalities (5) corresponding to $j \in N \setminus p$ imply $\overline{\nu}(N \setminus i) = 1$ for all $i \in N \setminus p$. If $S \in \Omega_{[2,n-2]}$ and $S \not\ni p$ then the corresponding inequality in (6) (after inserting the values $\overline{\nu}(N \setminus i)$, $i \in N \setminus S$, therein) takes the form $\overline{\nu}(S) \le 0$. For such S, we have $\overline{\nu}(S) = 0$. If $S \in \Omega_{[2,n-2]}$ and $S \ni p$ then the corresponding inequality in (6) takes the form $\overline{\nu}(S) \le 1$. For such S, $\overline{\nu}(S)$ can be equal to 0 or 1. So we have $\overline{\nu} \in E^p$; i.e., the inclusion

$$\mathrm{Ex}_I 1\mathrm{CO}^n \subseteq \bigcup_{k\in N} E^k$$

is proved.

Every game $\nu \in E^k$, $k \in N$, has $2^{n-1} + n + 1$ fixed components (for the empty, maximal, one-element, $(n-1)$-element coalitions and coalitions $S \in \Omega_{[2,n-2]}$ not containing player k). Consequently, the number of nonfixed components is equal to $2^{n-1} - n - 1$. Nonfixed components can take two values, and the number of the sets E^k equals n. The second assertion of the theorem is proved.

Theorem 4 is proved. □

Corollary 1. $\mathrm{Ex}_I 1\mathrm{CO}^n$ *consists of nonconvex big boss games. All games* $\nu \in E^k$ *have singleton core*

$$C(\nu) = \{e^k\}, \qquad e^k \in \mathbb{R}^n, \qquad e_i^k = \begin{cases} 1, & i = k, \\ 0, & i \ne k. \end{cases}$$

Proof. Every game $\overline{\nu} \in E^k$ and the corresponding vector $m^{\overline{\nu}}$ are nonnegative. The boss property (for agent k) and the union property are fulfilled. Hence, the integer extreme points of $1\mathrm{CO}^n$ are big boss games among which there are monotonic and nonmonotonic games. Take $i, j \in N$, $i \ne j \ne k$. By (7),

$$\overline{\nu}(N \setminus i) = \overline{\nu}(N \setminus j) = 1, \qquad \overline{\nu}(i,j) = 0.$$

The inequality

$$\overline{\nu}(N \setminus i) + \overline{\nu}(N \setminus j) \le \overline{\nu}(N) + \overline{\nu}(i,j)$$

fails; i.e., the game $\overline{\nu}$ is nonconvex; $C(\overline{\nu}) = \mathrm{conv}\{e^j\}_{j\in \mathrm{Veto}(\overline{\nu})}$ since $\overline{\nu}$ is a simple game. The relation $\mathrm{Veto}(\overline{\nu}) = \{k\}$ implies $C(\overline{\nu}) = \{e^k\}$.

Corollary 1 is proved. □

Remark 1. In accordance with the unique core imputation of each game ν belonging to the convex hull of E^k, all profit of the cooperation goes to agent k (the boss) despite his zero own opportunities. In a big boss game this imputation, called *tyrannic*, coincides with the bargaining set, kernel, τ-value, AL-value, any selector of the core. In this case, the consensus value of the game or the coalition consensus value look "fairer" (see [5]).

Remark 2. The sets of the integer extreme points of the polytopes of convex and 1-convex games contain the big boss games. However, by Corollary 1, $\mathrm{Ex}_I\mathrm{CO}^n \cap \mathrm{Ex}_I 1\mathrm{CO}^n = \varnothing$.

Theorem 5. $\mathrm{Ex}_{NI}1\mathrm{CO}^n$ *consists of the games* ν*, where*

$$\nu(S) = \begin{cases} (|S|-1)/(n-1), & |S| \in \{1, n-1, n\}, \\ 0 \ \ or \ \ (|S|-1)/(n-1), & otherwise. \end{cases} \tag{8}$$

$|\mathrm{Ex}_{NI}1\mathrm{CO}^n| = 4^{(2^{n-1}-n-1)}$.

Proof. Each of the games defined in (8) belongs to $1\mathrm{CO}^n$ and satisfies the system of 2^n-2-n linearly independent equations

$$\sum_{i\in N\setminus j} \nu(N\setminus i) = n-2, \quad j \in N, \quad \nu(S) = 0, \qquad S \in \Omega^0 \subseteq \Omega_{[2,n-2]},$$
$$-\nu(S) + \sum_{i\in N\setminus S} \nu(N\setminus i) = n-|S|-1, \qquad S \in \Omega_{[2,n-2]} \setminus \Omega^0.$$

Consequently, all these games are contained in $\mathrm{Ex}_{NI}1\mathrm{CO}^n$. Take a game $\overline{\nu} \in \mathrm{Ex}_{NI}1\mathrm{CO}^n$ and suppose that it coincides with no game defined in (8). If there exists a coalition $T \in \Omega_{[2,n-2]}$ such that $\overline{\nu}(T) > 0$ and $\overline{\nu}$ satisfies inequality (6) corresponding to T as a strict one then the games $\dot{\nu}$ and $\ddot{\nu}$, where

$$\dot{\nu}(T) = \overline{\nu}(T) + \beta, \qquad \dot{\nu}(T) = \overline{\nu}(T) - \beta,$$
$$\dot{\nu}(S) = \ddot{\nu}(S) = \overline{\nu}(S), \qquad S \subseteq N\setminus T,$$
$$\beta = \min\Big\{\overline{\nu}(T), \sum_{i\in N\setminus T} \overline{\nu}(N\setminus i) - \overline{\nu}(T) - n + |T| + 1\Big\},$$

belong to $1\mathrm{CO}^n$ and $\overline{\nu} = (\dot{\nu} + \ddot{\nu})/2$; a contradiction. Hence,

$$\overline{\nu}(S)\Big(-\overline{\nu}(S) + \sum_{i\in N\setminus S} \overline{\nu}(N\setminus i) - n + |S| + 1\Big) = 0, \qquad S \in \Omega_{[2,n-2]}. \tag{9}$$

Consider the possible cases:

Case 1 : $\overline{\nu}(N\setminus p) = 0$ for some $p \in N$. Then, by (4) and (5), $\overline{\nu}(N\setminus i) = 1$ for all $i \in N\setminus p$. From (9) we deduce that $\overline{\nu}$ is an integer game; a contradiction.

Case 2 : $\overline{\nu}(N\setminus i) > 0$ for all $i \in N$.

Case 2.1 : $\overline{\nu}$ satisfies all inequalities in (5) as equalities. Then $\overline{\nu}$ coincides with one of the games defined by (8); a contradiction.

Case 2.2 : There exists $r \in N$ such that

$$\sum_{i\in N\setminus r} \overline{\nu}(N\setminus i) > n-2.$$

System (4)–(6) contains 2^n-n-1 inequalities and 2^n-n-2 variables. Therefore,

$$\sum_{i\in N} \overline{\nu}(N\setminus i) = n-1, \qquad \sum_{i\in N\setminus j} \overline{\nu}(N\setminus i) = n-2, \qquad j \in N\setminus r,$$

yielding $\overline{\nu}(N\setminus r) = 0$. This contradicts the assumption $\overline{\nu}(N\setminus i) > 0$, $i \in N$.

We have proved that $\overline{\nu}$ coincides with one of the games defined by (8). Each of these games has $2+2n$ fixed components. The number of nonfixed components is equal to $2(2^{n-1}-n-1)$, and each of them can take two values. The proof of the second claim of the theorem is complete.

Theorem 5 is proved. □

Corollary 2. *Let* $\nu \in \mathrm{Ex}_{NI}1\mathrm{CO}^n$. *Then*

$$C(\nu) = \mathrm{conv}\{g^r\}_{r\in N}, \qquad g_i^r = \begin{cases} 1/(n-1), & i \neq r, \\ 0, & i = r. \end{cases}$$

Proof. To the games defined by (8), there correspond the identical vectors m^ν, where $m_i^\nu = 1/(n-1)$ for $i \in N$. Inserting them in the known formulas for the extreme points of the dual imputations set $I^*(\nu)$ and reckoning with the equality $C(\nu) = I^*(\nu)$, we obtain the desired representation of the core. Corollary 2 is proved. □

Corollary 3. $\mathrm{Ex}_{NI}1\mathrm{CO}^n$ *contains the symmetric convex game* ν^1 *of Theorem* 2. *The game* ν^1 *is an exactification of each* $\nu \in \mathrm{Ex}_{NI}1\mathrm{CO}^n$.

Proof. To every balanced game ν, there corresponds a unique exact game ν^E with the same core as the initial one,

$$\nu^E(N) = \nu(N), \qquad \nu^E(S) = \min\{x(S) : x \in C(\nu)\}, \qquad S \in \Omega.$$

For the game $\nu \in \mathrm{Ex}_{NI}1\mathrm{CO}^n$, we have

$$\min_{x\in C(\nu)} x(S) = \frac{|S|-1}{n-1}, \qquad S \in \Omega;$$

i.e., $\nu^E = \nu^1$. □

Remark 3. The nonconvex balanced TU-games to which there corresponds a convex exact game have special properties. It follows from Corollary 3 that the lexicore of noninteger extremal games of the polytope $1\mathrm{CO}^n$ coincides with their core and the dual imputations set. The AL-values $AL(\nu)$ of all games $\nu \in \mathrm{Ex}_{NI}1\mathrm{CO}^n$ are equal to $AL(\nu^1)$ and also to the Shapley value of the convex game ν^1, i.e., belong to the core $C(\nu) = C(\nu^1)$. Moreover, all $\nu \in \mathrm{Ex}_{NI}1\mathrm{CO}^n$ have identical solutions invariant under game exactification.

Remark 4. Corollaries 1 and 2 imply that the binary relation $\nu \equiv \omega \leftrightarrow C(\nu) = C(\omega)$ partitions $\mathrm{Ex}1\mathrm{CO}^n$ into $n+1$ cosets; moreover, all noninteger extreme points belong to the same coset.

5. RELATIONS BETWEEN POLYTOPES

The systems defining the polytopes $1\mathrm{CO}^n$ and MB_k^n, $k \in N$, contain inequalities opposite to the convexity conditions. However, they intersect the polytope of convex games. It is proved in [4] that

$$\mathrm{CO}^n \cap 1\mathrm{CO}^n = \{\nu^1\},$$

where ν^1 is the symmetric game of Theorem 2. Thus, the only $(0,1)$-normalized game can be simultaneously 1-convex and convex. By Theorem 4 and Corollary 1, the polytope of 1-convex games intersects all polytopes of monotonic big boss games:

$$1\mathrm{CO}^n \cap \mathrm{MB}_k^n \neq \varnothing, \qquad k \in N.$$

Each unanimity game u_T of a two-element coalition $T = \{i,j\}$ is a clan game with clan T. It is not hard to check that u_T belongs to MB_i^n and MB_j^n; i.e., a big boss game can have two bosses. This property implying that all polytopes of monotonic big boss games intersect pairwise:

$$\mathrm{MB}_i^n \cap \mathrm{MB}_j^n \neq \varnothing, \qquad i,j \in N, \quad i \neq j,$$

is not reflected in the literature. The description for the integer points of the polytope MB_k^n in [3] implies that a game $\nu \in \mathrm{MB}_k^n$ can have at most two bosses. The convexity of the unanimity games u_T, $T = \{i,j\}$, $i \neq j$, and their membership in the class of big boss games confirms the well-known fact that

$$\mathrm{CO}^n \cap \mathrm{MB}_k^n \neq \varnothing, \qquad k \in N.$$

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