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A General Approach to the Calculation of Stability Radii for the Max-Cut Problem with Multiple Criteria

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Abstract—Under consideration is the multiobjective version of the maximum cut problem. The formulas together with the lower and upper exact bounds of stability radii are obtained for solutions of this problem as well as for the various types of stability of the problem under assumption that the Hölder metrics are given on the spaces of a disturbing parameter. It is proved that the problem of finding the radii of every type of stability is intractable unless P=NP.

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INTRODUCTION

Study of stability in multiobjective discrete optimization problems is carried out mainly in the two directions [10]: qualitative and quantitative.

The qualitative direction focuses on generating the conditions under which the set of the effective solutions to the problem has some inherent predetermined property that characterizes the problem stability to small perturbations of the initial data. Several results in this direction correspond to finding the necessary and sufficient conditions that guarantee various types of stability of the multiobjective Boolean and integer problems with various optimality principles (for example, see [6]).

The quantitative direction is connected with obtaining some bounds of the acceptable changes in the input data that preserve certain predetermined properties of optimal solutions (see [4, 9]), and the development of the algorithms for calculating these bounds (see [8, 12, 13]). The key concept here is the definition of stability radius which implies the radius of the largest neighborhood of the initial data in the parameter space of the problem for which every perturbed problem with the set of parameters from this neighborhood is in some sense "close" to the original problem.

This article belongs to the field of quantitative research. We deal with finding a general approach to study various types of stability of multiobjective combinatorial problems. This approach allows us to describe various types of stability and quantitative relationships between them. In this paper, we introduce some characteristics of the multiple-criteria maximum cut problem which show how two feasible solutions can be "close" to each other in the sense of stability. Basing on these characteristics, we formulate statements on stability radii. Also, we established the fact of intractability of the problem of finding the radii of every type of stability in the case of P≠NP. Note that in [2] and [3] the problem was under study of stability of solutions of the multiobjective version of the maximum cut problem in the case of an unlimited region of admissible perturbations.

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1. STATEMENT OF THE PROBLEM AND MAIN DEFINITIONS

Consider a simple connected marked (n,m)-graph $G=(V,E)$ with the sets of vertices $V=\{v_1,v_2,\dots,v_n\}$, $n\geq 3$, and edges $E=\{e_1,e_2,\dots,e_m\}$, $m\geq 2$. Let a partition of V be given into two nonempty subsets S and $\overline{S}$. Then the subset of the edges of G whose endvertices lie in different subsets is called a *cut* of the graph and is denoted by $(S,\overline{S})$.

If to each edge $\{v_i,v_j\}\in E$ there is assigned a negative number $w_{\{i,j\}}$, called the *weight* of $\{v_i,v_j\}$, then the single-criterion maximum cut problem of an undirected graph G consists of finding some cut $(S,\overline{S})$ such that the sum of all edge weights is maximized. This problem is easily reduced to a problem of Boolean quadratic programming. Namely,to each cut $(S,\overline{S})$ of G we associate the Boolean vector $x=(x_1,x_2,\dots,x_n)^\top\in\mathbb{E}^n=\{0,1\}^n$ with the entries

$$x_i=\begin{cases}1, & \text{if } v_i\in S,\\ 0, & \text{if } v_i\in\overline{S}.\end{cases}$$

It is clear that, for every vector $x\in X=E^n\setminus\{\mathbf{0},\mathbf{1}\}$, there is a cut $(S,\overline{S})$ with subsets $S=\{v_i\in V\mid x_i=1\}$ and $\overline{S}=\{v_i\in V\mid x_i=0\}$. In particular, $(\overline{S},S)$ corresponds to $\overline{x}=\mathbf{1}-x\in X$. Thus, each vector $x\in X$ can be naturally considered as a cut of the graph (or a feasible solution to the problem). In what follows, we will call $x\in X$ a *cut* or a *solution* using these words as synonyms.

To obtain a multiobjective version of the problem, to each edge $\{v_i,v_j\}\in E$ we assign the vector

$$\left(w^1_{\{i,j\}},w^2_{\{i,j\}},\dots,w^s_{\{i,j\}}\right)^\top,$$

where $w^k_{\{i,j\}}\in\mathbb{R}_+=[0,+\infty)$ is the weight of $\{v_i,v_j\}$ corresponding to the criteria $k\in N_s$. According to the enumeration of edges of G, from the m columns we form the matrix $W=\left[w^k_{\{i,j\}}\right]\in\mathbb{R}^{s\times m}_+$ with rows $W_k\in\mathbb{R}^m_+$, $k\in N_s=\{1,2,\dots,s\}$. It is easy that the quadratic function

$$f_k(x,W_k)=\sum_{\{v_i,v_j\}\in E} w^k_{\{i,j\}}(x_i-x_j)^2$$

on the set of cuts X is the total weight of the cut x by the criterion k. In result, we have the *s-criteria version of the maximum cut problem*

$$Z^s(W):f(x,W)=\big(f_1(x,W_1),f_2(x,W_2),\dots,f_s(x,W_s)\big)\ \to\ \max_{x\in X}$$

consisting in the finding the *Pareto set*; i.e., the *set of effective solutions* (the *effective cuts*) $P^s(W)=\{x\in X\mid \mathrm{Dom}(x,W)=\varnothing\}$, where

$$\mathrm{Dom}(x,W)=\left\{x'\in X\mid f(x,W)\leq f(x',W)\ \&\ f(x,W)\neq f(x',W)\right\}.$$

Since $f(x,W)=f(\overline{x},W)$, we have $x\in P^s(W)$ if and only if $\overline{x}\in P^s(W)$; so $|P^s(W)|$ is even. Considering the enumeration of edges of G, we continue to use a simpler method of indexing the elements of W; namely, we assume that $W=[w_{kl}]\in\mathbb{R}^{s\times m}_+$. Then $W_k=(w_{k1},w_{k2},\dots,w_{km})$, $k\in N_s$.

2. VARIOUS TYPES OF STABILITY OF THE PROBLEM AND OF SOLUTIONS

We will study the various types of stability of Problem $Z^s(W)$ and its solutions to perturbations of parameters (the elements of W) and the vector function $f(x,W)$. For this purpose, on the spaces of criteria $\mathbb{R}^s$ and solutions $\mathbb{R}^m$ we define, in general, different Hölder norms $\|\cdot\|_p$ and $\|\cdot\|_r$ respectively, where $p,r\in[1,\infty]$. Let us remind that the Hölder norm $\|\cdot\|_p$ of a vector $z=(z_1,z_2,\dots,z_d)$ of dimension $d\in\mathbb{N}$ is

$$\|z\|_p=\begin{cases}\sqrt[p]{\sum\limits_{i\in N_d}|z_i|^p}, & \text{if } 1\leq p<\infty,\\ \max\limits_{i\in N_d}|z_i|, & \text{if } p=\infty.\end{cases}$$

On the space of solutions $\mathbb{R}^m$, together with the norm $\|\cdot\|_r$ we consider the dual norm $\|\cdot\|_{r'}$, where r and r' are related by $1/r + 1/r' = 1$. By the norm of a matrix W, we understand the norm of the vector composed of the norms of rows of W; i.e., $\|W\| = \|(\|W_1\|_r, \|W_2\|_r, \dots, \|W_s\|_r)\|_p$. We will model a perturbation of the entries of W by adding to W some perturbing matrix

$$U \in \Omega(\varepsilon) = \{U \in \mathbb{R}^{s\times m} \mid W + U \in \mathbb{R}_+^{s\times m} \;\&\; \|U\| < \varepsilon\}, \qquad \varepsilon > 0.$$

In addition, we consider the five traditional types of stability of the problem [4, 6]:

Problem $Z^s(W)$ is called T_1-*stable* (*strongly stable* in terminology of [4]) if

$$\Xi_1 = \{\varepsilon > 0 \mid \forall U \in \Omega(\varepsilon)\ (P^s(W) \cap P^s(W+U) \neq \varnothing)\} \neq \varnothing,$$

T_2-*stable* (*strongly quasi-stable*) if

$$\Xi_2 = \{\varepsilon > 0 \mid \exists x^* \in X\ \forall U \in \Omega(\varepsilon)\ (x^* \in P^s(W+U))\} \neq \varnothing,$$

T_3-*stable* (*stable*) if

$$\Xi_3 = \{\varepsilon > 0 \mid \forall U \in \Omega(\varepsilon)\ (P^s(W+U) \subseteq P^s(W))\} \neq \varnothing,$$

T_4-*stable* (*quasi-stable*) if

$$\Xi_4 = \{\varepsilon > 0 \mid \forall U \in \Omega(\varepsilon)\ (P^s(W) \subseteq P^s(W+U))\} \neq \varnothing,$$

and T_5-*stable* (*stable*) if

$$\Xi_5 = \{\varepsilon > 0 \mid \forall U \in \Omega(\varepsilon)\ (P^s(W) = P^s(W+U))\} \neq \varnothing.$$

In what follows, we will denote the set of solutions to Problem $Z^s(W)$ which are not efficient (i.e., ineffective solutions) by $\overline{P}^s(W) = X \setminus P^s(W)$. Together with stability of the problem, the stability of solutions is also under study (for instance, see [2, 6, 9]).

An effective solution $x^0 \in P^s(W)$ is called *stable* if

$$\Theta_P = \{\varepsilon > 0 \mid \forall U \in \Omega(\varepsilon)\ (x^0 \in P^s(W+U))\} \neq \varnothing.$$

An ineffective solution $x^0 \in \overline{P}^s(W)$ is called *stable* if

$$\Theta_{\overline{P}} = \{\varepsilon > 0 \mid \forall U \in \Omega(\varepsilon)\ (x^0 \in \overline{P}^s(W+U))\} \neq \varnothing.$$

Note that the relationships between the stability of the solutions of a problem of integer linear programming (ILP) and its T_2–T_5-stability was established in [6]. In this article, we obtain similar relationships between the radii of stability of Problem $Z^s(W)$.

Assume that $\sup\varnothing = 0$. The T_i-*stability radius* $\rho_i(W)$ of Problem $Z^s(W)$ with $i \in N_5$, we call $\rho_i(W) = \sup \Xi_i$. Let Π be one of the sets $P^s(W)$ or $\overline{P}^s(W)$. We call $\rho(x^0, W) = \sup \Theta_\Pi$ the *stability radius of the solution* $x^0 \in \Pi$ of Problem $Z^s(W)$. By these definitions, the radius of T_i-stability of a problem is positive if and only if the problem is T_i-stable; and the radius of stability of a solution is positive if and only if the solution is stable. Note that if $P^s(W) = X$ then the sets Ξ_1 and Ξ_3 are unbounded; therefore, $\rho_1(W) = \rho_3(W) = +\infty$. Owing to the definitions of stability radii of solutions and problems, it is not difficult to understand that the following are true:

$$\rho_1(W) \geq \max\{\rho_2(W),\ \rho_3(W)\}, \qquad \rho_2(W) = \max_{x \in P^s(W)} \rho(x, W), \tag{1}$$

$$\rho_3(W) = \min_{x \in \overline{P}^s(W)} \rho(x, W), \qquad \rho_4(W) = \min_{x \in P^s(W)} \rho(x, W), \tag{2}$$

$$\rho_5(W) = \min_{x \in X} \rho(x, W) = \min\{\rho_3(W), \rho_4(W)\}. \tag{3}$$

3. STABILITY RADII OF SOLUTIONS

Let us introduce the characteristics that indicate how "far" the two admitted solutions of Problem $Z^s(W)$ are separated in the sense of a stability radius. Put

$$\delta^\star(x, x', W) = \big(\delta_1^\star(x, x', W_1),\ \delta_2^\star(x, x', W_2),\ \dots,\ \delta_s^\star(x, x', W_s)\big),$$

where $\star$ is either $\geq$ or $>$,

$$\delta_k^\star(x, x', W_k) = \sup\{\varepsilon > 0 \mid \forall u \in \Omega_k(\varepsilon)\ (f_k(x, W_k + u) \star f_k(x', W_k + u))\},$$

$$\Omega_k(\varepsilon) = \left\{u \in \mathbb{R}^m \mid W_k + u \in \mathbb{R}_+^m \,\&\, \|u\|_r < \varepsilon\right\}, \qquad k \in N_s.$$

It is obvious that, for every $k \in N_s$,

$$\delta_k^{\geq}(x, x', W_k) \geq \delta_k^{>}(x, x', W_k) \geq 0. \tag{4}$$

In Theorem 1 we reveal the specific content of $\delta_k^\star(x, x', W_k)$, $k \in N_s$, for Problem $Z^s(W)$. The general idea of this theorem is that the greatest perturbation of the inequality $f_k(x, W_k) \star f_k(x', W_k)$ occurs when the absolute values of certain entries of the vector u are as close to each other as possible.

In order to formulate Theorem 1 we introduce some notations: With every solution $x \in X$ and every edge $e_l = \{v_i, v_j\} \in E$ we associate the number $\gamma_l(x) = |x_i - x_j|$. It is easy that $\gamma_l(x) = \gamma_l(\overline{x})$. Given x and x', put

$$\sigma(x, x') = (\sigma_1(x, x'), \sigma_2(x, x'), \dots, \sigma_m(x, x'))^\top,$$

where $\sigma_l(x, x') = \gamma_l(x) - \gamma_l(x')$, $l \in N_m$. Given $x \in X$, put $N(x) = \{l \in N_m \mid \gamma_l(x) = 1\}$. Let

$$\mu(x, x', \alpha_k) = (\mu_1(x, x', \alpha_k), \mu_2(x, x', \alpha_k), \dots, \mu_m(x, x', \alpha_k))^\top,$$

where $\alpha_k \geq 0$ and

$$\mu_l(x, x', \alpha_k) = \begin{cases} 0, & \text{if } \ l \in N(x) \setminus N(x') \ \text{ and } \ w_{kl} < \alpha_k, \\ \sigma_l(x, x'), & \text{otherwise.} \end{cases}, \qquad l \in N_m.$$

If $f_{k^*}(x, W_{k^*}) > f_{k^*}(x', W_{k^*})$ for some $k^* \in N_s$ then $N(x) \not\subseteq N(x')$. Then the set $N(x) \setminus N(x')$ contains $t \geq 1$ elements. For each $k \in N_s$, arrange w_{kl}, $l \in N(x) \setminus N(x')$, in nondecreasing order: $0 \leq w_{kl_1} \leq w_{kl_2} \leq \dots \leq w_{kl_t}$. We assume also that $w_{kl_0} = 0$. Note that if $f_k(x, W_k) > f_k(x', W_k)$ then for all $i \in N_t$ we have

$$w_{kl_0} < \frac{W_k \mu(x, x', w_{kl_i})}{\|\mu(x, x', w_{kl_i})\|_1} \leq \frac{\sum\limits_{j=i}^{t} w_{kl_j}}{\|\mu(x, x', w_{kl_i})\|_1} \leq \frac{(t - i + 1) w_{kl_t}}{\|\mu(x, x', w_{kl_i})\|_1} \leq w_{kl_t}.$$

Therefore, we have

Proposition 1. *If $f_k(x, W_k) > f_k(x', W_k)$ then there exists a unique index $q \in N_t$ such that*

$$w_{kl_{q-1}} < \frac{W_k \mu(x, x', w_{kl_q})}{\|\mu(x, x', w_{kl_q})\|_1} \leq w_{kl_q}.$$

In what follows, we assume that $q = q(k)$ is the index defined by Proposition 1. Note that q can also be found from the equation

$$q = \arg\max\left\{\frac{\sum_{j=i}^{t} w_{kl_j}}{\|\mu(x, x', w_{kl_i})\|_1} \;\Big|\; i \in N_t\right\}.$$

Put

$$g_k(x, x', i) = \frac{W_k \mu(x, x', w_{kl_q})}{\|\mu(x, x', w_{kl_q})\|_i}$$

and let $\widetilde{W}_k$ be the vector with entries $\widetilde{w}_{kl}$, $l \in N_m$, where

$$\widetilde{w}_{kl} = \begin{cases} w_{kl}, & \text{if } l \in N(x) \setminus N(x') \text{ and } w_{kl} < g_k(x, x', 1), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1. *Given $x, x' \in X$, $k \in N_s$, and $r \in [1, \infty]$, we have*

$$\delta_k^{>}(x, x', W_k) = \begin{cases} \|(g_k(x, x', r'),\ \|\widetilde{W}_k\|_r)\|_r, & \text{if } f_k(x, W_k) > f_k(x', W_k), \\ 0, & \text{if } f_k(x, W_k) \le f_k(x', W_k), \end{cases}$$

$$\delta_k^{\geq}(x, x', W_k) = \begin{cases} \delta_k^{>}(x, x', W_k), & \text{if } N(x) \not\supseteq N(x'), \\ +\infty, & \text{if } N(x) \supseteq N(x'). \end{cases}$$

Proof. For brevity, denote $\|(g_k(x, x', r'),\ \|\widetilde{W}_k\|_r)\|_r$ by φ. Let us consider the three possible cases:

Case 1: $N(x) \supseteq N(x')$. Since $f_k(x, W_k) \ge f_k(x', W_k)$ for every vector $W_k \in \mathbb{R}^m_+$; therefore,

$$\delta_k^{\geq}(x, x', W_k) = +\infty.$$

Moreover, if $f_k(x, W_k) = f_k(x', W_k)$ then, obviously, $\delta_k^{>}(x, x', W_k) = 0$. Thereby, it remains to prove that $\delta_k^{>}(x, x', W_k) = \varphi$ for $f_k(x, W_k) > f_k(x', W_k)$. Note that in this event φ is the norm $\|\cdot\|_r$ of the vector composed from the elements w_{lk}, $l \in N(x) \setminus N(x')$. On the other hand, the vector $u^* \in \mathbb{R}^m$ with the entries

$$u_l^* = \begin{cases} -w_{lk}, & \text{if } l \in N(x) \setminus N(x'), \\ 0, & \text{otherwise} \end{cases}$$

has minimal norm among all vectors u that generate the equality $f_k(x, W_k + u) = f_k(x', W_k + u)$, and this norm is equal to φ. Consequently, $\delta_k^{>}(x, x', W_k) = \varphi$ if $f_k(x, W_k) > f_k(x', W_k)$.

Case 2: $N(x) \not\supseteq N(x')$ and $f_k(x, W_k) \le f_k(x', W_k)$. Then, obviously, $\delta_k^{>}(x, x', W_k) = 0$; and if $f_k(x, W_k) < f_k(x', W_k)$ then $\delta_k^{\geq}(x, x', W_k) = 0$. So, we need to show that for $f_k(x, W_k) = f_k(x', W_k)$ we have $\delta_k^{\geq}(x, x', W_k) = 0$.

Let $0 < \lambda < \varepsilon$ and $l_0 \in N(x') \setminus N(x)$. Define the entries u_l^*, $l \in N_m$, of u^* as

$$u_l^* = \begin{cases} \lambda, & \text{if } l = l_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|u^*\|_r = \lambda$, $u^* \in \Omega_k(\varepsilon)$, and

$$f_k(x, W_k + u^*) = f_k(x, W_k) = f_k(x', W_k) < f_k(x', W_k) + \lambda = f_k(x', W_k + u^*).$$

Hence, for all $\varepsilon > 0$, there is a perturbing vector $u^* \in \Omega_k(\varepsilon)$ such that $f_k(x, W_k + u^*) < f_k(x', W_k + u^*)$; i.e., $\delta_k^{\geq}(x, x', W_k) = 0$.

Case 3: $N(x) \not\supseteq N(x')$ and $f_k(x, W_k) > f_k(x', W_k)$. Note that in this case the sets $N(x) \setminus N(x')$ and $N(x') \setminus N(x)$ are nonempty. Using this, prove that $\delta_k^{>}(x, x', W_k) = \delta_k^{\geq}(x, x', W_k) = \varphi$.

Consider the perturbing vector $u^* = u^*(\lambda) \in \mathbb{R}^m$ with entries $u_l^*(\lambda)$, $l \in N_m$, such that

$$u_l^*(\lambda) = \begin{cases} -w_{kl}, & \text{if } l \in N(x) \setminus N(x') \text{ and } w_{kl} < g_k(x, x', 1), \\ -g_k(x, x', 1), & \text{if } l \in N(x) \setminus N(x') \text{ and } w_{kl} \ge g_k(x, x', 1), \\ g_k(x, x', 1) + \lambda, & \text{if } l \in N(x') \setminus N(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then, given $\lambda \geq 0$, we obtain

$$W_k + u^*(\lambda) \in \mathbb{R}^m_+, \tag{5}$$

$$\begin{aligned} f_k(x, W_k + u^*) - f_k(x', W_k + u^*) &= f_k(x, W_k) - f_k(x', W_k) \\ &- \sum_{i=0}^{q-1} w_{kl_i} - g_k(x, x', 1)\|\mu(x, x', w_{kl_q})\|_1 - \lambda|N(x') \setminus N(x)| \\ &= W_k\mu(x, x', w_{kl_q}) - g_k(x, x', 1)\|\mu(x, x', w_{kl_q})\|_1 - \lambda|N(x') \setminus N(x)|. \end{aligned}$$

Therefore, if $\lambda = 0$ then $f_k(x, W_k + u^*) = f_k(x', W_k + u^*)$; and, by $N(x') \setminus N(x) \neq \varnothing$, for all $\lambda > 0$, we have

$$f_k(x, W_k + u^*) < f_k(x', W_k + u^*). \tag{6}$$

On the other hand, since $g_k(x, x', 1)\|\mu(x, x', w_{kl_q})\|_r = g_k(x, x', r')$, we find

$$\|u^*(0)\|_r = \left\|\left(g_k(x, x', 1)\|\mu(x, x', w_{kl_q})\|_r, \|\widetilde{W}_k\|_r\right)\right\|_r = \varphi.$$

Moreover, it is clear that, for all $\varepsilon > \varphi$, there is $\lambda^* > 0$ such that $u^*(\lambda^*) \in \Omega_k(\varepsilon)$.

Hence, taking into account (5) and (6), we conclude that for every $\varepsilon > \varphi$ there exists $u^* \in \Omega_k(\varepsilon)$ generating the inequality $f_k(x, W_k + u^*) < f_k(x', W_k + u^*)$.

Consequently, $\delta_k^{\geq}(x, x', W_k) \leq \varphi$, which, by (4), implies

$$\delta_k^{>}(x, x', W_k) \leq \delta_k^{\geq}(x, x', W_k) \leq \varphi.$$

It remains to show that $\delta_k^{\geq}(x, x', W_k) \geq \delta_k^{>}(x, x', W_k) \geq \varphi$. For this purpose, owing to (4), it suffices to prove that $\delta_k^{>}(x, x', W_k) \geq \varphi$. Since for $r = 1$ (i.e., in the case of metrics $\|\cdot\|_1$ on $\mathbb{R}^m$) we have

$$\varphi = \|(W_k\mu(x, x', w_{kl_q}), \|\widetilde{W}_k\|_1)\|_1 = f_k(x, W_k) - f_k(x', W_k);$$

therefore, for every $u \in \Omega_k(\varphi)$ we infer that

$$\begin{aligned} f_k(x, W_k + u) - f_k(x', W_k + u) &= f_k(x, W_k) - f_k(x', W_k) + f_k(x, u) - f_k(x', u) \\ &\geq f_k(x, W_k) - f_k(x', W_k) - \|u\|_1 > f_k(x, W_k) - f_k(x', W_k) - \varphi = 0. \end{aligned}$$

Consequently, $\delta_k^{>}(x, x', W_k) \geq \varphi$ for $r = 1$.

We now prove the inequality $\delta_k^{>}(x, x', W_k) \geq \varphi$ for $r \in (1, \infty]$. Suppose the contrary. Among all $u' \in \Omega_k(\varphi)$ satisfying $f_k(x, W_k + u') \leq f_k(x', W_k + u')$, we choose u^0 having the minimal norm.

Moreover, without loss of generality, we assume that $u_l^0 \leq 0$ for $l \in N(x) \setminus N(x')$, while $u_l^0 \geq 0$ for $l \in N(x') \setminus N(x)$; since, otherwise, it can be replaced by the elements of opposite signs, preserving both the vector norm and the inequality $f_k(x, W_k + u^0) \leq f_k(x', W_k + u^0)$.

Let $\widetilde{u}^0$ be the vector with entries $u_{l_i}^0$, $i \in \{0, 1, \ldots, q-1\}$, where $u_{l_0}^0$ equals zero. Let $\widehat{u}^0$ be obtained of the vector u^0 whose all entries $u_{l_i}^0$ with $i \in N_{q-1}$ are replaced with zeros. Then $\|u^0\|_r = \|(\|\widehat{u}^0\|_r, \|\widetilde{u}^0\|_r)\|_r$, and therefore, since $\|u^0\|_r < \varphi$, at least one of the inequalities $\|\widehat{u}^0\|_r < g_k(x, x', r')$ or $\|\widetilde{u}^0\|_r < \|\widetilde{W}_k\|_r$ has to be true. However, the first one is not available. Moreover, we have

$$\|\widehat{u}^0\|_r > g_k(x, x', r') \tag{7}$$

because otherwise, by the Hölder inequality, we deduce

$$\begin{aligned} f_k(x, W_k + u^0) - f_k(x', W_k + u^0) &= \sum_{i=0}^{q-1} \left(w_{kl_i} + u_{l_i}^0\right) + (W_k + \widehat{u}^0)\mu(x, x', w_{kl_q}) \\ &\geq \sum_{i=0}^{q-1} \left(w_{kl_i} + u_{l_i}^0\right) + W_k\mu(x, x', w_{kl_q}) - \|\widehat{u}^0\|_r\|\mu(x, x', w_{kl_q})\|_{r'} > 0. \end{aligned}$$

Inequality (7) indicates that there is some $h \in N_{q-1} \neq \varnothing$ such that $u^0_{l_h} > -w_{kl_h}$; and, among the entries of $\widehat{u}^0$, there is at least one with the absolute value greater than $g_k(x, x', 1)$. Let M be the set of the entries of $\widehat{u}^0$ with the maximum absolute value. To each of these elements, we add or subtract some $\alpha > 0$ so that their absolute values decrease; however, they still belong to M. Then $u^0_{l_h}$ reduces by β, where $0 < \beta \leq w_{kl_h} + u^0_{l_h}$. Hence, we can construct the vector $u^* = u^*(\alpha, \beta)$. Since $w_{kl_h} < g_k(x, x', 1)$, we can choose α and β so that

$$\|u^0\|_r > \|u^*\|_r, \qquad f_k(x, u^0) - f_k(x', u^0) \geq f_k(x, u^*) - f_k(x', u^*).$$

This means that u^* satisfies $f_k(x, W_k + u^*) \leq f_k(x', W_k + u^*)$ and has a norm less than that of u^0. This contradiction has convinced us of the validity of inequality $\delta^>_k(x, x', W_k) \geq \varphi$ for $r \in (1, \infty]$.

The proof of Theorem 1 is complete. □

Given some real $y \in \mathbb{R}$, put $[y]^+ = \max\{0, y\}$.

Note that in some cases, the formula of $\delta^>_k(x, x', W_k)$ is essentially simplified. In particular, the two corollaries are valid:

Corollary 1. *If each row W_k, $k \in N_s$, of the matrix W consists of equal numbers then*

$$\delta^>_k(x, x', W_k) = \frac{[W_k\sigma(x, x')]^+}{\|\sigma(x, x')\|_{r'}}.$$

Corollary 2. *If $\mathbb{R}^m$ is endowed with the norm $\|\cdot\|_1$ then, for every $k \in N_s$, we have*

$$\delta^>_k(x, x', W_k) = [W_k\sigma(x, x')]^+.$$

The connection between the values $\delta^\star_k(x, x^0, W_k)$, $k \in N_s$, and the stability radii of the solutions of Problem $Z^s(W)$ is described by the following easy verifiable statements:

Proposition 2. *Let $x^0 \in P^s(W)$. Then*

$$\rho(x^0, W) = \min_{x \in X \setminus \{x^0, \overline{x^0}\}} \|\delta^{\geq}(x^0, x, W)\|_p.$$

Proposition 3. *Let $x^0 \in \overline{P}^s(W)$. Then*

$$\rho(x^0, W) \geq \max_{x \in \mathrm{Dom}(x^0, W)} \min\Big\{ \min_{k \in N_s} \delta^{\geq}_k(x, x^0, W_k),\ \|\delta^>(x, x^0, W)\|_p \Big\}.$$

We have to note that, from a theoretical point of view, the determination of the stability radius of an effective solution is simpler than in the case of an ineffective solution. This is caused by the fact that for loss of optimality of an effective solution of the original problem $Z^s(W)$ it suffices to have one dominant solution of the perturbed problem $Z^s(W + U)$. But an ineffective solution of $Z^s(W)$ becomes effective only if all its dominating solutions ceased to dominate in the perturbed problem $Z^s(W + U)$.

Of course, this remark is true not only for the problem under study, but also for every general multiobjective optimization problem with a Pareto optimality principle. That is why most results on determination of the stability radius of the effective solutions (as well as the T_2- and T_4-stability of a problem) are exact formulas (for instance, see [2] and [9]), while the results on the T_3-stability of the problem have only achievable estimates (for example, see [3] and [4]).

It is obvious that the effective solution x^0 has the infinite radius of stability if and only if $N(x^0) \supseteq N(x)$ for all $x \in X$; i.e., the cut x^0 contains all edges of the graph. It means that the infinite radius of stability can only be in one pair of effective solutions: x^0 and $\overline{x^0}$, and this is possible only in a bipartite graph. The ineffective solutions always have finite radius of stability. Moreover, we have

Theorem 2. *Let $x^0 \in P^s(W)$ and let at least one solution $x^* \in X$ be such that $N(x^0) \not\supseteq N(x^*)$. Then $\rho(x^0, W) \leq \|W\|$.*

Let $x^0 \in \overline{P}^s(W)$. *Then* $\rho(x^0, W) \le \|W\|$. *Moreover, if for every solution* $x \in \mathrm{Dom}(x^0, W)$ *we have* $N(x^0) \not\supseteq N(x)$ *then*

$$\rho(x^0, W) \le \min_{k \in N_s} \|W_k\|_r.$$

Proof. Consider $x^0 \in P^s(W)$. Since $N(x^0) \not\supseteq N(x^*)$, there exists some $l^* \in N(x^*) \setminus N(x^0)$. Therefore, it is possible to construct the perturbing matrix $U^0 = U^0(\lambda)$ with the entries

$$u^0_{kl} = \begin{cases} \lambda - w_{kl}, & \text{if} \quad l = l^*, \quad k \in N_s, \\ -w_{kl}, & \text{if} \quad l \in N_m \setminus \{l^*\}, \quad k \in N_s. \end{cases}$$

Then, for every $\lambda > 0$, we have

$$f_k\big(x^0, W_k + U^0_k\big) = 0 < \lambda = f_k\big(x^*, W_k + U^0_k\big)$$

for each index $k \in N_s$. These show that $x^0 \notin P^s(W + U^0)$. It is also easy that, for every $\varepsilon > \|W\|$, we can find a positive λ^* such that $U^*(\lambda^*) \in \Omega(\varepsilon)$. Thus, $\rho(x^0, W) \le \|W\|$.

Let $x^0 \in \overline{P}^s(W)$. The inequality $\rho(x^0, W) \le \|W\|$ is easy to check if we take $-W$ as the disturbing matrix U. It remains to consider the case when, given an arbitrary solution $x \in \mathrm{Dom}(x^0, W)$, we show that $N(x^0) \not\supseteq N(x)$.

Let $k^* \in N_s$ be the index at which the minimum of $\|W_k\|_r$ is attained. Let us construct the perturbing matrix $U^* = U^*(\lambda)$ with entries

$$u^*_{kl} = \begin{cases} -w_{kl} + \lambda, & \text{if} \quad l \in N(x^0), \quad k = k^*, \\ -w_{kl}, & \text{if} \quad l \in N_m \setminus N(x^0), \quad k = k^*, \\ 0, & \text{otherwise}. \end{cases}$$

Then $x^0 \in P^s(W + U^*(\lambda))$ for every $\lambda > 0$. Moreover, it is obvious that, for every

$$\varepsilon > \min_{k \in N_s} \|W_k\|_r,$$

there is some positive λ^* such that $U^*(\lambda^*) \in \Omega(\varepsilon)$. Consequently,

$$\rho(x^0, W) \le \min_{k \in N_s} \|W_k\|_r.$$

The proof of Theorem 2 is complete. □

4. STABILITY RADII OF THE PROBLEM

Using Theorems 1 and 2, Propositions 2 and 3, together with (1) and (2), it is easy to obtain the following formula and estimates for the radii of T_1–T_5-stability of Problem $Z^s(W)$:

Theorem 3. *Let* $P^s(W) \ne X$. *Then*

$$\rho_1(W) \ge \max_{x \in P^s(W)} \min_{x' \in \overline{P}^s(W)} \|\delta^{\ge}(x, x', W)\|_p,$$

$$\rho_1(W) \ge \min_{x \in \overline{P}^s(W)} \max_{x' \in \mathrm{Dom}(x, W)} \min_{k \in N_s} \delta^{\ge}_k(x', x, W_k).$$

Wherein $\rho_1(W) = +\infty$ *if and only if for every* $x \in \overline{P}^s(W)$ *there exists* $x^* \in P^s(W)$ *such that* $N(x^*) \supset N(x)$. *Otherwise,* $\rho_1(W) \le \|W\|$.

Theorem 4. *We have*

$$\rho_2(W) = \max_{x \in P^s(W)} \min_{x' \in X \setminus \{x, \overline{x}\}} \|\delta^{\ge}(x, x', W)\|_p.$$

Moreover, $\rho_2(W) = +\infty$ *if and only if there exists* $x^0 \in P^s(W)$ *such that* $N(x^0) \supseteq N(x)$ *for all* $x \in X$. *Otherwise,* $\rho_2(W) \leq \|W\|$.

Theorem 5. *Let* $P^s(W) \neq X$. *Then* $\rho_3(W) \leq \|W\|$ *and*

$$\rho_3(W) \geq \min_{x \in \overline{P}^s(W)} \max_{x' \in \mathrm{Dom}(x,W)} \min\Big\{ \min_{k \in N_s} \delta_k^{\geq}(x', x, W_k),\ \|\delta^{>}(x', x, W)\|_p \Big\}.$$

Moreover,

$$\rho_3(W) \leq \min_{k \in N_s} \|W_k\|_r$$

if and only if there is a solution $x^* \in \overline{P}^s(W)$ *such that* $N(x) \not\supseteq N(x^*)$ *for all* $x \in \mathrm{Dom}(x^*, W)$.

Theorem 6. *We have*

$$\rho_4(W) = \min_{x \in P^s(W)} \min_{x' \in X \setminus \{x, \overline{x}\}} \|\delta^{\geq}(x, x', W)\|_p.$$

Moreover, $\rho_4(W) = +\infty$ *if and only if* $P^s(W) = \{x^0, \overline{x^0}\}$ *and* $N(x^0) \supseteq N(x)$ *for all* $x \in X$. *Otherwise,* $\rho_4(W) \leq \|W\|$.

Note that, by (3) and Theorems 5 and 6, it is also easy to obtain some estimates for the radius of T_5-stability of Problem $Z^s(W)$.

Note also that, in Theorems 3 and 5, instead of $\mathrm{Dom}(x, W)$ we can use $\mathrm{Dom}(x, W) \cap P^s(W)$, which does not change the values of the lower bounds for the corresponding radii of stability, but may decrease combinatorial exhaustion in their calculation.

If $P^s(W) = \{x^0, \overline{x^0}\}$ then we can specify the exact formulas for all types of stability of Problem $Z^s(W)$. Since in this case the radii of T_1-, T_2-, and T_4-stability are equal; therefore, by Theorem 6, we have

Corollary 3. *If* $P^s(W) = \{x^0, \overline{x^0}\}$ *then*

$$\rho_1(W) = \rho_2(W) = \rho_4(W) = \rho(x^0, W) = \min_{x \in X \setminus \{x^0, \overline{x^0}\}} \|\delta^{\geq}(x^0, x, W)\|_p.$$

Corollary 4. *If* $P^s(W) = \{x^0, \overline{x^0}\}$ *then*

$$\rho_3(W) = \rho_5(W) = \min_{x \in X \setminus \{x^0, \overline{x^0}\}} \min\Big\{ \min_{k \in N_s} \delta_k^{\geq}(x^0, x, W_k),\ \|\delta^{>}(x^0, x, W)\|_p \Big\}.$$

Proof. For brevity, denote the right-hand side of the formula of Corollary 4 by ψ. By the definitions of $\delta^{>}(x^0, x, W)$ and $\delta_k^{\geq}(x^0, x, W_k)$, we have $f(x, W + U) \neq f(x^0, W + U)$ for every $x \in X \setminus \{x^0, \overline{x^0}\}$ and every perturbing matrix $U \in \Omega(\psi)$; and the inequality $f_k(x, W_k + U_k) > f_k(x^0, W_k + U_k)$ is false for any $k \in N_s$. Therefore, $\rho_3(W) \leq \psi$, which, by Theorem 5, gives $\rho_3(W) = \psi$.

On the other hand, owing to Corollary 3, it is easy that $\rho_4(W) \geq \psi$. Thereby, it follows from (3) that $\rho_3(W) = \rho_5(W)$. This completes the proof. $\square$

In the single-criterion case, Theorems 4–6 turn out into the following propositions:

Corollary 5. *Let* $s = 1$. *If there exists the solution* $x^0 \in P^1(W)$ *such that for every* $x \in P^1(W)$ *the inclusion* $N(x^0) \supseteq N(x)$ *is true then*

$$0 < \rho_2(W) = \rho(x^0, W) = \min_{x \in \overline{P^1}(W)} \delta_1^{\geq}(x^0, x, W_1).$$

Otherwise, $\rho_2(W) = 0$.

Corollary 6. *Let* $s = 1$ *and* $P^1(W) \neq X$. *Then*

$$0 < \min_{x \in \overline{P^1}(W)} \max_{x' \in P^1(W)} \delta_1^{>}(x', x, W_1) \leq \rho_3(W) \leq \|W_1\|_r.$$

Corollary 7. *Let* $s = 1$. *If* $P^1(W) = \{x^0, \overline{x^0}\}$ *then*

$$0 < \rho_4(W) = \rho(x^0, W) = \min_{x \in X \setminus \{x^0, \overline{x^0}\}} \delta_1^{\geq}(x^0, x, W_1).$$

Otherwise, $\rho_4(W) = 0$.

By Corollaries 6 and 7, we conclude that $\rho_3(W) \leq \rho_4(W)$ for $P^1(W) = \{x^0, \overline{x^0}\}$. Therefore, we have

Corollary 8. *Let* $s = 1$. *If* $P^1(W) = \{x^0, \overline{x^0}\}$ *then*

$$0 < \rho_5(W) = \min_{x \in X \setminus \{x^0, \overline{x^0}\}} \delta_1^{>}(x^0, x, W_1) \leq \|W_1\|_r.$$

Otherwise, $\rho_5(W) = 0$.

Let us note that all results of Sections 3 and 4 are valid also for Problem $Z^s_{\text{mod}}(W)$, where $W \in \mathbb{R}^{s \times m}$, with the set of perturbation matrices

$$\Omega(\varepsilon) = \left\{U \in \mathbb{R}^{s \times m} \mid \|U\| < \varepsilon\right\}$$

and the particular criteria of the form MAXSUM MODUL:

$$f_k(x, W_k) = \sum_{l \in N_m} |w_{kl}| x_l \ \rightarrow \ \max_{x \in X}, \quad k \in N_s.$$

5. COMPLEXITY OF CALCULATION OF STABILITY RADII

Following [1], we call a problem *intractable* if there is no polynomial algorithm of solution. In [13], the single-criterion problem was considered of finding some acceptable changes of the edge weights under which the optimal solution selected in advance remains optimal. It was proved that, for a wide class of combinatorial problems assuming P≠NP, it is impossible to construct a polynomial-time algorithm for finding these acceptable changes. We show that, despite the fact that the stability radii give less information about the tolerance of the edges, the problem of their finding is intractable for P≠NP either.

Let $Z^1(\mathbf{1})$ be some single-criterion problem of searching the maximum cut in the graph G whose each edge has unit weight. It is well known that Problem $Z^1(\mathbf{1})$ is NP-hard. Moreover, in [11], the NP-completeness is found of Problem $Z^1_{\text{in}}(x, \mathbf{1})$ inverse to $Z^1(\mathbf{1})$ and consisting in verification whether this cut x be maximal.

The following algorithm shows how we can find a solution of $Z^1(\mathbf{1})$ by calculating the stability radii of optimal solutions for the sequence that consists of at most m^2 Problems $Z^1(W)$. During the execution of the algorithm, the edge weights will be modified until each of them becomes equal 1. Let us denote this variable vector of weights by w.

Algorithm 1. The reduction of Problem $Z^1(\mathbf{1})$ to the problem of finding the stability radius of the optimal solution.

Step 0. Choose a cut x^0 such that

$$\forall x \in X \ (N(x) \not\supset N(x^0)). \tag{8}$$

If $N(x^0) = N_m$ then Problem $Z^1(\mathbf{1})$ is solved and x^0 is its maximum cut.

Step 1. Put the weights of the edges of the cut x^0 equal to 1, and the weight of the other edges, 0 (in this case, $\rho(x^0, w)$ will be positive).

Step 2. If $\rho(x^0, w) > 0$ and there is at least one edge of zero weight in Problem $Z^1(w)$ then pass to Step 3; otherwise, to Step 4.

Step 3. Choose arbitrarily an edge with zero weight and give it the weight equal to 1. Pass to Step 2.

Step 4. If there is no edges with zero weight in Problem $Z^1(w)$ then $Z^1(\mathbf{1})$ is solved and x^0 is a maximum cut.

Step 5. Choose arbitrarily an edge e with zero weight. Verify whether increasing the weight of e to 1 implies the optimality loss of the cut x^0 or not. If yes, then pass to Step 6. If not, then put the weight of e equal to 1 and go to Step 4.

Step 6. Assuming that the edge e has weight 1, we find a new maximum cut x^*. Then the weight of all edges of this cut (including the edge e) put equal to 1. Renaming x^* by x^0, pass to Step 2.

The estimates for the stability radii in Propositions 2 and 3 allow us to conclude that if the edges of the graph have weights 0 or 1 then any positive stability radius is at least $1/m$ for any norm $\|\cdot\|_r$, $r \in [1, \infty]$. This we will use in the proof of Theorems 7 and 8 without further notice.

Theorem 7. *For every norm $\|\cdot\|_r$ on $\mathbb{R}^m$, Algorithm 1 solves Problem $Z^1(\mathbf{1})$ with $O(m^2\zeta)$ operations, where $O(\zeta)$ is the complexity of calculating the stability radii of an optimal solution.*

Proof. First of all, we verify that Step 0 can be performed in every Problem $Z^1(\mathbf{1})$; in other words, it is always possible to choose some cut x^0 satisfying (8). To this end, select in G a spanning tree T and paint all its tops with white and black colors. Let each vertex obtaining odd label by searching in the width be painted white, and each vertex with even label, with black. Such a partition of the graph vertices defines the cut x^0 that contains all edges of T (among them). Moreover, each cut $x \in X \setminus \{x^0, \overline{x^0}\}$ defines repainting of the tops such that at least one edge e of the tree T is incident to the two vertices of the same color. This means that e is not included into the cut x. Thus, the cut x^0 is not included in any other cut as proper subset; i.e., x^0 meets (8). At that, Step 0 can be done in $O(m)$ operations.

Implementation of Steps 1–4 does not cause difficulties. Let us show now how we can perform Step 5. Let w^0 denote the vector of weights of the edges which are formed by time of Step 5 execution; and let w^1 be the vector w^0 in which we assume the weight of the edge e to be 1. At the same time $\rho(x^0, w^0) = 0$. We have to check whether the cut $x^0 \in P^1(w^0)$ is maximal in Problem $Z^1(w^1)$. Since Step 3 was still performed at least once, there is $m' \geq 1$ edges with weight 1 not belonging to x^0. We decrease the weights of all such edges by $1/2m$. Then, since x^0 satisfies (8); by Proposition 2 the stability radius $\rho(x^0, w)$ is positive. Calculate $\rho(x^0, w)$. Then we will increase the weight of e by $1/2m$ step by step until the stability radius be reduced or the weight of e be greater than $m'/2m$.

Consider the two possible cases: Note that, since $\rho(x^0, w^0) = 0$, by Proposition 2, the set of equivalent solutions $Q(x^0, w^0) = \{x \in X \mid f(x^0, w^0) = f(x, w^0)\}$ includes some cut different from x^0 and $\overline{x^0}$.

Case 1: The weight of e became equal to $m'/2m$, and the stability radius did not change. Then the edge e does not belong to any cut of $Q(x^0, w^0)$. Therefore, the increase of the weight of e to 1 will preserve the inequality $f(x^0, w) \geq f(x, w)$ for all $x \in X$. Thus, the cut x^0 is still maximal in Problem $Z^1(w^1)$. Put w equal to w^1 and go to Step 4.

Case 2: The weight of e became equal to $m^*/2m \leq m'/2m$, and the stability radius decreased. Then e belongs to at least one cut of $Q(x^0, w^0) \setminus \{x^0, \overline{x^0}\}$. Hence, every increase of the weight of e in Problem $Z^1(w^0)$ results in the optimality loss of x^0. Go to Step 6 without any modification of w.

It is easy that Step 5 can be done in $O(m\zeta)$ operations.

Finally, let us discuss the execution of Step 6. If we came to this step then the cut x^0 is not maximal in Problem $Z^1(w^1)$ anymore. Show how we can find a new maximal cut x^* in this event. Even so, we will keep the optimality of x^0.

Attach the weight $(m^* - 1)/2m$ to e and keep the weights of all other edges the same as after execution of Step 5. Since e belongs to every maximal cut in Problem $Z^1(w^1)$, we include e in x^*. Thereafter, we will check each of the edges $e' \in E \setminus \{e\}$ whether it belongs to x^*. To this end, we will use the next property of Problem $Z^1(w)$: Let the edge e' belong to maximal cut in Problem $Z^1(w^1)$. Then a sufficiently small increase of its weight implies some decrease of the stability radius $\rho(x^0, w)$ if e' did not belong to x^0. Otherwise, the stability radius $\rho(x^0, w)$ remains the same.

First, we check the edges that do not belong to x^0. If the stability radius decreases under increasing an edge weight by $1/2m^2$ then we include this edge into x^* and keep its weight increased; and if the stability radius does not change then we return the initial weight to the edge and do not include it into x^*.

Next, we check the edges of the cut x^0. If the stability radius does not change under increasing an edge weight by $1/2m^2$ then we include this edge into x^* and keep its weight increased; and if the stability radius increases then we return the initial weight to the edge and do not include it into x^*.

It is easy to understand that the cut x^*, obtained in result of checking all edges from $E \setminus \{e\}$, is maximal in Problem $Z^1(w^1)$ and it is not included in any other cut as a proper subset. Therefore, we can use x^* further as the cut x^0 in Algorithm 1. Obviously, Step 6 can be done in $O(m\zeta)$ operations.

When all edges gain weight 1, we will find a maximal cut in Problem $Z^1(\mathbf{1})$. Since at least one of the edges gains weight 1 under transition from the step with a higher number to the step with a lower number, such a transition will be done at most m times in the process of Algorithm 1 execution. It is easy to see that the most time-consuming stages of Algorithm 1 are Steps 5 and 6 of complexity $O(m\zeta)$. Thus, Algorithm 1 has complexity $O(m^2\zeta)$.

The proof of Theorem 7 is complete. □

Corollary 9. *If* P≠NP *then the problems of finding the radii of* T_2-, T_4-, *and* T_5-*stability of the single-criterion Problem* $Z^1(W)$ *are intractable.*

Indeed, by Theorem 7, to prove Corollary 9 we have only to ascertain that in Algorithm 1 we can use $\rho_2(w)$, $\rho_4(w)$, and $\rho_5(w)$ instead of $\rho(x^0, w)$. Obviously, the cut x^0 meets the condition (8) and none of the maximal cuts $x \in P^1(w) \setminus \{x^0, \overline{x^0}\}$ is included into x^0. Therefore, owing to Corollaries 5, 7, and 8, we conclude that

$$\rho(x^0, w) = \rho_2(w) = \rho_4(w) = \rho_5(w).$$

The next algorithm shows how we can solve Problem $Z^1_{\text{in}}(x, \mathbf{1})$ by finding the stability radii of the nonoptimal solutions for a series of at most m Problems $Z^1(W)$. Over the execution of the algorithm, the weights of the edges will be modified until each of them becomes equal to 1. As before, we will denote this varying vector of weights by w.

Algorithm 2. Reduction of Problem $Z^1_{\text{in}}(x, \mathbf{1})$ to the problem of finding a stability radius of a nonoptimal solution.

Step 0. If cut x^0 includes all edges of the graph then x^0 is maximal. *Stop.*

Step 1. To each edge of x^0 we attach the zero weight, while to each of the other edges, the weight 1.

Step 2. If there is at least one edge with zero weight in Problem $Z^1_{\text{in}}(x, w)$ then go to Step 3. Otherwise, x^0 is nonoptimal.

Step 3. Choose an edge e of zero weight and attach the weight $1 - 1/2m$ to e. If $\rho(x^0, w) \le 1/2m$ then x^0 is the optimal cut. If $\rho(x^0, w) > 1/2m$ then go to Step 4.

Step 4. Attach the weight 1 to e and go to Step 2.

Theorem 8. *Given an arbitrary norm* $\|\cdot\|_r$ *on* $\mathbb{R}^m$, *Algorithm* 2 *solves Problem* $Z^1_{\text{in}}(x, \mathbf{1})$ *in* $O(m\eta)$ *operations, where* $O(m\eta)$ *is the complexity of finding the stability radius of a nonoptimal solution.*

Proof. Skip the trivial case when x^0 includes all edges of the graph and assume that, running Algorithm 2, we come to Step 3. Let w^* denote the vector of the edge weights obtained if the weight of the edge e is $1 - 1/2m$ and let w^1 be the vector w^* in which the weight of e is 1.

It is easy that x^0 cannot be maximal in Problem $Z^1_{\text{in}}(x, w^*)$. However, if x^0 becomes maximal in $Z^1_{\text{in}}(x, w^1)$ then its stability radius is at most $1/2m$ since, otherwise, the increase of the weight of e to 1 would keep the cut x^0 nonoptimal. Thus, $x^0 \in P^1(w^1)$ if and only if $\rho(x^0, w^*) \le 1/2m$ holds at Step 3. Moreover, obviously, further change of the zero weights to the unit values does not lead to the optimality loss of the cut x^0. Hence, if $\rho(x^0, w^*) \le 1/2m$ then x^0 is the maximal cut in Problem $Z^1_{\text{in}}(x, \mathbf{1})$.

It is clear that Step 3 can be done in $O(\eta)$ operations, and Algorithm 2 itself, in $O(m\eta)$ operations.

The proof of Theorem 8 is complete. □

Note that, by analogy with Algorithm 2, it is rather easy to construct an algorithm for reducing Problem $Z^1_{\text{in}}(x, \mathbf{1})$ to the problem of finding the radius of T_1- or T_3-stability.

Summing up all the above, we conclude that, under assumption of P≠NP, the problem of finding some of the stability radii of Problem $Z^1(W)$ on maximal cut of a graph is intractable. The same can be proposed for the s-criteria Problem $Z^s(W)$ ($s \geq 2$) since finding the stability radii of $Z^s(W)$ is not easier than in the case of the single-criterion $Z^1(W)$. In particular, given an arbitrary stability radius, it is easy to provide a bijective correspondence between the single-criterion and multi-criteria problems such that the information on the stability radius of the multi-objective problem guarantees the information on the stability radius of the single-criterion problem as well. For instance, given a single-criterion problem, if we add to it arbitrary many criteria with all edges of zero weights then the sets of optimal solutions and the radii $\rho_1(W)$, $\rho_2(W)$, $\rho_4(W)$, and the stability radius $\rho(x^0, W)$ of the optimal solution x^0 for the single-criterion and multiple-criteria problems coincide.

Thus, for the single-criterion and multiple-criteria problems of a maximal cut of a graph, it is highly improbable that any polynomial algorithm exists for computing an arbitrary stability radius. Therefore, the prospective directions for constructing some efficient algorithms computing the stability radius of Problem $Z^s(W)$, $s \geq 1$, could be approximate probabilistic and genetic algorithms. Some progress is also possible by reduction of the definition of stability radius by analogy to [5, 7], for example.

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