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Solvability of a Mixed Boundary Value Problem for Stationary Equations of Magnetohydrodynamics of a Viscous Heat-Conducting Liquid

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Abstract—Under study is some boundary value problem for stationary equations of magnetohydrodynamics of a viscous heat-conducting liquid considered together with the Dirichlet condition for the velocity and mixed boundary conditions for the electromagnetic field and temperature. Some sufficient conditions are established on the initial data providing the global solvability of this problem and the local uniqueness of the solution.

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1. STATEMENT OF A MIXED BOUNDARY VALUE PROBLEM

Simulation of the flows of an electrically conducting liquid in real technical devices leads often to the necessity of studying the magnetohydrodynamic processes in the domains with the boundaries whose different parts have different electrophysical properties. A typical situation is the case when one part of the boundary of some technical device is perfectly conducting, whereas the other is a dielectric. Studying the flows of a conductive liquid in the domains with boundaries of this type leads to the necessity of investigating the boundary value problems for the equations of magnetohydrodynamics (MHD) under mixed boundary conditions for the electromagnetic field. This type of a mixed boundary value problem for a stationary model of MHD of a viscous heat-conducting liquid is considered in this paper.

Let Ω be a bounded domain in $\mathbb{R}^3$ with the boundary $\Sigma = \partial\Omega$ consisting of two parts Σ_ν and Σ_τ or Σ_D and Σ_N. In Ω, we consider a boundary value problem for stationary equations of magnetic hydrodynamics [1]

$$\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \varkappa\,\mathrm{rot}\,\mathbf{H}\times\mathbf{H} = \mathbf{f} - \beta T\mathbf{G}, \qquad \mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1}$$

$$\nu_1\mathrm{rot}\,\mathbf{H} - \mathbf{E} + \varkappa\mathbf{H}\times\mathbf{u} = \nu_1\mathbf{j}, \qquad \mathrm{div}\,\mathbf{H} = 0, \qquad \mathrm{rot}\,\mathbf{E} = \mathbf{0} \qquad \text{in } \Omega, \tag{2}$$

$$-\lambda\Delta T + \mathbf{u}\cdot\nabla T = f \qquad \text{in } \Omega, \tag{3}$$

which describe the motion of an incompressible viscous heat-conducting and electrically conducting liquid in Ω under the following boundary conditions:

$$\mathbf{u}|_{\partial\Omega} = \mathbf{g}, \qquad \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = 0, \qquad \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}, \qquad \mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}, \tag{4}$$

$$T|_{\Sigma_D} = 0, \qquad \lambda\left(\frac{\partial T}{\partial n} + \alpha T\right)\bigg|_{\Sigma_N} = \chi. \tag{5}$$

Here $\mathbf{u}$ is the velocity vector; $\mathbf{H}$ is the magnetic field vector; $\mathbf{E} = \mathbf{E}'/\rho_0$ and $p = P/\rho_0$, where $\mathbf{E}'$ is the electric field vector; P is the pressure; $\rho_0 = \mathrm{const}$ is the density of the liquid; T is the temperature,

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$\varkappa = \mu/\rho_0$; $\nu_1 = 1/\rho_0\sigma = \varkappa\nu_m$, σ, μ, ν, ν_m, and λ are the constant coefficients of electrical conductivity, magnetic permeability, kinematic and magnetic viscosities, and heat conductivity, respectively; $\mathbf{n}$ is the unit outer normal of $\partial\Omega$; $\mathbf{f}$ is the volume density of external forces; $\mathbf{j}$ is the density of external currents; f is the density of heat sources; G is the acceleration of gravity; β is the variable coefficient of thermal expansion; $\mathbf{g}$ is a function on the boundary of Σ; while χ and α are functions on Σ_N.

Below we will refer to the problem (1)–(5) for given functions $\mathbf{f}$, $\mathbf{j}$, f, $\mathbf{g}$, α, and χ as *Problem I*. Note that all quantities in (1)–(5) are dimensional, and their physical dimensions are written in the units of the International System of Units. Physically, the boundary conditions for the electromagnetic field in (4) correspond to a situation where a part Σ_τ of the boundary $\partial\Omega$ is a perfect conductor, whereas another part $\Sigma_\nu \subset \partial\Omega$ is a perfect dielectric. The model (1)–(3) is fundamental in solving a number of applied problems and, in particular, the problems of cooling the nuclear reactor by liquid metals in the nuclear industry [2, 3].

In the particular case when $f = 0$, $\chi = 0$, and $T = 0$, the problem (1)–(5) is transformed into the boundary value problem (1), (2), and (4) (for $T = 0$) for stationary equations of magnetic hydrodynamics of a viscous non-heat-conducting liquid, considered under mixed boundary conditions for the electromagnetic field. This problem was studied in [4] as well as in [5] under inhomogeneous mixed boundary conditions for the electromagnetic field, where its global solvability is proved under certain restrictions on the region Ω and the subdivision $\partial\Omega$ into parts Σ_τ and Σ_ν. In another special case corresponding to the situation when $\Sigma_\nu = \varnothing$ (i.e., the entire boundary $\Sigma = \Sigma_\tau$ of the flow domain Ω is perfectly conducting), solvability of the problem (1)–(5) was studied in [6, 7]. For other boundary conditions on $\mathbf{E}$ and $\mathbf{H}$ simulating the matching conditions on the surface between two media, the model (1)–(3) was studied in [8] where the local solvability of the corresponding boundary value problem was proved (for small data). Let us note also the articles [9, 10] where the solvability is studied of the boundary value problems and the optimization problems for the generalized model of the type (1)–(3) in which the main parameters of the medium depend on temperature and spatial variables. In the case when $T = 0$ and, moreover, $\Sigma = \Sigma_\tau$, the boundary value problem (1), (2), and (4) (for $T = 0$) for the MHD model of a viscous incompressible liquid has been studied since the pioneering work by V. A. Solonnikov [11] and in a number of other studies (for instance, see [12–16]).

The aim of the article is to analyze the solvability of the mixed boundary value problem (1)–(5). Within this goal, we establish some sufficient conditions on the data which provide global solvability and local uniqueness of the solution. In analyzing the solvability of the problem (1)–(5) we substantially use the results on the solvability of mixed boundary value problems for the static Maxwell equations and div-rot systems, established in [17–19].

2. FUNCTION SPACES. PRELIMINARY RESULTS

We will use the Sobolev spaces $H^s(D)$, $s \in \mathbb{R}$, $H^0(D) \equiv L^2(D)$, where D denotes the domain Ω, the boundary $\partial\Omega$ of Ω, or certain part Σ_0 of Ω. The corresponding spaces of vector functions will be denoted by $H^s(D)^3$ and $L^2(D)^3$. Norms and scalar products in $L^2(\Omega)$ and $L^2(\Omega)^3$ will be denoted by $\|\cdot\|_\Omega$ and $(\cdot,\cdot)$. Let $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ denote the norm and seminorm on $H^1(\Omega)$ or $H^1(\Omega)^3$. Given an arbitrary Hilbert space H, let H^* stand for the dual space of H.

We assume that the domain Ω and the partitions of its boundary $\Sigma = \partial\Omega$ into the parts Σ_τ and Σ_ν or Σ_D and Σ_N satisfy the following conditions:

(i) Ω is a bounded domain in $\mathbb{R}^3$, whereas its boundary $\partial\Omega$ consists of finitely many disjoint closed C^2-surfaces each of which has finite area.

(ii) Σ_τ is a nonempty open subset of $\partial\Omega$ consisting of $M+1$ disjoint nonempty open components $\{\sigma_0, \sigma_1, \dots, \sigma_M\}$, $M \geq 1$, and there exists a positive number d_0 such that the distance

$$\operatorname{dist}(\sigma_i, \sigma_j) \geq d_0 > 0, \qquad i \neq j, \qquad i, j = 0, 1, \dots, M.$$

The boundary of each component σ_i is either the empty set or some $C^{1,1}$-curve. Put

$$\Sigma_\nu = \partial\Omega \setminus \overline{\Sigma}_\tau.$$

(iii) The open parts Σ_D and Σ_N of the boundary $\partial\Omega$ satisfy

$$\Sigma_D \in C^{0,1}, \qquad \Sigma_D \neq \varnothing, \qquad \Sigma_N \in C^{0,1}, \qquad \Sigma_D \cap \Sigma_N = \varnothing, \qquad \partial\Omega = \overline{\Sigma}_D \cup \overline{\Sigma}_N.$$

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable compactly supported in Ω functions and let $H_0^1(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. Let

$$V = \{\mathbf{v} \in H_0^1(\Omega)^3 : \operatorname{div} \mathbf{v} = 0\}, \quad H^{-1}(\Omega)^3 = \left(H_0^1(\Omega)^3\right)^*, \quad L_0^2(\Omega) = \{p \in L^2(\Omega) : (p,1) = 0\},$$

$$H(\operatorname{rot}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{rot} \mathbf{v} \in L^2(\Omega)^3\}, \qquad H^0(\operatorname{rot}, \Omega) = \{\mathbf{h} \in H(\operatorname{rot}, \Omega) : \operatorname{rot} \mathbf{h} = 0\},$$

$$H^1(\Omega, \Sigma_\tau) = \{\varphi \in H^1(\Omega) : \varphi|_{\Sigma_\tau} = 0\}, \qquad \mathcal{T} = H^1(\Omega, \Sigma_D) = \{\varphi \in H^1(\Omega) : \varphi|_{\Sigma_D} = 0\},$$

$$C_{\Sigma_\tau 0}(\overline{\Omega})^3 := \{\mathbf{v} \in C^0(\overline{\Omega})^3 : \mathbf{v} \cdot \mathbf{n}|_{\Sigma_\tau} = 0,\ \mathbf{v} \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\}, \qquad L_+^2(\Sigma_N) = \{\alpha \in L^2(\Sigma_N) : \alpha \geq 0\}.$$

In addition to the above-introduced spaces, we use the space

$$H_{DC}(\Omega) = \{\mathbf{h} \in H(\operatorname{rot}, \Omega) : \operatorname{div} \mathbf{h} \in L^2(\Omega)\}$$

endowed with the Hilbert norm

$$\|\mathbf{h}\|_{DC}^2 := \|\mathbf{h}\|_\Omega^2 + \|\operatorname{rot} \mathbf{h}\|_\Omega^2 + \|\operatorname{div} \mathbf{h}\|_\Omega^2. \tag{6}$$

Every vector $\mathbf{v}$ defined on the boundary $\partial\Omega$ (or on its part $\Sigma \subset \partial\Omega$) can be represented as the sum of its normal and tangential components $\mathbf{v}_n$ and $\mathbf{v}_T$:

$$\mathbf{v} = \mathbf{v}_n + \mathbf{v}_T.$$

These components are defined by the formulas

$$\mathbf{v}_n = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \equiv v_n \mathbf{n}, \qquad \mathbf{v}_T = \mathbf{v} - \mathbf{v}_n \equiv (\mathbf{n} \times \mathbf{v}) \times \mathbf{n}.$$

Here the scalar $v_n = \mathbf{v} \cdot \mathbf{n}$ is the normal component of the vector field $\mathbf{v}$; and $\mathbf{v} \times \mathbf{n}$ is the tangent vector which is orthogonal to both the normal $\mathbf{n}$ and the vector $\mathbf{v}_T$. Obviously, $\mathbf{v}_T = \mathbf{0}$ on Σ if and only if $\mathbf{v} \times \mathbf{n}|_\Sigma = \mathbf{0}$. As usual, the subscript T in the designation of one of the spaces $H_T^1(\Omega)^3$ or $H_T^{1/2}(\Sigma_0)^3$ means that the corresponding space consists of the vector functions tangential on Σ.

In what follows we use the following Green's formulas [20, 21]:

$$\int_\Omega \Delta u v\, dx = -\int_\Omega \nabla u \cdot \nabla v\, d\sigma + \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, d\sigma \quad \text{for all} \quad u \in H^2(\Omega),\ v \in H^1(\Omega), \tag{7}$$

$$\int_\Omega \mathbf{v} \cdot \operatorname{grad} \varphi\, dx + \int_\Omega \operatorname{div} \mathbf{v} \varphi\, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \varphi\, d\sigma \quad \text{for all} \quad \mathbf{v} \in H^1(\Omega)^3,\ \varphi \in H^1(\Omega), \tag{8}$$

$$\int_\Omega (\mathbf{v} \cdot \operatorname{rot} \mathbf{w} - \mathbf{w} \cdot \operatorname{rot} \mathbf{v})\, dx = \int_{\partial\Omega} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{w}_T\, d\sigma \quad \text{for all} \quad \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3. \tag{9}$$

In the case when $\varphi \in H^1(\Omega, \Sigma_\tau)$ or $\mathbf{w} \in C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3$, the right-hand sides of (8) or (9) take the form $\int_{\Sigma_\nu} \mathbf{v} \cdot \mathbf{n} \varphi\, d\sigma$ or $\int_{\Sigma_\tau} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{w}_T\, d\sigma$. Based on this fact and (8), we will say, following [18], that the function $\mathbf{v} \in H_{DC}(\Omega)$ satisfies the condition $\mathbf{v} \cdot \mathbf{n} = 0$ *weakly* on Σ_ν if

$$\int_\Omega (\mathbf{v} \cdot \operatorname{grad} \varphi + \operatorname{div} \mathbf{v} \varphi)\, dx = 0 \quad \text{for all} \quad \varphi \in H^1(\Omega, \Sigma_\tau).$$

In the same fashion, based on (9), we will say that the function $\mathbf{v} \in H(\operatorname{rot}, \Omega)$ satisfies the condition $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ *weakly* on Σ_τ if

$$\int_\Omega (\mathbf{v} \cdot \operatorname{rot} \mathbf{w} - \mathbf{w} \cdot \operatorname{rot} \mathbf{v})\, dx = 0 \quad \text{for all} \quad \mathbf{w} \in C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3.$$

Let $H_{DC\Sigma_\tau}(\Omega)$ denote the closure of $C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3$ with respect to the norm $\|\cdot\|_{DC}$ defined in (6). We introduce the spaces

$$\mathcal{H}_{\Sigma_\tau}(\Omega) = \big\{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{h} = 0,\ \operatorname{rot}\mathbf{h} = \mathbf{0}\ \text{ in }\ \Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\tau} = 0,\ \mathbf{h}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\big\},$$

$$\mathcal{H}_{\Sigma_\nu}(\Omega) = \big\{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{h} = 0,\ \operatorname{rot}\mathbf{h} = \mathbf{0}\ \text{ in }\ \Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\nu} = 0,\ \mathbf{h}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}\big\}, \tag{10}$$

$$V_{\Sigma_\tau}(\Omega) = \big\{\mathbf{v} \in H_{DC\Sigma_\tau}(\Omega) : \operatorname{div}\mathbf{v} = 0\ \text{ in }\ \Omega\big\} \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp.$$

In [18], many important properties of the above-introduced spaces were proved. It is convenient for us to formulate these properties as

Theorem 1. *Let conditions* (i) *and* (ii) *hold. Then*

(1) *the spaces $\mathcal{H}_{\Sigma_\tau}(\Omega)$ and $\mathcal{H}_{\Sigma_\nu}(\Omega)$ are finite-dimensional;*

(2) *the continuous embedding $H_{DC\Sigma_\tau}(\Omega) \subset H^1(\Omega)^3$ holds, whereas the norm $\|\cdot\|_{DC}$ is equivalent on $H_{DC\Sigma_\tau}(\Omega)$ to the norm $\|\cdot\|_{1,\Omega}$;*

(3) *there exists a constant δ_1 depending on Ω and Σ_τ such that the coercivity inequality holds:*

$$\|\operatorname{rot}\mathbf{h}\|^2 \geq \delta_1\|\mathbf{h}\|_{1,\Omega}^2 \quad \textit{for all}\quad \mathbf{h} \in V_{\Sigma_\tau}(\Omega); \tag{11}$$

(4) *we have the orthogonal decomposition*

$$L^2(\Omega)^3 = \nabla H^1(\Omega,\Sigma_\tau) \oplus \operatorname{rot} H_{DC\Sigma_\tau}(\Omega) \oplus \mathcal{H}_{\Sigma_\nu}(\Omega). \tag{12}$$

The relation (12) means that every vector $\mathbf{h} \in L^2(\Omega)^3$ can be represented in the form

$$\mathbf{h} = \nabla\varphi + \operatorname{rot}\mathbf{v} + \mathbf{e}.$$

Here $\varphi \in H^1(\Omega,\Sigma_\tau)$, $\mathbf{v} \in H_{DC\Sigma_\tau}(\Omega)$, and $\mathbf{e} \in \mathcal{H}_{\Sigma_\nu}(\Omega)$ are some functions uniquely defined by the vector $\mathbf{h}$.

In [19], it is also proved that $\nabla\varphi \times \mathbf{n} = 0$ weakly on Σ_τ for every function $\varphi \in H^1(\Omega,\Sigma_\tau)$) and the relation holds

$$\operatorname{rot} H_{DC\Sigma_\tau}(\Omega) \equiv \operatorname{rot} V_{\Sigma_\tau}(\Omega). \tag{13}$$

Along with Theorem 1, we also use some properties of bilinear and trilinear forms associated with linear and nonlinear terms in the equations (6)–(8). We formulate them as Lemma 1 whose proof follows from the results in [18–21]:

Lemma 1. *Under conditions* (i) *and* (ii)*, there exist some positive constants $\delta_0 = \delta_0(\Omega)$, $\delta_2 = \delta_2(\Omega)$, $C_1 = C_1(\Omega)$, $\gamma_i = \gamma_i(\Omega)$, $i = 0,1,2,3$, $\beta = \beta(\Omega)$, and $\beta_1 = \beta_1(\Omega)$ such that*

$$(\nabla\mathbf{v},\nabla\mathbf{v}) \geq \delta_0\|\mathbf{v}\|_{1,\Omega}^2 \quad \textit{for all}\quad \mathbf{v} \in H_0^1(\Omega)^3, \tag{14}$$

$$|((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})| \leq \gamma_0'\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{L^4(\Omega)^3} \leq \gamma_0\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega} \quad \textit{for all}\quad \mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega)^3, \tag{15}$$

$$\|\operatorname{rot}\mathbf{H}\|_\Omega \leq C_1\|\mathbf{H}\|_{1,\Omega} \quad \textit{for all}\quad \mathbf{H} \in H^1(\Omega)^3, \tag{16}$$

$$|(\operatorname{rot}\mathbf{u}\times\mathbf{v},\mathbf{w})| \leq \gamma_1'\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{L^4(\Omega)^3} \leq \gamma_1\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega} \quad \textit{for all}\quad \mathbf{u},\mathbf{v},\mathbf{w} \in V_{\Sigma_\tau}(\Omega), \tag{17}$$

$$(\nabla T,\nabla T) \geq \delta_2\|T\|_{1,\Omega}^2 \quad \textit{for all}\quad T \in \mathcal{T}, \tag{18}$$

$$|(\mathbf{u}\cdot\nabla S,T)| \leq \gamma_2\|\mathbf{u}\|_{1,\Omega}\|S\|_{1,\Omega}\|T\|_{1,\Omega} \quad \textit{for all}\quad \mathbf{u} \in H^1(\Omega)^3,\quad S \in \mathcal{T},\quad T \in \mathcal{T}, \tag{19}$$

$$|(\mathbf{b}T,\mathbf{v})| \leq \|\mathbf{b}\|_\Omega\|\mathbf{v}\|_{L^4(\Omega)^3}\|T\|_{L^4(\Omega)} \leq \beta_1\|T\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega} \quad \textit{for all}\quad T \in H^1(\Omega),\quad \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{20}$$

$$|(\alpha T,S)_{\Sigma_N}|\le\gamma_3\|\alpha\|_{\Sigma_N}\|T\|_{1,\Omega}\|S\|_{1,\Omega},\qquad |(\chi,T)_{\Sigma_N}|\le\gamma_3\|\chi\|_{\Sigma_N}\|T\|_{1,\Omega},\quad S\in\mathcal{T},\qquad T\in\mathcal{T},\tag{21}$$

$$\sup_{\mathbf{v}\in H_0^1(\Omega)^3,\mathbf{v}\ne 0}\frac{-(\operatorname{div}\mathbf{v},p)}{\|\mathbf{v}\|_{1,\Omega}}\ge\beta\|p\|_\Omega\qquad \textit{for all}\ \ p\in L_0^2(\Omega).\tag{22}$$

Moreover, the following are true:

$$((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})=-((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})\quad \textit{for all}\quad \mathbf{u}\in H^1(\Omega)^3,\quad \mathbf{v}\in H_0^1(\Omega)^3,\quad \mathbf{w}\in H^1(\Omega)^3,\quad \operatorname{div}\mathbf{u}=0,\tag{23}$$

$$(\mathbf{H}\times\mathbf{u},\operatorname{rot}\boldsymbol{\Psi})=(\operatorname{rot}\boldsymbol{\Psi}\times\mathbf{H},\mathbf{u})=-(\operatorname{rot}\boldsymbol{\Psi}\times\mathbf{u},\mathbf{H})\qquad \textit{for all}\ \ \boldsymbol{\Psi},\mathbf{H},\mathbf{u}\in H^1(\Omega)^3,\tag{24}$$

$$(\mathbf{u}\cdot\nabla T,T)=0\qquad \textit{for all}\ \ \mathbf{u}\in H_T^1(\Omega)^3,\ \ T\in\mathcal{T}.\tag{25}$$

In studying Problem I, an important role will be played by the product of the spaces

$$X=H_0^1(\Omega)^3\times V_{\Sigma_\tau}(\Omega),\qquad Z=V\times V_{\Sigma_\tau}(\Omega),$$

as well as the dual spaces

$$X^*=H^{-1}(\Omega)^3\times V_{\Sigma_\tau}(\Omega)^*,\qquad Z^*=V^*\times V_{\Sigma_\tau}(\Omega)^*;$$

here X and Z are some Hilbert spaces endowed with the usual graph norm

$$\|(\mathbf{u},H)\|_X=\big(\|\mathbf{u}\|_{1,\Omega}^2+\varkappa\|\mathbf{H}\|_{1,\Omega}^2\big)^{1/2}.$$

Recall that $\varkappa=\mu\rho_0^{-1}\equiv\nu_1/\nu_m$ is a dimensional parameter in the first equation in (2). The parameter $\varkappa$ is introduced into the norm $\|(\mathbf{u},H)\|_X$ in order to equalize the dimensions of both summands involved.

The elements of the space X^* (or Z^*) have the form $\mathbf{F}=(\mathbf{f},\mathbf{q})$, where $\mathbf{f}\in H^{-1}(\Omega)^3$ and $\mathbf{q}\in V_{\Sigma_\tau}(\Omega)^*$ (or $\mathbf{f}\in V^*$ and $\mathbf{q}\in V_{\Sigma_\tau}(\Omega)^*$), and, by definition,

$$\langle(\mathbf{f},\mathbf{q}),(\mathbf{v},\boldsymbol{\Psi})\rangle_{X^*\times X}=\langle\mathbf{f},\mathbf{v}\rangle_{H^{-1}(\Omega)^3\times H_0^1(\Omega)^3}+\langle\mathbf{q},\boldsymbol{\Psi}\rangle_{V_{\Sigma_\tau}(\Omega)^*\times V_{\Sigma_\tau}(\Omega)}.$$

It is easy to verify (for instance, see [21, p. 283]) that

$$\|\mathbf{F}\|_{X^*}:=\|(\mathbf{f},\mathbf{q})\|_{X^*}\le\|\mathbf{f}\|_{-1,\Omega}+\varkappa^{-1/2}\|\mathbf{q}\|_{V_{\Sigma_\tau}(\Omega)^*}.\tag{26}$$

Moreover, the inequality $\|\mathbf{F}\|_{Z^*}\le\|\mathbf{F}\|_{X^*}$ holds for all $\mathbf{F}\in X^*$.

Let us define the following two bilinear forms:

$$\tilde a(T,S)=\lambda(\nabla T,\nabla S)+\lambda(\alpha T,S)_{\Sigma_N}\qquad \text{for}\ \ \alpha\in L_+^2(\Sigma_N),\tag{27}$$

$$a((\mathbf{u},\mathbf{H}),(\mathbf{v},\boldsymbol{\Psi}))=\nu(\nabla\mathbf{u},\nabla\mathbf{v})+\nu_1(\operatorname{rot}\mathbf{H},\operatorname{rot}\boldsymbol{\Psi})\equiv\nu(\nabla\mathbf{u},\nabla\mathbf{v})+\nu_m\varkappa(\operatorname{rot}H,\operatorname{rot}\Psi).\tag{28}$$

By (18), (11), and (14), the bilinear form $\tilde a$ is continuous on $H^1(\Omega)$ and coercive on $\mathcal{T}$, whereas the form a is continuous on $H^1(\Omega)^3\times H^1(\Omega)^3$ and coercive on X. Moreover, the following estimates are true:

$$|\tilde a(T,S)|\le(1+\gamma_3\|\alpha\|_{L^2(\Sigma_N)})\|T\|_{1,\Omega}\|S\|_{1,\Omega},\tilde a(S,S)\ge\delta_2\|S\|_{1,\Omega}^2\quad \text{for all}\quad S\in\mathcal{T},\quad T\in\mathcal{T},\tag{29}$$

$$a((\mathbf{v},\boldsymbol{\Psi}),(\mathbf{v},\boldsymbol{\Psi}))\ge\nu_*\big(\|\mathbf{v}\|_{1,\Omega}^2+\varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2\big)\quad \text{for all}\quad (\mathbf{v},\boldsymbol{\Psi})\in X,\quad \nu_*=\min(\delta_0\nu,\delta_1\nu_m).\tag{30}$$

The next result proved in [4] will be used in Section 3 in the derivation of the weak formulation of Problem I not containing the electric field $\mathbf{E}$:

Lemma 2. *Suppose that under the conditions* (i) *and* (ii) $\mathbf{E}\in H(\operatorname{rot},\Omega)$ *and* $\mathbf{E}\times\mathbf{n}|_{\Sigma_\tau}=\mathbf{0}$. *Then*

$$(\mathbf{E},\operatorname{rot}\boldsymbol{\Psi})=(\operatorname{rot}\mathbf{E},\boldsymbol{\Psi})\quad \textit{for all}\ \ \boldsymbol{\Psi}\in H_{DC\Sigma_\tau}(\Omega).\tag{31}$$

Let us emphasize that if in addition to the conditions of Lemma 2 we have $\operatorname{rot}\mathbf{E}=\mathbf{0}$ then (31) assumes the form

$$(\mathbf{E},\operatorname{rot}\boldsymbol{\Psi})=0\ \ \text{for all}\ \ \boldsymbol{\Psi}\in H_{DC\Sigma_\tau}(\Omega).\tag{32}$$

3. SOLVABILITY OF PROBLEM I

Suppose that in addition to (i)–(iii) the following hold:

(iv) $\mathbf{f} \in H^{-1}(\Omega)^3$, $\mathbf{j} \in L^2(\Omega)^3$, $f \in \mathcal{T}^*$, $\mathbf{b} \equiv \beta\mathbf{G} \in L^2(\Omega)^3$, $\mathbf{g} \in H_T^{1/2}(\Sigma)$, $\chi \in L^2(\Sigma_N)$, and $\alpha \in L^2_+(\Sigma_N)$.

Let us define the functionals $\mathbf{F} : X \to \mathbb{R}$ and $l : \mathcal{T} \to \mathbb{R}$ by formulas

$$\langle \mathbf{F}, (\mathbf{v}, \mathbf{\Psi})\rangle = \langle \mathbf{f}, \mathbf{v}\rangle + \nu_1(\mathbf{j}, \operatorname{rot} \mathbf{\Psi}), \qquad \langle l, S\rangle = (f, S) + (\chi, S)_{\Sigma_N}. \tag{33}$$

It follows from conditions (iv) together with (16), (21), and (26) that $\mathbf{F} \in X^*$ and $l \in \mathcal{T}^*$, whereas

$$\|\mathbf{F}\|_{X^*} \le M := \|\mathbf{f}\|_{-1,\Omega} + C_1\nu_1\varkappa^{-1/2}\|\mathbf{j}\|_\Omega, \qquad \|l\|_{\mathcal{T}^*} \le \|f\|_{\mathcal{T}^*} + \gamma_3\|\chi\|_{\Sigma_N}. \tag{34}$$

Suppose that the quintuple

$$(\mathbf{u}, \mathbf{H}, p, T, \mathbf{E}) \in (C^2(\overline{\Omega})^3 \times (C^1(\overline{\Omega})^3 \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp) \times C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times C^1(\overline{\Omega})^3$$

is a classical solution of Problem I. Multiply the first equation in (1) by the function $\mathbf{v} \in H_0^1(\Omega)^3$, the first equation in (2), by $\operatorname{rot} \mathbf{\Psi}$, where $\mathbf{\Psi} \in V_{\Sigma_\tau}(\Omega)$, integrate the obtained result over Ω, and use the Green's formulas (7)–(9) together with (24) and (32). In result, we infer

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - \varkappa(\operatorname{rot}\mathbf{H} \times \mathbf{H}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v}\rangle - (\mathbf{b}T, \mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in H_0^1(\Omega)^3, \tag{35}$$

$$\nu_1(\operatorname{rot}\mathbf{H}, \operatorname{rot}\mathbf{\Psi}) + \varkappa(\operatorname{rot}\mathbf{\Psi} \times \mathbf{H}, \mathbf{u}) = \nu_1(\mathbf{j}, \operatorname{rot}\mathbf{\Psi}) \quad \text{for all} \quad \mathbf{\Psi} \in V_{\Sigma_\tau}(\Omega). \tag{36}$$

Likewise, we multiply (35) by $S \in \mathcal{T}$, integrate over Ω, and apply the Green's formulas (7) and (8). We obtain

$$\lambda(\nabla T, \nabla S) + \lambda(\alpha T, S)_{\Sigma_N} + (\mathbf{u}\cdot\nabla T, S) = \langle l, S\rangle \quad \text{for all} \quad S \in \mathcal{T}, \tag{37}$$

where the functional l is defined in (33). Adding (35) and (36), we arrive at a weak formulation of Problem I. It consists in determining the quadruple

$$(\mathbf{u}, \mathbf{H}, p, T) \in H^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \times L_0^2(\Omega) \times \mathcal{T}$$

satisfying (37) and the following relations:

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \nu_1(\operatorname{rot}\mathbf{H}, \operatorname{rot}\mathbf{\Psi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, p) + \varkappa(\operatorname{rot}\mathbf{\Psi} \times \mathbf{H}, \mathbf{u}) - \varkappa(\operatorname{rot}\mathbf{H} \times \mathbf{H}, \mathbf{v}) + (\mathbf{b}T, \mathbf{v}) = \langle \mathbf{F}, (\mathbf{v}, \mathbf{\Psi})\rangle \quad \text{for all} \quad (\mathbf{v}, \mathbf{\Psi}) \in X \equiv H_0^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega), \tag{38}$$

$$\operatorname{div}\mathbf{u} = 0 \ \text{ in } \ \Omega, \qquad \mathbf{u} = \mathbf{g} \ \text{ on } \ \Gamma. \tag{39}$$

In (37) and (38) $\mathbf{F}$ and l are the functionals in (33). The quadruple $(\mathbf{u}, \mathbf{H}, p, T)$ will be called the *weak solution of Problem* I. Considering the restriction of (38) to the space $Z \subset X$, we note that the triple $(\mathbf{u}, \mathbf{H}, T)$ satisfies the identity

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \nu_1(\operatorname{rot}\mathbf{H}, \operatorname{rot}\mathbf{\Psi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) + \varkappa\,[(\operatorname{rot}\mathbf{\Psi} \times \mathbf{H}, \mathbf{u}) - (\operatorname{rot}\mathbf{H} \times \mathbf{H}, \mathbf{v})] + (\mathbf{b}T, \mathbf{v}) = \langle \mathbf{F}, (\mathbf{v}, \mathbf{\Psi})\rangle \quad \text{for all} \quad (\mathbf{v}, \mathbf{\Psi}) \in Z. \tag{40}$$

Formula (40) does not contain a pair $(p, \mathbf{E})$. However, $(p, \mathbf{E})$ can be reconstructed uniquely by the triple $(\mathbf{u}, \mathbf{H}, T) \in Z \times \mathcal{T}$ satisfying (40) so that (38) and all relationships in (2) together with (4) hold. Indeed, we have the following

Lemma 3. *Suppose that, under fulfillment of conditions* (i)–(iii)*,*

$$(\mathbf{u}, \mathbf{H}, T) \in H^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \times \mathcal{T}$$

is a solution of problem (37), (39), *and* (40). *Then there exist* $p \in L^2_0(\Omega)$ *and* $\mathbf{E} \in H(\mathrm{rot}, \Omega)$ *such that the quadruple* $(\mathbf{u}, \mathbf{H}, p, T)$ *is a weak solution of Problem* I, *while the quintuple* $(\mathbf{u}, \mathbf{H}, p, T, \mathbf{E})$ *satisfies the boundary conditions in* (4) *in the sense of traces and the equations in* (2) *almost everywhere in* Ω. *Moreover, the first equation in* (1), *the equation* (3), *and the second boundary condition in* (5) *are true in the following sense:*

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \varkappa \,\mathrm{rot}\,\mathbf{H}\times\mathbf{H} = \mathbf{f} - \mathbf{b}T \quad in \;\; \mathcal{D}'(\Omega)^3, \tag{41}$$

$$-\lambda\Delta T + \mathbf{u}\cdot\nabla T = f \quad in \;\; \mathcal{D}'(\Omega), \tag{42}$$

$$\lambda\left(\frac{\partial T}{\partial n} + \alpha T\right)\bigg|_{\Sigma_N} = \chi \quad in \;\; H^{-1/2}(\Sigma_N). \tag{43}$$

Proof. Suppose that a triple $(\mathbf{u}, \mathbf{H}, T) \in H^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \times \mathcal{T}$ is a solution of (37), (39), and (40). The recovering of pressure $p \in L^2_0(\Omega)$ so that (35) holds is carried out, as usual, via the de Rham theorem and the inf-sup condition (22) (see the details in [20, 21]). Setting $\mathbf{v} = \mathbf{0}$ in (40), we obtain (36). Adding (35) and (36), we arrive at (38). Choosing $\mathbf{v} \in \mathcal{D}(\Omega)^3$ in (35) and $S \in \mathcal{D}(\Omega)$ in (37), we arrive at (41) and (42), while the fact that (43) holds is proved as in [21, p. 50].

It remains to prove the existence of $\mathbf{E} \in H(\mathrm{rot}, \Omega)$ such that, together with the pair $(\mathbf{u}, \mathbf{H})$, satisfies all relations in (2) and the boundary condition $\mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}$. To this end, we consider (36) which, by (13) and (24), can be rewritten as

$$(\nu_1 \mathrm{rot}\,\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j},\ \mathrm{rot}\,\boldsymbol{\Psi}) = 0 \quad \text{for all} \;\; \boldsymbol{\Psi} \in H_{DC\Sigma_\tau}(\Omega). \tag{44}$$

The condition (44) means that the vector

$$\mathbf{A} \equiv \nu_1 \mathrm{rot}\,\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j}$$

is orthogonal to rot $\boldsymbol{\Psi}$, where $\boldsymbol{\Psi} \in H_{DC\Sigma_\tau}$ is an arbitrary function. By (12), this can be true if and only if

$$\nu_1 \mathrm{rot}\,\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j} = \nabla\varphi + \mathbf{e}.$$

Here $\varphi \in H^1(\Omega, \Sigma_\tau)$ is the scalar potential, $\mathbf{e} \in \mathcal{H}_{\Sigma_\nu}(\Omega)$ is a vector (harmonic vector potential), whereas the pair $(\varphi, \mathbf{e})$ is uniquely determined by the vector $\mathbf{A}$. Putting $\mathbf{E} = \nabla\varphi + \mathbf{e}$, we note that $\mathbf{E}$ together with the pair $(\mathbf{u}, \mathbf{H})$ satisfy

$$\nu_1 \mathrm{rot}\,\mathbf{H} - \mathbf{E} + \varkappa\mathbf{H}\times\mathbf{u} = \nu_1\mathbf{j} \quad \text{almost everywhere in} \;\; \Omega.$$

Moreover, since $\mathbf{e} \in \mathcal{H}_{\Sigma_\nu}(\Omega)$ and $\varphi \in H^1(\Omega, \Sigma_\tau)$; therefore, we have

$$\mathrm{rot}\,\mathbf{e} = \mathbf{0}, \qquad \mathbf{e}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}, \qquad \nabla\varphi\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}.$$

This means that rot $\mathbf{E} = \mathbf{0}$ almost everywhere in Ω and $\mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}$.

The proof of Lemma 3 is complete. □

Lemma 3 implies that the proof of the existence of a weak solution $(\mathbf{u}, \mathbf{H}, p, T)$ of Problem I is reduced to proving the existence of a solution

$$(\mathbf{u}, \mathbf{H}, T) \in H^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \times \mathcal{T}$$

of the problem (37), (39), and (40). To prove the existence of a solution of the latter problem, we need the following lemma from [14] on the existence of a lifting (a proper extension inside the domain Ω) of the boundary function $\mathbf{g} \in H^{1/2}_T(\Sigma)^3$:

Lemma 4. *If condition* (i) *is fulfilled then, for every function* $\mathbf{g} \in H^{1/2}_T(\Sigma)^3$ *and every number* $\varepsilon > 0$, *there exists a vector-function* $\mathbf{u}_\varepsilon \in H^1_T(\Omega)^3$ *such that* $\mathrm{div}\,\mathbf{u}_\varepsilon = 0$ *in* Ω, $\mathbf{u}_\varepsilon = \mathbf{g}$ *on* Σ, *and following estimates hold:*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega} \le C_\varepsilon\|\mathbf{g}\|_{1/2,\Sigma}, \qquad \|\mathbf{u}_\varepsilon\|_{L^4(\Omega)^3} \le \varepsilon\|\mathbf{g}\|_{1/2,\Sigma}. \tag{45}$$

Here the constant C_ε *depends on* ε *and* Ω.

By Lemma 4, we choose ε to be sufficiently small; namely, assuming that $\|\mathbf{g}\|_{1/2,\Sigma} > 0$, we put $\varepsilon = \varepsilon_0$, where

$$\varepsilon_0 = \nu_* \min\big(1/\gamma_0',\ 1/\gamma_1'\big)/2\|\mathbf{g}\|_{1/2,\Sigma}, \qquad \nu_* = \min(\delta_0\nu, \delta_1\nu_m). \tag{46}$$

Here γ_0' and γ_1' are the constants introduced in (15) and (17). We have the following

Theorem 2. *When the conditions* (i)–(iv) *are fulfilled, there exists a weak solution*

$$(\mathbf{u}, \mathbf{H}, p, T) \in H^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \times L_0^2(\Omega) \times \mathcal{T}$$

of Problem I*, and the following are fulfilled:*

$$\|\mathbf{u}\|_{1,\Omega} \le M_{\mathbf{u}}, \qquad \|\mathbf{H}\|_{1,\Omega} \le M_{\mathbf{H}}, \qquad \|T\|_{1,\Omega} \le M_T, \qquad \|p\|_\Omega \le M_p. \tag{47}$$

Here $M_{\mathbf{u}}$, $M_{\mathbf{H}}$, M_T, and M_p are defined as

$$M_{\mathbf{u}} = M_* + C_{\varepsilon_0}\|\mathbf{g}\|_{1/2,\Sigma}, \qquad M_* = \frac{2}{\nu_*}(M + M_{\mathbf{g}} + \beta_1 M_T),$$

$$M_{\mathbf{H}} = \varkappa^{-1/2} M_* = \frac{1}{\nu_*\sqrt{\varkappa}}(M + M_{\mathbf{g}} + \beta_1 M_T), \qquad M_T = \frac{1}{\delta_2\lambda}\big(\|f\|_{\mathcal{T}^*} + \gamma_3\|\chi\|_{\Sigma_N}\big), \tag{48}$$

$$M_p = \beta^{-1}\big[(\nu + \gamma_0 M_{\mathbf{u}})M_{\mathbf{u}} + \gamma_1\varkappa M_{\mathbf{H}}^2 + \beta_1 M_T + \|\mathbf{f}\|_{-1,\Omega}\big],$$

where M and β, β_1, δ_2, γ_3 are introduced in (34) *and Lemma* 1*, respectively; C_{ε_0} is the constant introduced in Lemma* 4 *and corresponds to ε_0 in* (46)*, whereas $M_{\mathbf{g}}$ is defined as*

$$M_{\mathbf{g}} = C_{\varepsilon_0}\|\mathbf{g}\|_{1/2,\Sigma} + \gamma_0'\varepsilon_0 C_{\varepsilon_0}\|\mathbf{g}\|_{1/2,\Sigma}^2. \tag{49}$$

If, additionally, the functions $\mathbf{f}$, $\mathbf{j}$, f, $\mathbf{g}$, *and χ are small in the sense that*

$$\gamma_0 M_{\mathbf{u}} + \gamma_1\left(\frac{\sqrt{\varkappa}}{2}\right)M_{\mathbf{H}} + \frac{\beta_1\gamma_2}{\delta_2\lambda}M_T < \delta_0\nu, \qquad \gamma_1 M_{\mathbf{u}} + \gamma_1\left(\frac{\sqrt{\varkappa}}{2}\right)M_{\mathbf{H}} < \delta_1\nu_m, \tag{50}$$

then the weak solution is unique. In (50)*, δ_0, δ_1, γ_0, and γ_1 are the constants introduced in Lemma* 1 *and* (11)*, while ν and ν_m are the coefficients of kinematic and magnetic viscosities.*

Proof. To prove the existence of a weak solution of Problem I, by Lemma 3, it suffices to prove the existence of a solution $(\mathbf{u}, \mathbf{H}, T) \in Z \times \mathcal{T}$ of the problem (37), (39), and (40). On the basis of the boundary condition in (39) for the velocity $\mathbf{u}$, we will try to find the component $\mathbf{u}$ of the solution $(\mathbf{u}, \mathbf{H}, T)$ of the problem (37), (39), and (40) in the form

$$\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}. \tag{51}$$

Here $\mathbf{u}_0 \equiv \mathbf{u}_{\varepsilon_0}$ is the lifting from Lemma 4 of the boundary function $\mathbf{g}$, whereas $\tilde{\mathbf{u}} \in V$ is a new unknown function. Substituting (51) in (37) and (40), we arrive at the identities

$$\lambda(\nabla T, \nabla S) + \lambda(\alpha T, S)_{\Sigma_N} + \big((\mathbf{u}_0 + \tilde{\mathbf{u}})\cdot\nabla T, S\big) = \langle l, S\rangle \qquad \text{for all} \quad S \in \mathcal{T}, \tag{52}$$

$$\begin{aligned} a((\tilde{\mathbf{u}}, \mathbf{H}), (\mathbf{v}, \boldsymbol{\Psi})) &+ ((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{u}_0, \mathbf{v}) + ((\mathbf{u}_0\cdot\nabla)\tilde{\mathbf{u}}, \mathbf{v}) + ((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}, \mathbf{v}) \\ &+ \varkappa[(\mathrm{rot}\,\boldsymbol{\Psi}\times\mathbf{H}, \tilde{\mathbf{u}}) + \varkappa(\mathrm{rot}\,\boldsymbol{\Psi}\times\mathbf{H}, \mathbf{u}_0) - (\mathrm{rot}\,\mathbf{H}\times\mathbf{H}, \mathbf{v})] + (\mathbf{b}T, \mathbf{v}) \\ &= \langle \mathbf{F}_1, (\mathbf{v}, \boldsymbol{\Psi})\rangle \qquad \text{for all} \quad (\mathbf{v}, \boldsymbol{\Psi}) \in Z. \end{aligned} \tag{53}$$

Here $\mathbf{F}_1 : X \to \mathbb{R}$ is a linear functional defined by the formula

$$\langle \mathbf{F}_1, (\mathbf{v}, \boldsymbol{\Psi})\rangle = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle - \nu(\nabla\mathbf{u}_0, \nabla\mathbf{v}) - ((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0, \mathbf{v}) \quad \text{for all} \quad (\mathbf{v}, \boldsymbol{\Psi}) \in Z. \tag{54}$$

To prove the existence of a solution $(\tilde{\mathbf{u}}, \mathbf{H}, T) \in V \times V_{\Sigma_\tau}(\Omega) \times \mathcal{T}$ of the problem (52), (53) we apply the Schauder fixed point theorem. For this purpose, we define a mapping $G : Z \to Z$ acting according to the formula $G(\mathbf{w}, h) = (\tilde{\mathbf{u}}, \mathbf{H})$, where the pair $(\tilde{\mathbf{u}}, \mathbf{H}) \in Z$ is a solution of the linear problem

$$a((\tilde{\mathbf{u}}, \mathbf{H}), (\mathbf{v}, \boldsymbol{\Psi})) + a_{\mathbf{w},\mathbf{h}}\big((\tilde{\mathbf{u}}, \mathbf{H}),\ (\mathbf{v}, \boldsymbol{\Psi})\big) = \langle \mathbf{F}_1, (\mathbf{v}, \boldsymbol{\Psi})\rangle - (\mathbf{b}T, \mathbf{v}) \quad \text{for all} \quad (\mathbf{v}, \boldsymbol{\Psi}) \in Z \tag{55}$$

obtained by linearization of (52) and (53). Here the bilinear forms a and $a_{\mathbf{w},\mathbf{h}} : X \times X \to \mathbb{R}$ are defined, respectively, by the relation (28) and

$$a_{\mathbf{w},\mathbf{h}}\big((\tilde{\mathbf{u}}, \mathbf{H}), (\mathbf{v}, \boldsymbol{\Psi})\big) = \big((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{u}_0, \mathbf{v}\big) + \big((\mathbf{u}_0 \cdot \nabla)\tilde{\mathbf{u}}, \mathbf{v}\big) + \big((\mathbf{w} \cdot \nabla)\tilde{\mathbf{u}}, \mathbf{v}\big) + \varkappa\big[\big(\operatorname{rot}\boldsymbol{\Psi} \times \mathbf{h}, \tilde{\mathbf{u}}\big) + \varkappa\big(\operatorname{rot}\boldsymbol{\Psi} \times \mathbf{H}, \mathbf{u}_0\big) - \big(\operatorname{rot}\mathbf{H} \times \mathbf{h}, \mathbf{v}\big)\big], \quad (56)$$

whereas $T \equiv T_{\mathbf{w}}$ in (55) is a solution of the linear problem

$$\tilde{a}(T, S) \equiv \lambda(\nabla T, \nabla S) + \lambda(\alpha T, S)_{\Sigma_N} + ((\mathbf{u}_0 + \mathbf{w}) \cdot \nabla T, S) = \langle l, S \rangle \quad \text{for all} \quad S \in \mathcal{T}. \quad (57)$$

It is easy to see that the mapping G is correctly defined. Indeed, let us first consider problem (57). A simple analysis using (25) and (29) shows that the form $\tilde{a}(\cdot,\cdot)$ on the left-hand side of (57) is continuous and coercive on $\mathcal{T}$ with the constant $\delta_2\lambda$, whereas the functional l on the right-hand side of (57) defined in (33) belongs to $\mathcal{T}^*$ and satisfies the second estimate in (34). Then it follows from the Lax–Milgram theorem (for instance, see [21, p. 33]) that, for every vector $\mathbf{w} \in V$, a solution $T = T_{\mathbf{w}}$ of problem (57) exists, is unique, and satisfies the following estimate:

$$\|T_{\mathbf{w}}\|_{1,\Omega} \le M_T \equiv \frac{1}{\delta_2\lambda}\big(\|f\|_{\mathcal{T}^*} + \gamma_3\|\chi\|_{\Sigma_N}\big). \quad (58)$$

Now consider problem (55), where we put $T = T_{\mathbf{w}}$. It follows from the results of Section 2 that the form $a(\cdot,\cdot)$ is continuous and coercive on Z, whereas the bilinear form $a_{\mathbf{w},\mathbf{h}}$ defined in (56) is continuous on X and $\nu_*/2$ is "small" on Z since, by (23), (15), (17), and (46), we have

$$\begin{aligned} |a_{\mathbf{w},\mathbf{h}}((\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{v}, \boldsymbol{\Psi}))| &= \big|\big((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{u}_0\big) + \varkappa\big(\operatorname{rot}\boldsymbol{\Psi} \times \boldsymbol{\Psi}, \mathbf{u}_0\big)\big| \\ &\le \gamma_0'\|\mathbf{v}\|_{1,\Omega}^2\|\mathbf{u}_0\|_{L^4(\Omega)^3} + \gamma_1'\varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2\|\mathbf{u}_0\|_{L^4(\Omega)^3} \\ &\le \gamma_0'\varepsilon_0\|\mathbf{g}\|_{1/2,\Sigma}\|\mathbf{v}\|_{1,\Omega}^2 + \gamma_1'\varepsilon_0\|\mathbf{g}\|_{1/2,\Sigma}\varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2 \\ &\le \frac{\nu_*}{2}\big(\|\mathbf{v}\|_{1,\Omega}^2 + \varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2\big) \qquad \text{for all} \quad (\mathbf{v}, \boldsymbol{\Psi}) \in Z. \end{aligned}$$

It follows that the bilinear form $a(\cdot,\cdot) + a_{\mathbf{w},\mathbf{h}}(\cdot,\cdot)$ on the left-hand side of (55) is continuous and coercive on Z with the constant $\nu_*/2$. Moreover, the right-hand side of (55) for the specified $T_{\mathbf{w}} \in \mathcal{T}$ is the value of the continuous functional at the element $(\mathbf{v}, \boldsymbol{\Psi}) \in Z$ since, by (54), (34), (15), (20), (45), and (58), we have

$$\begin{aligned} &|\langle \mathbf{F}_1, (\mathbf{v}, \boldsymbol{\Psi})\rangle - (\mathbf{b}T_{\mathbf{w}}, \mathbf{v})| \\ &\quad \le M\|(\mathbf{v}, \boldsymbol{\Psi})\|_X + (C_{\varepsilon_0}\nu\|\mathbf{g}\|_{1/2,\Sigma} + \gamma_0'\varepsilon_0 C_{\varepsilon_0}\|\mathbf{g}\|_{1/2,\Sigma}^2 + \beta_1 M_T)\|\mathbf{v}\|_{1,\Omega} \\ &\quad \le (M + M_{\mathbf{g}} + \beta_1 M_T)\|(\mathbf{v}, \boldsymbol{\Psi})\|_X \quad \text{for all} \quad (\mathbf{v}, \boldsymbol{\Psi}) \in Z. \quad (59) \end{aligned}$$

Here M and $M_{\mathbf{g}}$ are the constants defined in (34) and (49), respectively. Then it follows from the Lax–Milgram theorem and (59) that, for every pair $(\mathbf{w}, \mathbf{h}) \in Z$, the solution $(\mathbf{u}, \mathbf{H}) \in Z$ of (55) exists, is unique, and satisfies the following estimate independently of $(\mathbf{w}, \mathbf{h})$:

$$\|(\tilde{\mathbf{u}}, \mathbf{H})\|_X \equiv \big(\|\tilde{\mathbf{u}}\|_{1,\Omega}^2 + \varkappa\|\mathbf{H}\|_{1,\Omega}^2\big)^{1/2} \le M_* \equiv \frac{2}{\nu_*}(M + M_{\mathbf{g}} + \beta_1 M_T). \quad (60)$$

Put $r = M_*$ and introduce the ball $B_r = \{(\mathbf{v}, \boldsymbol{\Psi}) \in Z : \|(\mathbf{v}, \boldsymbol{\Psi})\|_X \le r\}$ in the space Z. It follows from (60) that the introduced operator G maps B_r into itself. Arguing as in [7], we can show that the operator G is compact and continuous on B_r. Then the Schauder Theorem implies that G has a fixed point $(\tilde{\mathbf{u}}, \mathbf{H}) = G(\tilde{\mathbf{u}}, \mathbf{H}) \in B_r$. The specified fixed point $(\tilde{\mathbf{u}}, \mathbf{H}) \in Z$ together with the solution $T_{\tilde{\mathbf{u}}} \in \mathcal{T}$ of the problem (57) for $\mathbf{w} = \tilde{\mathbf{u}}$ constitute the desired solution of the problem (52), (53), while the estimates (60) and (58) hold for it. Then the triple $(\mathbf{u}, \mathbf{H}, T)$, where $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, is the desired solution of the problem (37), (39), and (40), whereas the first three estimates in (47) are obvious consequences of the estimates (60), (58), and (45).

Let us prove the estimate in (47) for the pressure p, which together with the sought triple $(\mathbf{u}, \mathbf{H}, T)$ satisfies the identity (38) due to Lemma 3. To do this, we use the inf-sup condition (22), because of which, for the indicated function $p \in L^2_0(\Omega)$ and arbitrary $\delta > 0$, there exists a function $\mathbf{v}_0 \in \mathbf{H}^1_0(\Omega)$, $\mathbf{v}_0 \neq \mathbf{0}$, such that the following inequality with the constant $\beta_0 = \beta - \delta > 0$ is true:

$$-(\operatorname{div}\mathbf{v}_0, p) \geq \beta_0 \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega.$$

Put $\mathbf{v} = \mathbf{v}_0$ and $\mathbf{\Psi} = \mathbf{0}$ in (38). We have

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}_0) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}_0) - (\operatorname{div}\mathbf{v}_0, p) - \varkappa(\operatorname{rot}\mathbf{H}\times\mathbf{H}, \mathbf{v}_0) + (\mathbf{b}T, \mathbf{v}_0) = \langle \mathbf{f}, \mathbf{v}_0\rangle. \quad (61)$$

Using the previous estimate for $-(\operatorname{div}\mathbf{v}_0, p)$ together with (15), (17), and (20), we derive from (61) that

$$\beta_0\|\mathbf{v}_0\|_{1,\Omega}\|p\|_\Omega \leq -(\operatorname{div}\mathbf{v}_0, p) \leq \nu\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega} + \gamma_0\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{u}\|^2_{1,\Omega}$$
$$+ \gamma_1\varkappa\|\mathbf{H}\|^2_{1,\Omega}\|\mathbf{v}_0\|_{1,\Omega} + \beta_1\|T\|_{1,\Omega}\|\mathbf{v}_0\|_{1,\Omega} + \|\mathbf{f}\|_{-1,\Omega}\|\mathbf{v}_0\|_{1,\Omega}. \quad (62)$$

Cancelling out $\|\mathbf{v}_0\|_{1,\Omega} \neq 0$ and using the already proved estimates in (47) and (60), we derive from (62) the following:

$$\|p\|_\Omega \leq \beta_0^{-1}\big(\nu\|\mathbf{u}\|_{1,\Omega} + \gamma_0\|\mathbf{u}\|^2_{1,\Omega} + \gamma_1\varkappa\|\mathbf{H}\|^2_{1,\Omega} + \beta_1\|T\|_{1,\Omega} + \|\mathbf{f}\|_{-1,\Omega}\big)$$
$$\leq \beta_0^{-1}\big[(\nu + \gamma_0 M_{\mathbf{u}})M_{\mathbf{u}} + \gamma_1\varkappa M^2_{\mathbf{H}} + \beta_1 M_T + \|\mathbf{f}\|_{-1,\Omega}\big]. \quad (63)$$

The last estimate in (47) follows from (63) due to arbitrariness of $\delta > 0$.

Finally, note that the uniqueness of a weak solution of Problem I is proved according to the standard scheme (for example, see [21, Chap. 7]) provided the conditions (40) are fulfilled.

The proof of Theorem 2 is complete. □

Remark. We proved the solvability of the boundary value problem (1)–(5) for stationary equations of magnetic hydrodynamics of a viscous heat-conducting liquid, considered under the homogeneous mixed boundary conditions for the electromagnetic field and the homogeneous Dirichlet condition on a portion Σ_D of the boundary Σ for the temperature.

In the case when the homogeneous condition $T|_{\Sigma_D} = 0$ in (5) is replaced by its inhomogeneous analogue $T|_{\Sigma_D} = \psi \in H^{1/2}(\Gamma_D)$, the solvability of the corresponding inhomogeneous boundary value problem can be proved in a similar way using the theorem on the existence of proper lifting for the temperature [21, Theorem 9.3, p. 403].

At the same time, the case when the mixed boundary conditions used in (4) for the electromagnetic field are inhomogeneous is more complicated and requires a special consideration.

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