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The Hopf–Cole Transformation and Multidimensional Representations of Solutions to Evolution Equations

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Abstract—We consider some new identities and representations of solutions to the second-order partial differential equations connected with the Hopf–Cole transformation.

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In this paper, we present some new identities and representations for solutions to the second-order differential equations that generalize those available [1-3] and go on with the study of the general problem (formulated in [2]) of the constructive construction of partial differential equations from a given class of functions determining the solutions of these equations (see also [1–7]).

Theorems 1–3 of the article are directly related to the Hopf–Cole transformation which makes it possible to obtain some multidimensional representations of the solutions and coefficients of Burgers-type nonlinear equations. Theorem 4 is a new version of the application of a fundamental system of solutions to a second-order linear ordinary differential equation with variable parameter for obtaining a representation of the solution and the coefficients of a second-order multidimensional equation with considerable arbitrariness.

We first give results for a Burgers-type multidimensional equation with parameter p applying the Hopf–Cole transformation and its generalization. As is known [8, 9], the Hopf–Cole transformation consists in that if a function $F(y,t,p) \neq 0$ is a solution to the parabolic equation with parameter p

$$\frac{\partial F}{\partial t} = p\,\frac{\partial^2 F}{\partial y^2}$$

then the function $w(y,t,p) = -2p\,\frac{\partial F}{\partial y}\Big/F$ satisfies the nonlinear Burgers equation

$$\frac{\partial w}{\partial t} = p\,\frac{\partial^2 w}{\partial y^2} - w\,\frac{\partial w}{\partial y}$$

with the same parameter p. Note that the variable parameter makes it possible to obtain, as $p \to 0$, solutions to the classical Hopf equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial y} = 0,$$

which is important for applications.

Suppose that $a_{ij}(x,t)$, $a_{ij} = a_{ji}$, are continuous functions $i,j = 1,2,\ldots,n$, $x \in D \subset \mathbb{R}^n$, and $t_0 < t < t_1$, where D is a domain in the real Euclidean space $\mathbb{R}^n$.

A generalization of the Hopf–Cole transformation to the multidimensional case is given by

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Theorem 1. *Let $F(y,t,p) \neq 0$ be a solution to the parabolic equation with parameter p*

$$\frac{\partial F}{\partial t} = p\,\frac{\partial^2 F}{\partial y^2}, \qquad y \in \mathbb{R}^n, \qquad t_0 < t < t_1,$$

and let $v(x,t)$ be a twice continuously differentiable function such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial v}{\partial x_i}\,\frac{\partial v}{\partial x_j} \neq 0,$$

which is arbitrary in all other respects. Then

$$w(x,t,p) = -2p\,\frac{\partial F}{\partial y}\Big/ F\Big|_{y=v(x,t)}$$

satisfies the following equation independent of F:

$$\left(\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right)\frac{\partial w}{\partial t} = p\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i \partial x_j} - w\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}$$
$$+\left[\frac{\sum\limits_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial t} - p\sum\limits_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i\partial x_j}}{\sum\limits_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}}\right]\sum_{k,l=1}^{n} a_{kl}(x,t)\frac{\partial w}{\partial x_k}\frac{\partial v}{\partial x_l}. \tag{1}$$

The proof is immediate from the following relations obtained by the representation

$$w(y,t,p) = -2p\frac{\partial F}{\partial y}\Big/ F\Big|_{y=v(x,t)}$$

with account taken of the equation $\frac{\partial F}{\partial t} = p\frac{\partial^2 F}{\partial y^2}$ and the equalities

$$\frac{\partial w}{\partial t} = -2p\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial v}{\partial t} + \frac{\partial^2 F}{\partial y \partial t}\right)\Big/ F + 2p\left(\frac{\partial F}{\partial y}\frac{\partial v}{\partial t} + \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial y}\Big/ F^2,$$

$$\frac{\partial w}{\partial x_i} = \left(-2p\frac{\partial^2 F}{\partial y^2}\Big/ F + 2p\left(\frac{\partial F}{\partial y}\right)^2\Big/ F^2\right)\frac{\partial v}{\partial x_i},$$

$$w\frac{\partial w}{\partial x_i} = -\frac{2p}{F}\,\frac{\partial F}{\partial y}\left(-2p\frac{\partial^2 F}{\partial y^2}\Big/ F + 2p\left(\frac{\partial F}{\partial y}\right)^2\Big/ F^2\right)\frac{\partial v}{\partial x_i},$$

$$\frac{\partial^2 w}{\partial x_i \partial x_j} = \frac{\partial^2 v}{\partial x_i \partial x_j}\left(-2p\frac{\partial^2 F}{\partial y^2}\Big/ F + 2p\left(\frac{\partial F}{\partial y}\right)^2\Big/ F^2\right)$$
$$+\left(-2p\frac{\partial^3 F}{\partial y^3}\Big/ F + 6p\frac{\partial^2 F}{\partial y^2}\frac{\partial F}{\partial y}\Big/ F^2 - 4p\left(\frac{\partial F}{\partial y}\right)^3\Big/ F^3\right)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}.$$

For $a_{ij} = \delta_{ij}$, $n = 1$, and $v(x,t) = x$, from equation (1) we obtain the one-dimensional Burgers equation. Assume in what follows that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} \neq 0, \qquad x \in D \subset \mathbb{R}^n, \qquad t_1 < t < t_2,$$

without further specification. Of interest are the more general solutions w; namely, if we suppose that $w(x,t,p)$ is the root of the equation

$$g(w(x,t,p),p) = -2p\frac{\partial F}{\partial y}\Big/ F\Big|_{y=v(x,t)},$$

where $g(z,p)$ is a twice continuously differentiable function, $\frac{\partial g}{\partial z}(z,p)\neq 0$, and F is three times continuously differentiable then equation (1) has a substantially nonlinear and more general form which is convenient for applications.

We have

Theorem 2. *Suppose that $g(z,p)$ and $v(x,t)$ are twice continuously differentiable functions, while $F(y,t,p)\neq 0$ is an arbitrary three times continuously differentiable function,*

$$\alpha < p < \beta, \qquad z\in\mathbb{R}^1, \qquad y\in\mathbb{R}^1, \qquad x\in D\subset\mathbb{R}^n, \qquad t_1 < t < t_2.$$

Then the function $w(x,t,p)$ that is the root of the equation

$$g(w(x,t,p),p) = -2p\frac{\partial F}{\partial y}\Big/ F\Big|_{y=v(x,t)}$$

satisfies the equation

$$\begin{aligned}\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial t} &= p\frac{g''_{zz}(w,p)}{g'_z(w,p)}\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\\ &\quad + p\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i\partial x_j} - g(w,p)\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}\\ &\quad + \left[\frac{\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial t} - p\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i\partial x_j}}{\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}}\right]\sum_{k,l=1}^n a_{kl}(x,t)\frac{\partial w}{\partial x_k}\frac{\partial v}{\partial x_l}\\ &\quad - \frac{2p}{g'_z(w,p)}\left[\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t} - p\frac{\partial^2 F}{\partial y^2}\right)\Big/ F - \frac{\partial F}{\partial y}\left(\frac{\partial F}{\partial t} - p\frac{\partial^2 F}{\partial y^2}\right)\Big/ F^2\right]\Big|_{y=v(x,t)}\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}.\end{aligned} \tag{2}$$

Proof. Differentiating $g(w(x,t,p),p) = -2p\frac{\partial F}{\partial y}\Big/ F\Big|_{y=v(x,t)}$, we infer

$$g'_z\frac{\partial w}{\partial t} = -2p\left(\frac{\partial^2 F}{\partial y^2}\Big/ F - \left(\frac{\partial F}{\partial y}\right)^2\Big/ F^2\right)\frac{\partial v}{\partial t} - 2p\left(\frac{\partial^2 F}{\partial y\partial t}\Big/ F - \frac{\partial F}{\partial y}\frac{\partial F}{\partial t}\Big/ F^2\right), \tag{3}$$

$$g'_z\frac{\partial w}{\partial x_i} = -2p\left(\frac{\partial^2 F}{\partial y^2}\Big/ F - \left(\frac{\partial F}{\partial y}\right)^2\Big/ F^2\right)\frac{\partial v}{\partial x_i}, \tag{4}$$

$$\begin{aligned}g'_z\frac{\partial^2 w}{\partial x_i\partial x_j} + g''_z\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} &= -2p\left(\frac{\partial^3 F}{\partial y^3}\Big/ F - 3\frac{\partial^2 F}{\partial y^2}\frac{\partial F}{\partial y}\Big/ F^2 + 2\left(\frac{\partial F}{\partial y}\right)^3\Big/ F^3\right)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\\ &\quad - 2p\left(\frac{\partial^2 F}{\partial y^2}\Big/ F - \left(\frac{\partial F}{\partial y}\right)^2\Big/ F^2\right)\frac{\partial^2 v}{\partial x_i\partial x_j}.\end{aligned} \tag{5}$$

Inserting (3)–(5) into (2), we prove Theorem 2. □

Theorem 2 obviously implies

Theorem 3. *Suppose that $g(z,p)$ and $v(x,t)$ are twice continuously differentiable functions, $g'_z(z,p)\neq 0$, and $F(y,t,p)\neq 0$ is a solution to the parabolic equation $\dfrac{\partial F}{\partial t}=p\dfrac{\partial^2 F}{\partial y^2}$. Then the function $w(x,t,p)$ that is the root of the equation*

$$g(w(x,t,p),p)=-2p\frac{\partial F}{\partial y}\Big/F\Big|_{y=v(x,t)},$$

satisfies the following equation with the coefficients independent of F:

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial t}=p\frac{g''_{zz}(w,p)}{g'_z(w,p)}\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}$$
$$+p\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i\partial x_j}-g(w,p)\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}$$
$$+\left[\frac{\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial t}-p\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i\partial x_j}}{\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}}\right]\sum_{k,l=1}^{n}a_{kl}(x,t)\frac{\partial w}{\partial x_k}\frac{\partial v}{\partial x_l}.$$

Now, consider the questions of using the fundamental systems of solutions for representing the solutions and coefficients of the second-order multidimensional evolution equations. To this end, consider the second-order linear ordinary differential equation with meromorphic coefficients and variable parameter $p\geq 0$:

$$F''(z)+b(z)F'+(pa(z)+c(z))F(z)=0. \tag{6}$$

Let $Z_1(z,p)$, $Z_2(z,p)$ be a fundamental system of solutions to (6). Give an example of constructing $Z_1(z,p)$ and $Z_2(z,p)$. Consider the hypergeometric equation

$$z(1-z)F''+(\gamma-(\alpha+\beta+1)z)F'-\alpha\beta F(z)=0,\qquad \gamma>0, \tag{7}$$

with the fundamental system of solutions Z_1 and Z_2 [10]:

$$Z_1(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\frac{z^k}{k!},\qquad |z|<1,$$

$$Z_2(\alpha,\beta,\gamma,z)=z^{1-\gamma}Z_1(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,z),\qquad |z|<1.$$

If in equation (7) we assume the parameters $\gamma=a_0>0$ and $-(\alpha+\beta+1)=b_0$ to be fixed and the values $-\alpha\beta=p\geq 0$ to be variable then, for $z\neq 0$ and $|z|<1$, equation (7) can be rewritten as

$$F''+\frac{a_0+b_0z}{z(1-z)}F'+\frac{p}{z(1-z)}F(z)=0$$

for

$$b(z)=\frac{a_0+b_0z}{z(1-z)},\qquad a(z)=\frac{1}{z(1-z)},\qquad c(z)=0$$

with the explicitly computed $Z_1(z,p)$ and $Z_2(z,p)$.

As above, denote by $a_{ij}(x,t)$, $v(x,t)$, $a_{ij}=a_{ji}$, $x\in D\subset\mathbb{R}^n$, and $t_1<t<t_2$ respectively the continuously differentiable functions with the condition

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\neq 0,$$

and designate as $Q(p)$ and $R(p)$ with $p \geq 0$ the integrable functions decreasing at infinity rapidly enough. We have

Theorem 4. *Suppose that the function*

$$w(x,t) = \int_0^\infty [Q(p)Z_1(v(x,t),p) + R(p)Z_2(v(x,t),p)]e^{-pt}\,dp, \qquad x \in D \subset \mathbb{R}^n, \qquad t_1 < t < t_2,$$

twice differentiable under integral is well defined. Then it satisfies the equation

$$a(v)\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial t} = \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i \partial x_j} + c(v)\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}w$$
$$+\left[\frac{a(v)\sum\limits_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial t} - \sum\limits_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i\partial x_j}}{\sum\limits_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}} + b(v)\right]\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}. \tag{8}$$

Proof. Let $F(z,p)$ be an arbitrary solution to (6) with parameter p, $0 \leq p < \infty$, and let $\alpha(p)$ be a suitable function. Consider the function

$$u(x,t) = \int_0^\infty \alpha(p)F(v(x,t),p)e^{-pt}\,dp$$

and show that it satisfies (8). We have

$$\frac{\partial u}{\partial t} = \int_0^\infty \alpha\left[F'\frac{\partial v}{\partial t} - Fp\right]e^{-pt}\,dp, \qquad \frac{\partial u}{\partial x_i} = \int_0^\infty \alpha F'\frac{\partial v}{\partial x_i}e^{-pt}\,dp,$$
$$\frac{\partial^2 u}{\partial x_i\partial x_j} = \int_0^\infty \alpha\left[F''\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + F'\frac{\partial^2 v}{\partial x_i\partial x_j}\right]e^{-pt}\,dp.$$

Hence,

$$a(v)\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial t}\int_0^\infty \alpha F'\frac{\partial v}{\partial t}e^{-pt}\,dp - a(v)\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\int_0^\infty \alpha F p e^{-pt}\,dp$$
$$-\left\{\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\int_0^\infty \alpha F'' e^{-pt}\,dp + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i\partial x_j}\int_0^\infty \alpha e^{-pt}\,dp\right.$$
$$+\, c(v)\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\int_0^\infty \alpha F(v(x,t),p)e^{-pt}\,dp$$

$$+ \left[\frac{a(v) \sum\limits_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial t} - \sum\limits_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 v}{\partial x_i \partial x_j}}{\sum\limits_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}} + b(v) \right]$$

$$\times \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \int_0^\infty \alpha F' e^{-pt}\, dp \Bigg\}$$

$$= \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \int_0^\infty \alpha(p) \left[F''(z,p) + b(z)F'(z,p) + (pa(z) + c(z))F(z,p)\right]\big|_{z=v(x,t)}\, e^{-pt}\, dp.$$

Choosing $Q(p)$ and $R(p)$ as $\alpha(p)$ and Z_1 and Z_2 as F, we completely prove Theorem 4. $\square$

In view of the representation of a solution $w(x,t)$ to (6) by the formula of Theorem 4

$$w = \int_0^\infty [Q(p)Z_1(v(x,t),p) + R(p)Z_2(v(x,t),p)]e^{-pt}\, dp,$$

a possibility opens for applying the theory of second-order linear differential equations with some constant parameters. We confine ourselves to just one example.

Suppose that $b(z) = 0$, $a(z) = 1$, and $c(z) = 0$ in (6); i.e., $F'' + pF = 0$. We have

$$Z_1(z,p) = \sin \sqrt{p}\, z, \qquad Z_2(z,p) = \cos \sqrt{p}\, z.$$

Assume that

$$R(p) = \sum_{k=0}^{\infty} R_k \delta(p-k), \qquad Q(p) = \sum_{k=0}^{\infty} Q_k \delta(p-k),$$

R_k and Q_k are constant, $|R_k| \le 1/k^\alpha$, and $|Q_k| \le 1/k^\alpha$ with $\alpha > 1$. Then

$$w(x,t) = \sum_{k=0}^{\infty} [R_k \sin \sqrt{k} v(x,t) + Q_k \cos \sqrt{k} v(x,t)] e^{-kt}, \tag{10}$$

series (10) converges for $t \ge 0$, and equation (8) has the form

$$\sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 w}{\partial x_i \partial x_j}$$

$$+ \left[\frac{\sum\limits_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial t} - \sum\limits_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 v}{\partial x_i \partial x_j}}{\sum\limits_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}} \right] \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j}.$$

As was observed many times [4–7], the representations of solutions and coefficients found here can be used in the direct and inverse problems of mathematical physics.

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