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On Complexity of the Bilevel Location and Pricing Problems

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Abstract—We consider the bilevel mixed integer location and pricing problems. Each problem is determined by the optimization problems of the upper and lower levels of which the first describes the choice of location and pricing, while the second models the reaction of the customers on the upper-level solution. The article focuses on studying the computational complexity of bilevel problems with various pricing strategies: uniform, mill, and discriminatory pricing. We show that, for an arbitrary pricing strategy, the corresponding optimization problem is NP-hard in the strong sense, belongs to the class Poly-APX, and is complete in it with respect to AP-reducibility.

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INTRODUCTION

The processes of facility location and pricing are usually studied separately and independently of each other [3, 9, 13, 14, 17], mainly because they lie at different levels of planning. The processes of location are long-term, while the processes of pricing are part of short-term planning. As a consequence, in the majority of cases we choose firstly locations and then prices. However, already [15] pointed out that the separation of location and pricing is unacceptable because the facilities (shops) must be placed taking demand into account, which in some way depends on the prices. On the other hand, decisions on the best prices depend on the facility locations. Thus, the separation of location and pricing in the model can make it impossible to obtain the best variants of location and prices. Moreover, even in the situation when there is no need to know the exact prices on the product produced, but it suffices to know only the price range in the chosen niche of the market to match the competitors, the separation of location and pricing is also pointless [6]. Therefore, the modern approaches to the choice of an efficient mechanism for the interaction between the processes of facility location and pricing rest on their joint analysis in the framework of one model [5, 6, 11, 16].

But to estimate the quality of the solutions made it is also necessary to be able to adequately estimate the reaction of the market and, in particular, customers to the proposed variants of location and pricing. Aiming at that, we find it convenient to model the whole process as a bilevel programming problem [2, 12].

The article is organized as follows: in Section 1 we give the mathematical statements of the problems and in Section 2 present the results.

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1. STATEMENTS OF THE PROBLEMS

Consider the following Stackelberg game. The two types of players participate in the game: the Leader (manufacturer) and the Followers (customers). The manufacturer makes the first move by opening (placing) a facility to manufacture certain homogeneous products and puts a price on the product. Then each customer chooses an open facility at which his total expenses on buying the product and transportation are minimal and makes a purchase in the case that his budget allows these expenses. The goal of the game is to open the facilities and establish the prices for which the manufacturer's revenue (the sum of the prices on the goods bought by the customers) minus the expenses on opening the facilities is maximal.

We consider only the cooperative version of this game. Assume that if the expenses of some customer are minimal at several facilities then he chooses the nearest facility; that is, from the manufacturer's point of view, the chosen facility will have the greatest price of the product. This means that the customers always make choices maintaining the manufacturer's profit.

We confine the discussion to just three pricing strategies [15]: mill pricing, uniform pricing, and discriminatory pricing.

When the first pricing strategy is used, prices are set individually at each open facility. For the uniform pricing, the same price is set at all open facilities. The discriminatory pricing is the strategy which can infringe upon the interests of some groups of customers; that is, at each open facility different prices may be set for different customers.

Introduce the notation:

$I = \{1, \dots, n\}$ is the set of possible locations of open facilities;

$J = \{1, \dots, m\}$ is the set of customers;

$b_j \geq 0$ is the budget of customer j;

$c_{ij} \geq 0$ is the matrix of transportation expenses of customers;

$f_i \geq 0$ is the cost of opening a facility at location $i \in I$ (facility i);

$$y_i = \begin{cases} 1, & \text{if facility } i \text{ is open,} \\ 0, & \text{otherwise;} \end{cases} \qquad x_{ij} = \begin{cases} 1, & \text{if facility } i \text{ serves customer } j, \\ 0, & \text{otherwise.} \end{cases}$$

We use this notation in the formulations of all three problems we study: LDP (location and discriminator pricing) problem, LMP (location and mill pricing) problem, and LUP (location and uniform pricing) problem, which correspond to the three pricing strategies described above. In addition, introduce the following notation different for each formulation:

$p_{ij} \geq 0$ is the price of the product at facility i for customer j;

$p_i \geq 0$ is the price of the product at facility i (the same for all customers);

$p \geq 0$ is the price of the product (the same at all facilities for all customers).

Using this notation, express the Stackelberg game in the case of discriminatory pricing as the following (LDP) problem of bilevel quadratic programming:

$$\sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} - \sum_{i \in I} f_i y_i \;\to\; \max_{p,x,y},$$

$$p_{ij} \geq 0, \qquad y_i \in \{0,1\}, \qquad i \in I, \qquad j \in J,$$

where the vector x stands for the optimal solution to the lower-level problem:

$$\sum_{i \in I} \sum_{j \in J} (b_j - c_{ij} - p_{ij}) x_{ij} \to \max_x,$$

$$\sum_{i \in I} x_{ij} \leq 1, \qquad j \in J,$$

$$x_{ij} \leq y_i, \qquad i \in I, \qquad j \in J,$$

$$x_{ij} \in \{0,1\}, \qquad i \in I, \qquad j \in J.$$

The objective function of the problem determines the manufacturer's revenue. At the lower-level the objective function expresses the budget saved by customers, while the constraints ensure that each customer is served at most at one open facility. These constraints and the definition of objective function also imply that the customer makes a purchase provided that his budget allow it.

Basing on LDP problem, we describe LMP and LUP problems. Refer as the *location and mill pricing problem* (LMP) to the LDP problem with the variable p_i, instead of p_{ij}, and as the *location and uniform pricing problem* (LUP) to the LDP problem with the variable p instead of p_{ij}.

The cooperativity condition introduced above enables us to talk about optimal solutions to LDP, LMP, and LUP problems. In the case of equal expenses, each customer chooses the nearest facility in accordance with the matrix of transportation expenses, which enables the manufacturer to calculate his revenue explicitly.

Express the bilevel problem as a quadratic programming problem with mixed variables:

$$\sum_{i\in I}\sum_{j\in J} p_{ij}x_{ij} - \sum_{i\in I} f_i y_i \to \max_{p,x,y}, \tag{1}$$

$$\sum_{i\in I}(b_j - c_{ij} - p_{ij})x_{ij} \geq 0, \qquad j \in J, \tag{2}$$

$$\sum_{i\in I}(c_{ij} + p_{ij})x_{ij} \leq c_{kj} + p_{kj}, \qquad k \in I, \qquad j \in J, \tag{3}$$

$$\sum_{i\in I} x_{ij} \leq 1, \qquad j \in J, \tag{4}$$

$$x_{ij} \leq y_i, \qquad i \in I, \qquad j \in J, \tag{5}$$

$$p_{ij} \geq 0, \quad x_{ij}, y_i \in \{0,1\}, \qquad i \in I, \qquad j \in J. \tag{6}$$

The objective function (1) determines the manufacturer's revenue. The constraints (2) ensure that the customers stay within their budgets. The fulfillment of constraints (3) leads to the minimization of the total expenses of each customer on purchases and transportation. The constraints (4) mean that each customer can be served at most at one facility. The constraints (5) imply that customers can be served only at open facilities.

Keep the same acronym LDP for this reformulation of the bilevel problem. Similarly we obtain the equivalent one-level representations for the bilevel location problems with uniform pricing and mill pricing. Also we use the acronyms LMP and LUP for the corresponding one-level reformulations of these bilevel problems.

Assume henceforth that all input data f_i, b_j, and c_{ij} amounts to rational numbers.

2. COMPUTATIONAL COMPLEXITY

A key question in studying the complexity of finding an optimal or even an feasible solution to an optimization problem is the relation of this problem to the polynomial hierarchy of recognition problems. Usually we consider only the zeroth and first levels of this hierarchy, namely, the classes P, NP, and co-NP.

Theorem 1. *The* LDP*,* LMP*, and* LUP *problems are* NP*-hard problems in the strong sense.*

Proof. Consider the minimal covering problem [1] NP-hard in the strong sense.

Take a set $M = \{1,\dots,m\}$ and a tuple of its subsets $M_1,\dots,M_n$ such that $\bigcup_{i\in N} M_i = M$, where $N = \{1,\dots,n\}$. A collection $\widetilde{N} \subset N$ of subsets M_i, $i \in \widetilde{N}$ is called a *covering* of M whenever $\bigcup_{i\in\widetilde{N}} M_i = M$. Assign unit weight to each M_i. We need to find a covering of minimal weight.

Reduce the minimal covering problem to the LDP, LMP, and LUP problems, that is, construct the functions g and h such that g constructs from input t of the minimal covering problem some input v of the LDP, LMP, or LUP problem whose length is bounded by a polynomial in the length $|t|$ of t in polynomial time with respect to $|t|$, while h constructs from an optimal solution to the LDP, LMP, or LUP problem for input v an optimal solution to problem t in polynomial time with respect to $|t|$.

To begin with, construct from an arbitrary input of the minimal covering problem an input of the LDP, LMP, or LUP problem; that is, define g. Suppose that $I := N$ is the set of possible locations to open facilities and $J := M$ is the set of customers. Define the budget of each customer: $b_j := 2$ for $j \in J$. Put $c_{ij} := 1$ for $i \in I$ and $j \in M_i$. Otherwise, put transportation expenses equal to 2 and thus block the remaining transportation paths. Suppose that $f_i := 1$ for $i \in I$. Put $\widetilde{I} := \{i \in I \,:\, y_i = 1\}$.

Lemma 1. *The* LDP*,* LMP*, and* LUP *problems for the input* $g(t)$ *are equivalent to the problem* $|J| - |\widetilde{I}| \to \max$ *for* $\widetilde{I} \subseteq I$ *under the constraint* $\bigcup_{i\in\widetilde{I}} M_i = J$.

Proof. Verify firstly that there exist optimal solutions $(p^{\text{LDP}}, x^{\text{LDP}}, y^{\text{LDP}})$, $(p^{\text{LMP}}, x^{\text{LMP}}, y^{\text{LMP}})$, and $(p^{\text{LUP}}, x^{\text{LUP}}, y^{\text{LUP}})$ to the LDP, LMP, and LUP problems for the input $g(t)$ with $p_{ij}^{\text{LDP}} = p_i^{\text{LMP}} = p^{\text{LUP}} = 1$, for $i \in I$ and $j \in J$. Consider an arbitrary optimal solution $(p^{\text{LDP}}, x^{\text{LDP}}, y^{\text{LDP}})$ to the LDP problem with the input $g(t)$ and a tuple $(\widetilde{p}^{\text{LDP}}, x^{\text{LDP}}, y^{\text{LDP}})$, where $\widetilde{p}_{ij}^{\text{LDP}} = 1$ for $i \in I$ and $j \in J$. Checking all constraints, we easily see that this tuple is an feasible solution. The constraints of the LDP problem imply that if $x_{ij}^{\text{LDP}} = 1$ then $p_{ij}^{\text{LDP}} \le 1$. We obtain

$$w_{\text{LDP}}(p^{\text{LDP}}, x^{\text{LDP}}, y^{\text{LDP}}) = \sum_{i\in I}\sum_{j\in J} p_{ij}^{\text{LDP}} x_{ij}^{\text{LDP}} - \sum_{i\in I} y_i^{\text{LDP}}$$
$$\le \sum_{i\in I}\sum_{j\in J} x_{ij}^{\text{LDP}} - \sum_{i\in I} y_i^{\text{LDP}} = w_{\text{LDP}}(\widetilde{p}^{\text{LDP}}, x^{\text{LDP}}, y^{\text{LDP}}),$$

where w_{LDP} is the objective function (1). Hence, $(\widetilde{p}^{\text{LDP}}, x^{\text{LDP}}, y^{\text{LDP}})$ is an optimal solution. For the LMP and LUP problems the argument is similar.

It follows that in order to find an optimal solution to the LDP, LMP, and LUP problems for the input $g(t)$, the brute-force search of feasible solutions of the form $(1, \dots, 1, x, y)$ suffices. Furthermore, if some customer $j \in M_i$ is not served then assign him to facility i. If this facility is closed then open it. At that, the value of the objective function cannot decrease. Thus, these particular cases are equivalent to the problem:

$$|J| - |\widetilde{I}| \to \max_{\widetilde{I}\subseteq I} \quad \text{under the constraint} \quad \bigcup_{i\in\widetilde{I}} M_i = J.$$

The proof of Lemma 1 is complete. ☐

According to Lemma 1, the maximum of the particular cases described above of the problems under study is attained on $\widetilde{I} = \widetilde{N}^*$, a covering of minimal weight, that is, when $\{i \in I : y_i = 1\} = \widetilde{N}^*$.

This completes the proof of Theorem 1. ☐

There is an even simpler proof that the LMP problem is NP-hard. It suffices to consider its particular case in which all facilities are already constructed, that is, on assuming that $f_i = 0$. A reduction of the minimal covering problem to this particular case of the LMP problem is described in [3].

Consider the complete weighted graph K^{n+m} in which the vertices are customers and the possible locations to open facilities, while the weights of edges are transportation expenses. Denote LDP, LMP, and LUP on K^{n+m} by LDP^K, LMP^K, and LUP^K. Then Theorem 1, provided that $c_{ik} = 2$, where either $i, k \in I$ or $i, k \in J$, implies

Corollary 1. *The* LDP^K*,* LMP^K*, and* LUP^K *problems are* NP*-hard in the strong sense even if transportation expenses satisfy the triangle inequality.*

Introduce some notation that correspond to an arbitrary optimization problem A with a criterion for maximizing the objective function:

$L(A)$ is a set of instances of problem A (refer to an arbitrary example $t \in L(A)$ as a problem t);

$\text{OPT}_A(t)$ is the optimal value of the objective function in $t \in L(A)$;

$D_A(t)$ is the set of feasible solutions to $t \in L(A)$;

$F_A(t, s)$ is the value of the objective function in a problem $t \in L(A)$ on a solution $s \in D_A(t)$.

Theorem 1 implies, on assuming that P≠NP, that finding an optimal solution of either the LDP, or LMP, or LUP problem becomes rather laborious as the dimension grows. Then it makes sense to consider the question of finding a "good" feasible solution. Usually in this case we consider the complexity of the problem from the viewpoint of constructing an efficient algorithm for finding an approximate solution with a guaranteed accuracy estimate, that is, the position of the optimization problem in the hierarchy of approximation classes [7]

$$\text{PO} \subseteq \text{FPTAS} \subseteq \text{PTAS} \subseteq \text{APX} \subseteq \text{Log-APX} \subseteq \text{Poly-APX} \subseteq \text{Exp-APX} \subseteq \text{NPO}.$$

Each of these classes describes a certain quality of approximation which its constituent optimization problems enjoy. This hierarchy is used to describe the properties of problems in the class NPO. We can describe it meaningfully as the class of optimization problems for which the corresponding recognition problem is of class NP. The class PO is formed by the problems for each of which some exact polynomial algorithm for solution exists. The class FPTAS consists of the problems for which fully polynomial approximate schemes for solution exist, while the class PTAS is formed by the problems for which some polynomial approximate schemes for solution exist. The classes APX, Log-APX, Poly-APX, and Exp-APX consist of the problems for which there exist polynomial approximate algorithms for solution with constant, logarithmic, polynomial, and exponential estimates for the accuracy of error respectively. In the last three cases, the values of the above-mentioned functions depend on the length of the expression for the input data of the problem. For a formal definition, see [7, 8]. It is also known, on assuming that P≠NP, that the inclusions mentioned above among the classes are proper [7, 8, 10].

In the definition of this hierarchy, the error of a solution $s \in D_A(t)$ to the problem A with an input $t \in L(A)$ is determined as

$$R_A(t, s) = \max\left\{\frac{F_A(t, s)}{\mathrm{OPT}_A(t)}, \frac{\mathrm{OPT}_A(t)}{F_A(t, s)}\right\} \geq 1.$$

It is clear that if A is an optimization problem with a criterion for the maximization of the objective function then

$$R_A(t, s) = \frac{\mathrm{OPT}_A(t)}{F_A(t, s)}.$$

It is shown in [4] that, for a fixed location, the LMP problem belongs to the class Log-APX, while LDP and LUP problems are polynomially solvable. The following establishes in some sense an "upper bound" on the position of the problems under study in the approximation hierarchy:

Lemma 2. *The* LDP, LMP*, and* LUP *problems belong to the class* Poly-APX.

Proof. Consider particular cases of these problems. Denote by $\mathrm{L^1DP}$, $\mathrm{L^1MP}$ and $\mathrm{L^1UP}$ the corresponding problems with at most one open facility. It is shown in [3] that $\mathrm{L^1DP}$, $\mathrm{L^1MP}$, and $\mathrm{L^1UP}$ are polynomially solvable. We use the optimal solutions to these problems as feasible solutions to the original problems. Consider only LDP since the proofs for LMP and LUP are similar.

It is obvious that an arbitrary optimal solution to the $\mathrm{L^1DP}$ problem is an feasible solution to the LDP problem. Verify that

$$\mathrm{OPT_{LDP}}(t) \leq n * \mathrm{OPT_{L^1DP}}(t),$$

where $n = |I|$. Take an arbitrary feasible solution $(\widetilde{y}, \widetilde{p}, \widetilde{x})$ to the LDP problem. Put

$$i^* = \arg \max_{i \in I\,:\,\widetilde{y}_i = 1} \Big\{ \sum_j \widetilde{p}_{ij} \widetilde{x}_{ij} - f_i \Big\},$$

$$y_i^{i^*} = \begin{cases} 1, & \text{if } i = i^*, \\ 0, & \text{otherwise}, \end{cases} \qquad x_{ij}^{i^*} = \begin{cases} \widetilde{x}_{ij}, & \text{if } i = i^*, \\ 0, & \text{otherwise}. \end{cases}$$

Then $(y^{i^*}, \widetilde{p}, x^{i^*})$ is an feasible solution to the L^{1}DP problem. In addition, the following relation holds by definition:

$$F_{\mathrm{LDP}}(t, (\widetilde{y}, \widetilde{p}, \widetilde{x})) = \sum_{i,j} \widetilde{p}_{ij}\widetilde{x}_{ij} - \sum_i f_i \widetilde{y}_i \leq n * \Big(\sum_{i,j} \widetilde{p}_{ij} x_{ij}^{i^*} - \sum_i f_i y_i^{i^*}\Big) \leq n * \mathrm{OPT}_{\mathrm{L^1DP}}(t).$$

We infer that the optimal solution to the L^{1}DP problem is an feasible solution to the LDP problem with the accuracy estimate n. The proof of Lemma 2 is complete. □

Corollary 2. *The* LDP^K*,* LMP^K *and* LUP^K *problems belong to the class* Poly-APX *even if transportation expenses satisfy the triangle inequality.*

Lemma 2 presents some efficient algorithms for finding feasible solutions to the LDP, LMP, and LUP problems with accuracy estimates bounded by polynomials in the length of the input of the corresponding problems. However, this result does not guarantee the absence of polynomial algorithms with better estimates corresponding to the lower-lying classes of the hierarchy. Theorem 2 enables us to solve this problem.

Recall the definition of AP-reducibility [7]. Take two problems A and B of class NPO with a criterion for the maximization of the objective function. Say that A AP-*reduces* to B if and only if there exist two functions φ and ρ and a positive constant α such that

(i) $\varphi(t, r) \in L(B)$ for all $t \in L(A)$ and $r > 1$ and φ is computable in polynomial time with respect to the length $|t|$ of the input t and r;

(ii) $\rho(t, s, r) \in D_A(t)$ for all $t \in L(A)$, $r > 1$, and $s \in D_B(\varphi(t, r))$, and ρ is computable in polynomial time with respect to $|t|$, the length of the solution s, and r;

(iii) it follows from $R_B(\varphi(t, r), s) \leq r$ that $R_A(t, \rho(t, s, r)) \leq 1 + \alpha(r - 1)$ for all $t \in L(A)$, $r > 1$, and $s \in D_B(\varphi(t, r))$.

Recall the definition of closed subclass of the class NPO with respect to some reducibility. A subclass C of the class NPO is *closed under reducibility* Γ whenever the Γ-reducibility of a problem A to a problem $B \in C$ implies that $A \in C$. It is known, on assuming that P≠NP, that the inclusions of classes in the approximation hierarchy are proper, while the classes FPTAS, PTAS, APX, and Log-APX are closed under AP-reducibility [7, 8, 10]. Since AP-reducibility is transitive, we infer that, for every problem which is Poly-APX-complete with respect to AP-reducibility, on assuming that P≠NP there exist no polynomial approximate algorithms with better estimates for the errors of feasible solutions as compared to optimal solutions than the estimates bounded by a polynomial in the length of input.

Theorem 2. *The* LDP*,* LMP*, and* LUP *problems are* Poly-APX-*complete with respect to* AP-*reducibility.*

Proof. Consider the maximal independent set problem which is Poly-APX-complete with respect to AP-reducibility [7, 8, 10]. Take an arbitrary graph $G = (V, E)$ with vertex set V and edge set $E \subseteq V \times V$. Refer as an *independent set* in G to a subset of vertices $\widetilde{V} \subseteq V$ with $\widetilde{V} \times \widetilde{V} \cap E = \varnothing$. Then we can express the maximal independent set (MIS) problem as the optimization problem:

$$|\widetilde{V}| \to \max_{\widetilde{V} \subseteq V} \quad \text{with the constraint} \quad \widetilde{V} \times \widetilde{V} \cap E = \varnothing.$$

Consider the reduction of the MIS problem to the LDP problem which is easy to transform into reductions to the LMP and LUP problems.

Given $t \in L(\mathrm{MIS})$, construct $\varphi(t) \in L(\mathrm{LDP})$ as follows: Put $w(i) := |j \in V : (i, j) \in E|$, which is the number of vertices adjacent to vertex i. Given $i \in I$, consider the tuple of customers J_i such that $J_i \cap J_k = \varnothing$ for $k \in I \setminus i$, $J_i \cap E = \varnothing$, as well as $|J_i| = |I| - w(i) = n - w(i)$. Denote by $\varphi(t)$ the input in which

$$I := V, \qquad J := \Big(\bigcup_{i \in I} J_i\Big) \cup E,$$

the expenses to open an arbitrary facility are equal to $n-1$, the budget of every customer equals 2, while transportation expenses are defined as follows: at facility i, the transportation expenses c_{ij} are equal to 1 if $j \in J_i$ or $j=(i,k) \in E$ for some $k \in I$, and otherwise transportation expenses are equal to 2.

Take an feasible solution (y,p,x) to the LDP problem. Apply the following algorithm constructing from (y,p,x) an auxiliary feasible solution $(\widetilde{y},\widetilde{p},\widetilde{x})$ to LDP, constructing from which a solution to the MIS problem is easy. Put $\widetilde{p}_{ij} := 1$ for all $i \in I$ and $j \in J$, as well as $\widetilde{y} := y$ and $\widetilde{x} := x$. Inspect all facilities $i \in I$ one by one. If $\widetilde{y}_i = 1$ and facility i is unprofitable, that is,

$$\sum_{j \in J} \widetilde{p}_{ij}\widetilde{x}_{ij} - f_i \le 0,$$

then put $\widetilde{y}_i := 0$ and $\widetilde{x}_{ij} := 0$ for all $j \in J$.

By the choice of prices and inputs, the solution $(\widetilde{y},\widetilde{p},\widetilde{x})$ is feasible. It is also obvious that there is no difference precisely between location and pricing problems this solution refers to. Observe also that the set $\widetilde{V} = \{i \in I : \widetilde{y}_i = 1\}$ is independent, that is, an feasible solution to the MIS problem. Indeed, if there exists a pair of distinct facilities $i,k \in \widetilde{V}$ then, since customer (i,k) can only be served at one facility (assume for definiteness that facility i does not serve him), the profit at facility i equals

$$\sum_{j \in J} \widetilde{p}_{ij}\widetilde{x}_{ij} - f_i = \sum_{j \in J\setminus(i,k)} \widetilde{p}_{ij}\widetilde{x}_{ij} - f_i \le \sum_{s=\overline{1,n-1}} (1) - (n-1) = 0$$

by the definitions of transportation and opening expenses. This implies that $i \notin \widetilde{V}$. We infer that $\widetilde{V}$ is an independent set. The complexity of the algorithm described above equals $O(n^2)$, and so it is polynomial in the length of the input of the MIS problem. Therefore, we determined the function ρ of the definition of AP-reducibility. Put $\alpha = 1$. For the reduction of the MIS problem to the LDP problem described above to be AP-reduction, it remains to show that

$$\text{if } R_B(\varphi(t,r),s) \le r, \text{ then } R_A(t,\rho(t,s,r)) \le 1 + \alpha(r-1).$$

By construction, the solution $(\widetilde{y},\widetilde{p},\widetilde{x})$ satisfies

$$F_{\mathrm{LDP}}(\varphi(t),(y,p,x)) \le F_{\mathrm{LDP}}(\varphi(t),(\widetilde{y},\widetilde{p},\widetilde{x})).$$

The optimal value in the LDP problem equals $n|V^*| - (n-1)|V^*| = |V^*|$, where V^* is an optimal solution to the MIS problem. Thus, we obtain

$$r \ge R_{\mathrm{LDP}}(\varphi(t),(y,p,x)) \ge \frac{\mathrm{OPT}_{\mathrm{LDP}}(t)}{F_{\mathrm{LDP}}(\varphi(t),(\widetilde{y},\widetilde{p},\widetilde{x}))} = \frac{n|V^*| - (n-1/n)|V^*|}{n|\widetilde{V}| - (n-1/n)|\widetilde{V}|} = \frac{|V^*|}{|\widetilde{V}|}.$$

Since $\alpha = 1$, we obtain

$$R_{\mathrm{MIS}}(t,\rho(t,(y,p,x))) = \frac{|V^*|}{|\widetilde{V}|} \le r.$$

The argument for the LMP and LUP problems is similar.

Thus, every problem of class Poly-APX is AP-reducible to location and pricing problems under study, while Lemma 2 implies that they belong to the class Poly-APX.

The proof of Theorem 2 is complete. □

If we put $c_{ik} = 2$, where either $i,k \in I$ or $i,k \in J$, then Theorem 2 and Corollary 2 implies

Corollary 3. *The* LDP^K*,* LMP^K*, and* LUP^K *problems are* Poly-APX*-complete with respect to* AP*-reducibility even if transportation expenses satisfy the triangle inequality.*

CONCLUSION

The main result of this article is that the bilevel problems under study with uniform pricing, mill pricing, and discriminatory pricing are Poly-APX-complete with respect to AP-reducibility. Thus, on assuming that P≠NP, for this problems there cannot exist exact or approximate polynomial algorithms with some better estimates for the error of feasible solutions as compared to optimal solutions than the estimates bounded by a polynomial in the length of input.

Subsequently we expect to clarify the relations of these problems to the polynomial hierarchy.

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