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Interval-Stochastic Thermal Processes in Electronic Systems: Modeling in Practice

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Abstract—Mathematical and computer modeling of thermal processes, applied presently in thermal design of electronic systems, is based on the assumption that the factors determining the thermal processes are completely known and uniquely determined, that is, they are deterministic. Meanwhile, practice shows that the determining factors are of indeterminate interval-stochastic character. Moreover, thermal processes in electronic systems are nonstationary and nonlinearly depend on both the stochastic determining factors and the temperatures of electronics elements and environment. At present, the literature does not present methods of mathematical modeling of nonstationary, stochastic, nonlinear, interval-stochastic thermal processes in electronic systems to model thermal processes, which satisfy all the above-listed requirements to modeling adequacy. The present paper develops a method of mathematical and computer modeling of the nonstationary interval-stochastic nonlinear thermal processes in electronic systems. The method is based on obtaining equations describing the dynamics of time variation of statistical measures (expectations, variances, covariances) of temperature of electronic system elements with given statistical measures of the initial interval-stochastic determining factors. A practical example of applying the developed approach to a the real electronic system is given.

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1. INTRODUCTION

The author, in his summarized paper [2], has shown that the methods of mathematical and computer modeling applied nowadays in thermal design of electronic systems (ESs) are based on the assumption on the deterministic character of thermal processes in ES and, hence, they are not adequate to the real thermal processes in ES, which in practice leads to errors in designing and to undesirables consequences [3, 5]. This is caused by the fact that in practice the determining factors in ES are ambiguous, interval, and liable to significant spread. The interval ambiguity of the determining factors causes interval uncertainty of temperature distributions in the ES and its elements. This means that temperature value of each element in the real ESs is not exact and unique, but represents an interval of possible temperature values that may occur in ES functioning in practice. In [2, 7] it has been analyzed in detail that the causes of the ambiguous interval character of the determining factors are, first, statistical technological spread of ES parameters and elements in ES fabrication and assembling; second, random factors arising during ES functioning; and third, random environment parameters. The interval-stochastic ambiguity of the factors determining the thermal processes in the ES and its elements is fundamental and unavoidable regardless of the technologies used and output control.

In the present paper we demonstrate application of the methods developed in [2] for modeling the interval-stochastic temperature distributions in the real ESs (note, in this paper we use the same notations as in [2]).

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2. STOCHASTIC MATHEMATICAL MODEL OF THERMAL PROCESSES IN ES

As is known [1, 4], the stochastic processes are characterized by laws of distribution of all orders. They can be determined only in some pathologic cases that are not interesting for practice in thermal design of real ESs. At the same time, for modeling of thermal processes in ES it is unnecessary to know the distribution laws since the most important, for engineering practice, thermal process characteristics are statistical measures of stochastic temperatures $T_i(t,\omega)$, $i = 1, 2, \ldots, N+1$, of elements of the thermal ES model [2, Fig. 1], namely, expectations, variances, standard deviations, and covariances between different stochastic temperatures. For analyzing the stochastic thermal processes in ES it is, therefore, necessary to have equations for determining the mentioned statistical measures. Solving these equations, one will be able to find the desired temperatures of ES elements, determine the boundaries of intervals containing the real temperature values of ES elements.

The initial nonstationary stochastic equations of the thermal processes in ES are nonlinear, hence, direct determination of statistical measures of stochastic temperature distributions in ES is impossible. Meanwhile, random deviations of random functions involved in the equations from their means are quite small compared to their expected values. This makes it possible to apply methods of linearization with respect to centered random functions [4] to the initial nonlinear stochastic equations of thermal processes in ES [2]. As a result of linearization, one obtains equations that remain nonlinear with respect to mathematical expectations of nonstationary temperatures of the elements, but become linear with respect to the centered random functions. Such an approach enables one to find adequate finite equations that directly describe the nonstationary statistical measures of random temperatures of elements in the thermal ES model [2].

In the method developed in [2] the heat flows both between the elements and the elements and the environment, and also the rate of the fluid flow through the ES package (shell) are linearized with respect to the centered stochastic functions of ES element temperatures $T_i(t,\omega)$, $i = 1, 2, \ldots, N+1$, environment temperature $T_e(t,\omega)$, conductive heat conductivities $g_{ij}(\omega)$, and fluid flow rate $G_a(t,\omega)$:

$$\overset{\circ}{T}_i = \overset{\circ}{T}_i(t,\omega) = T_i(t,\omega) - \overline{T}_i(t), \quad \overset{\circ}{T}_e(t,\omega) = T_e(t,\omega) - \overline{T}_e(t),$$

$$\overset{\circ}{g}_{ij} = \overset{\circ}{g}_{ij}(\omega) = g_{ij}(\omega) - \overline{g}_{ij}, \quad \overset{\circ}{G}_a = \overset{\circ}{G}_a(t,\omega) = G_a(t,\omega) - \overline{G}_a(t),$$

where $\overline{T}_i(t)$, $\overline{T}_e(t)$, $\overline{g}_{ij}$, and $\overline{G}_a(t)$ are mathematical expectations of the corresponding stochastic functions.

Since for the centered random functions in the working range of ES temperatures, bounded from above by 125°C, the conditions $|\overset{\circ}{T}_i/\overline{T}_i| < 1$, $|\overset{\circ}{T}_e/\overline{T}_e| < 1$, $|\overset{\circ}{g}_{ij}/\overline{g}_{ij}| < 1$, and $|\overset{\circ}{G}_a/\overline{G}_a| < 1$ hold, the heat flows between the elements, the elements and the environment, and the flow rate are continuous functions without discontinuities and angular points and have continuous derivatives of different orders, then the heat flows and the fluid flow rate can be linearized by the Taylor expansion method with retaining terms of order not higher than one [2, 4].

The error estimate $\varepsilon = |\Delta\overset{\circ}{T}/\Delta\overline{T}|$ arising as a result of linearizing the heat flows of natural convection satisfies the inequality [2] $\varepsilon \leq \sqrt{2\delta_0/n(n+1)}$, where δ_0

is the relative error of replacing the convective flow by its approximate value, retaining the first-order infinitesimal terms in the Taylor expansion; $0 < n \leq 1$ is the degree of temperature difference ΔT^{n+1} to which the convective heat flow is proportional; for different convection laws $n = 1/4, 1/3, 1/8$, etc. For instance, for $\delta_0 \leq 5\%$ the admissible deviation is: $\varepsilon \leq 84\%$, for natural convection obeying the law of 1/8 power, $\varepsilon \leq 57\%$, for the law of 1/4 power, and $\varepsilon \leq 47\%$ for the law of 1/3 power. In absolute units, e.g., for the 1/4 convection law and $\Delta\overline{T} = 40$°C, this means that the admissible variation of the centered temperature difference will be $\Delta\overset{\circ}{T} \leq 23$°C. The linearization error estimate for the radiation heat flows shows [2] that the relative deviation $\varepsilon = |\overset{\circ}{T}, /\overline{T}|$ of the centered temperature $\overset{\circ}{T} = T - \overline{T}$ will satisfy the condition $\varepsilon \leq \sqrt{\delta_0/6}$. For example, for the linearization error $\delta_0 \leq 5\%$ the admissible relative deviation ε is 9%. That is, if the absolute element temperature is $\overline{T} = 400$ K (the limiting value for ES), then the admissible range of the random temperature deviation of the element in absolute units should not exceed 36°C. Analysis of the values of errors of heat flows, linearized by Taylor series with retention of terms only of the first infinitesimal order, evidences that the employed linearization method allows modeling of thermal processes in ES with accuracy sufficient for engineering practice.

Using the method developed in [2] we obtained a matrix system of stochastic equations with respect to the sought-for stochastic temperature vector $T(t,\omega) = (T_1(t,\omega), T_2(t,\omega), \ldots, T_{N+1}(t,\omega))'$ of elements of the thermal ES model ($(t,\omega) \in [0,\tau] \times \Omega$):

$$H\frac{dT(t,\omega)}{dt} + A(\overline{T},t)\overline{T}(t) + B(\overline{T},t)\overset{\circ}{T}(t,\omega) + \overset{\circ}{D}(\omega)\overline{T}(t)$$

$$= \overset{\circ}{T}_e(t,\omega)C(\overline{T},t) + \overline{T}_e(t)R(\overline{T},t) + \Phi(t,\omega), \quad T(t=0,\omega) = T_0(\omega), \tag{1}$$

where $\overline{T}(t) = (\overline{T}_1(t), \overline{T}_2(t), \ldots, \overline{T}_{N+1}(t))'$ is the $N+1$-vector of mean temperatures of the thermal model elements; $\overset{\circ}{T}(t,\omega) = T(t,\omega) - \overline{T}(t)$ is the stochastic $N+1$-vector of element temperatures, centered with respect to the vector $\overline{T}(t)$; $T_0(\omega)$ is the $N+1$-vector of stochastic initial conditions of element temperatures; H is the deterministic diagonal $(N+1)\times(N+1)$ matrix with elements $h_i = \rho_i c_i V_i$ in which V_i is the volume, ρ_i is density, and c_i is specific heat capacity of the ES element material; $A(\overline{T},t)$, $B(\overline{T},t)$ are deterministic square $(N+1)\times(N+1)$ matrices that are nonlinearly dependent on the mathematical expectations $\overline{T}(t)$; $\overset{\circ}{D}(\omega)$ is the stochastic square symmetric $(N+1)\times(N+1)$ matrix with elements $\overset{\circ}{d}_{ij}(\omega)$ that are equal to linear functions of random heat conductive conductivities $\overset{\circ}{g}_{ij}(\omega)$; $\Phi(t,\omega)$ is stochastic $N+1$-vector of powers of internal heat sources in the ES elements; $C(\overline{T},t)$, $R(\overline{T},t)$ are deterministic $N+1$-vectors that are nonlinearly dependent on the vector $\overline{T}(t)$; $(\,\cdot\,)'$ is transpose operation.

3. EQUATIONS FOR STATISTICAL MEASURES OF STOCHASTIC TEMPERATURES OF ES ELEMENTS

In the practice of modeling and thermal design of the electronic systems the main and most informative characteristics of stochastic element temperatures $T_i(t,\omega)$, $i = 1,2,\ldots,N+1$, are the following statistical measures ($E\{\,\cdot\,\}$ is an expectation operator):

—the mean vector $\overline{T}(t) = (\overline{T}_1(t), \overline{T}_2(t), \ldots, \overline{T}_{N+1}(t))'$, where $\overline{T}_i(t) = E\{T_i(t,\omega)\}$;

—covariance $(N+1)\times(N+1)$-matrix $K_{TT}(t)$ with (i,j)-elements equal to covariances

$$K_{Ti,Tj}(t) = E\{\overset{\circ}{T}_i(t,\omega)\overset{\circ}{T}_j(t,\omega)\}$$

between temperatures of different elements $i,j = 1,2,\ldots,N+1$;

—vector of variances

$$Var(t) = (Var_{T1}(t), Var_{T2}(t), \ldots, Var_{T,N+1}(t))^t,$$

where $Var_{Ti}(t) = E\{(\overset{\circ}{T}_i(t,\omega))^2\}$, which are equal to diagonal elements of the covariance matrix $K_{TT}(t)$, and also the vector of mean square deviations $\sigma(t) = (\sigma_{T1}(t), \sigma_{T2}(t), \ldots, \sigma_{T,N+1}(t))^t$, where $\sigma_{Ti}(t) = \sqrt{Var_{Ti}(t)}$.

Having determined the mentioned statistical measures, we can find the variation intervals of stochastic temperatures of the ES elements with the real values of their temperatures. Knowing the found mathematical expectations $\overline{T}_i(t)$ and the mean square deviations $\sigma_{Ti}(t)$ of stochastic temperatures of each element in the thermal model $T_i(t,\omega)$, $i = 1,2,\ldots,N+1$, one can find lower $T_{Bot,i}(t)$ and upper $T_{Up,i}(t)$ boundaries of temperature intervals $[T_{Bot,i}(t),\ T_{Up,i}(t)]$, which vary with time in the evolving thermal process in ES, namely, $T_{Bot,i}(t) = \overline{T}_i(t) - \chi\cdot\sigma_{Ti}(t)$, $T_{Up,i}(t) = \overline{T}_i(t) + \chi\cdot\sigma_{Ti}(t)$, where χ is a coefficient determining the width of the interval of possible values of stochastic ES element temperature, and depending on the adopted value of the probability P with which the real temperature values of the elements will be bounded by the found interval. The estimate χ is determined by the Chebyshev inequality [4]: $P\{\overset{\circ}{T}_i(t,\omega) \leq \chi \cdot \sigma_{Ti}(t)\} \geq 1 - 1/\chi^2$.

The initial data for the numerical calculations of the statistical measures, which have to be a priori known, are statistical measures of the following stochastic factors that completely determine the stochastic thermal process in ES:

—mathematical expectation $\overline{T}_e(t)$ and variance $Var_{Te}(t)$ of environmental stochastic temperature $T_e(t,\omega)$;

—mathematical expectations $\overline{\Phi}_i(t)$ and variances $Var_{\Phi i}(t)$ of stochastic powers of the internal heat sources of the elements $\Phi_i(t,\omega)$, $i = 1, 2, \ldots, N$ ($\Phi_{N+1} = 0$);

—mathematical expectations $\overline{g}_{ij}$ and variances Var_{gij} of random values of thermal conduction conductivities $g_{ij}(\omega)$, $i, j = 1, 2, \ldots, N$ ($g_{i,N+1} = g_{N+1,j} = 0$);

—mathematical expectations $\overline{T}_{0,i}$ and variances $Var_{T0,i}$ of initial stochastic temperatures of the elements $T_{0,i}(\omega)$, $i = 1, 2, \ldots, N+1$.

The equations obtained in [2] for determining the statistical measures of stochastic temperatures of the ES elements have the following form:

- matrix equation for determining the mean vector $\overline{T}(t)$

$$H\frac{d\overline{T}}{dt} + A(\overline{T},t)\overline{T}(t) = \overline{T}_e R(\overline{T},t) + \overline{\Phi}, \quad \overline{T}(t=0) = \overline{T}_0, \tag{2}$$

in which the mean environmental temperature $\overline{T}_e$, the powers of internal heat sources $\overline{\Phi} = (\overline{\Phi}_1, \overline{\Phi}_2, \ldots, \overline{\Phi}_{N+1})'$, and the initial temperatures of the elements $\overline{T}_0 = (\overline{T}_{0,1}, \overline{T}_{0,2}, \ldots, \overline{T}_{0,N+1})'$ are a priori known from the source data;

- equation for covariance matrix $K_{TT}(t) = E\{\overset{\circ}{T}(t,\omega)\overset{\circ}{T}'(t,\omega)\}$

$$\begin{aligned}&\frac{dK_{TT}(t)}{dt} + H^{-1}B(\overline{T},t)K_{TT}(t) - H^{-1}C(\overline{T},t)K'_{TTe}(t) + H^{-1}F(\overline{T},t)\\&+K_{TT}(t)B'(\overline{T},t)H^{-1} - K_{TTe}(t)\cdot C'(\overline{T},t)\cdot H^{-1} + F'(\overline{T},t)H^{-1}\\&= H^{-1}K_{\Phi T}(t) + K'_{\Phi T}(t)H^{-1}, \quad K_{TT}(t=0) = K_{TT,0},\end{aligned} \tag{3}$$

where $F(\overline{T},t) = E\{\overset{\circ}{D}(\omega)\overline{T}(t)\overset{\circ}{T}'(t,\omega)\}$ is an $(N+1)\times(N+1)$ matrix defined hereafter;

$$K_{T0} = E\{\overset{\circ}{T}_0(\omega)\overset{\circ}{T}'_0(\omega)\}$$

is the a priori known covariance $(N+1)\times(N+1)$ matrix of random initial temperatures $T_0(\omega)$ of the thermal model elements;

- equations for the covariance vector $K_{TTe}(t) = E\{\overset{\circ}{T}(t,\omega)\overset{\circ}{T}_e(\omega)\}$ and covariance matrix $K_{\Phi T}(t) = E\{\overset{\circ}{\Phi}(\omega)\overset{\circ}{T}'(t,\omega)\}$ involved in (3):

$$H\frac{dK_{TTe}(t)}{dt} + B(\overline{T},t)\cdot K_{TTe}(t) = C(\overline{T},t)\cdot Var_{Te}, \quad K_{TTe}(t=0) = 0, \tag{4}$$

$$\frac{dK_{\Phi T}(t)}{dt}H + K_{\Phi T}(t)B'(\overline{T},t) = K_{\Phi}, \quad K_{\Phi T}(t=0) = 0, \tag{5}$$

where K_{Φ} is a diagonal matrix with elements equal to variances of powers of the internal heat sources $Var_{\Phi i}(t)$, $i = 1, 2, \ldots, N+1$;

- equation for the matrix $M_k(t) = E\{\overset{\circ}{d}_k(\omega)\overset{\circ}{T}'_k(t,\omega)\}$ defining the matrix $F(\overline{T}, t) = \sum_{k=1}^{N+1} \overline{T}_k(t) \cdot M_k(t)$ involved in (3):

$$\frac{dM_k(t)}{dt} H + M_k(t) B'(\overline{T}, t) = L_k(\overline{T}, t), \quad M_k(t = 0) = 0, \tag{6}$$

where $L_k(\overline{T}, t) = E\{\overset{\circ}{d}_k(\omega)\overline{T}(t)\overset{\circ}{D}(\omega)\}$ is an $(N+1) \times (N+1)$ matrix with elements equal to linear functions of variances Var_{gij} of random quantities $\overset{\circ}{g}_{ij}(\omega)$, $i, j = 1, 2, \ldots, N$, and sought-for temperature expectations $\overline{T}_1(t), \overline{T}_2(t), \ldots, \overline{T}_{N+1}(t)$. Elements of the matrix $L_k^{(}\overline{T}, t)$ are determined from the known initial data for the random quantities $\overset{\circ}{g}_{ij}(\omega)$ and are calculated for each particular ES design.

Thus, the statistical measures of stochastic temperatures of the elements $T_i(t, \omega)$, $i = 1, 2, \ldots, N+1$, of the thermal ES model, namely, the mean vector $\overline{T}(t)$ and the covariance matrix $K_{TT}(t)$, are determined via solving differential matrix equations (2)–(6). At that, to obtain the solution with respect to the covariance matrix $K_{TT}(t)$, one has to solve also differential matrix equations for the covariance vector $K_{TTe}(t)$ (4), covariance matrix $K_{\Phi T}(t)$ (5), and matrix $M_k(t)$ (6) to find from it the matrix $F(\overline{T}, t)$. The numerical solution to the obtained equations can be obtained by any sufficiently accurate method (e.g., by Runge–Kutta method). Determination of the desired statistical measures for all ES elements at each time is easily programmable and does not require much computer time and RAM.

4. A PRACTICAL MODELING EXAMPLE

We will consider the developed mathematical methods and models using, as an example, ES (Fig. 1a) as an example, which represents an integrated microcircuit (IC) in a TQFP-32 package, the IC being soldered to a multilayer printed circuit board (PCB). The power consumed by the IC is dissipated in heat that is transported by conduction into the multilayer PCB structure via the IC package leads soldered to the PCB, and through the air gap between the IC and the PCB, and then spreads over the multilayer PCB structure. There occurs heat transfer from the surfaces of PCB and IC package to the environment as a result of convection (natural) and radiation. Parameters of the thermal process in the ES, such as thickness of the air gap between the IC and the PCB, IC power consumption, and temperature of

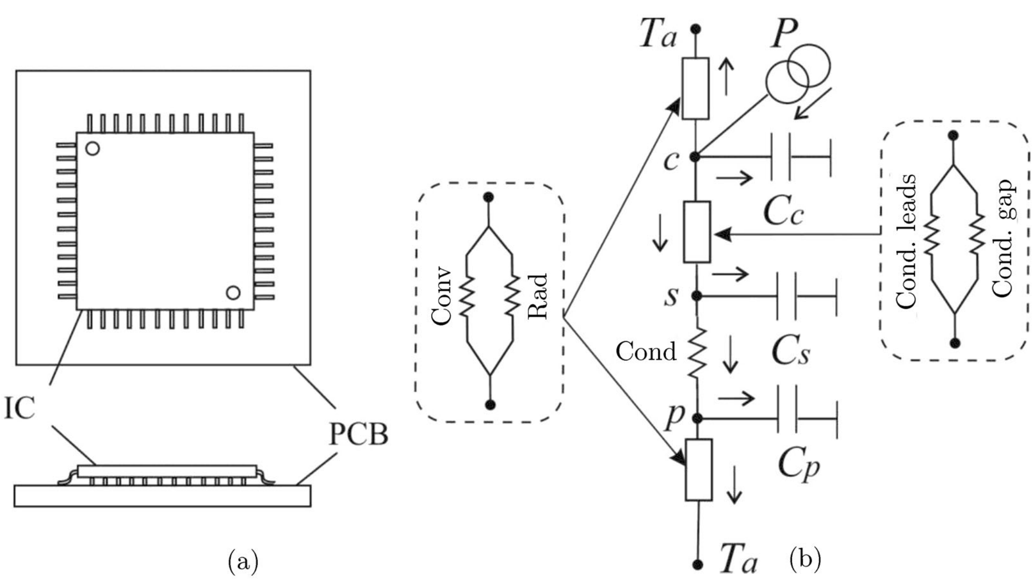

Fig. 1. Electronic system: (a) the integrated microcircuit (IC) welded to the printed circuit board (PCB); (b) its equivalent thermal circuit.

ES environment, vary from sample to sample (in a batch of "identical" ESs) and may take any values within their variation intervals whose width is caused by statistical spread of the adopted technology of IC production and mounting. The interval uncertainty of the input factors determines the interval character of the thermal processes in ES. Dynamics of development of the interval thermal processes and the values of boundaries of their variation at each time instant is determined by the obtained equations (2)–(6).

The interval-stochastic determining parameters of the thermal processes in ES are IC consumption power $P(\omega)$, thickness $\delta(\omega)$ of the air gap between IC and PCB, and ES environment temperature $T_a(\omega)$. The stochastic mathematical model describing the nonstationary interval-stochastic thermal processes in the thermal model of ES (Fig. 1b), namely, the nonstationary interval-stochastic temperatures of the IC package $T_c(t,\omega)$, PCB pad under the IC package $T_s(t,\omega)$, and PCB $T_p(t,\omega)$, has the following form ($(t,\omega) \in [0,\tau] \times \Omega$):

$$h_c \frac{dT_c(t,\omega)}{dt} + J_{ca}(T_c, T_a, t, \omega) + J_{cs}(T_c, T_s, g, t, \omega) = P(\omega),$$

$$h_s \frac{dT_s(t,\omega)}{dt} + J_{sp}(T_s, T_p, t, \omega) - J_{cs}(T_c, T_s, g, t, \omega) = 0,$$

$$h_p \frac{dT_p(t,\omega)}{dt} + J_{pa}(T_p, T_a, t, \omega) - J_{sp}(T_s, T_p, t, \omega) = 0,$$

where $h_c = \rho_c c_c V_c$ is the total heat capacity of the IC package of volume V_c, density ρ_c, and specific heat capacity c_c; $h_s = \rho_s c_s V_s$ is the total heat capacity of the board pad under the IC of volume V_s, density ρ_s, and specific heat capacity c_s; $h_p = \rho_p c_p V_p$ is the total heat capacity of the board of volume V_p, density ρ_p, and specific heat capacity c_p; $g(\omega) = \lambda_{gap} S_c / \delta(\omega)$ is the conduction conductivity of the IC–PCB gap, $\delta(\omega)$ is the random value of the gap thickness, λ_{gap} is the heat conductivity of the material in the gap (in our case, air), S_c is the gap area equal to the area of the IC package base turned to the board; $J_{sp}(T_s, T_p, t, \omega)$ is the heat flow of spread over the PCB; $J_{c(p)a}(T_{c(p)}, T_a, t, \omega) = J^{conv}_{c(p)a} + J^{rad}_{c(p)a}$ is the summarized heat flow of convection (natural) and radiation from the IC (PCB) package into the environment; $J_{cs}(T_c, T_s, t, \omega) = J^{cond}_{cs,lead} + J^{cond}_{cs,gap}$ is the summarized heat flow of convective heat transfer from the IC to the PCB via the package leads and the air gap.

The heat flows of natural convection, radiation, and conduction have the following form [6]:

$$J^{conv}_{c(p)a} = a_1 (T_{c(p)} - T_a)^{5/4}, \quad J^{rad}_{c(p)a} = a_2 (T^4_{c(p)} - J^4_a),$$

$$J^{cond}_{cs,lead} = \lambda_{lead} S_{lead} N_{lead} (T_c - T_s)/l_{lead}, \quad J^{cond}_{cs,gap} = g(\omega)(T_c - T_s),$$

where a_1 is a coefficient proportional to area of the heat-transferring surface of the IC (PCB) package and the coefficient of natural convection heat transfer to the medium (by the 1/4 law); a_2 is a coefficient proportional to area of the heat-transferring surface of the IC (PCB) package, emissivity, and angular irradiance coefficient; l_{lead}, λ_{lead}, S_{lead}, and N_{lead} are length, thermal conduction, section area, and the number of IC package leads, respectively.

By virtue of the nonlinear dependence of the heat flows on temperature of the ES elements and the environment, and also the stochastic gap conductivity $g(\omega)$, these dependences have to be linearized with respect to the centered stochastic temperatures $\overset{\circ}{T}_c(t,\omega)$, $\overset{\circ}{T}_s(t,\omega)$, $\overset{\circ}{T}_p(t,\omega)$, $\overset{\circ}{T}_a(\omega)$, and the heat conductive conductivity $\overset{\circ}{g}(\omega)$ according to the method [2]. As a result, instead of the initial equations of the mathematical model we obtain the following linear stochastic equations with respect to the centered stochastic temperatures $T_c(t,\omega)$, $T_s(t,\omega)$, and $T_p(t,\omega)$:

$$h_c \frac{dT_c(t,\omega)}{dt} + \overline{J}_{ca} + \overline{J}_{cs} + f_1 \overset{\circ}{T}_c(t,\omega) + f_2 \overset{\circ}{T}_s(t,\omega) + f_3 \overset{\circ}{T}_a(\omega) + f_4 \overset{\circ}{g}(\omega) = P(\omega),$$

$$h_s\frac{dT_s(t,\omega)}{dt}+\overline{J}_{sp}-\overline{J}_{cs}+f_5\overset{\circ}{T}_c(t,\omega)+f_6\overset{\circ}{T}_s(t,\omega)+f_7\overset{\circ}{T}_p(t,\omega)+f_8\overset{\circ}{T}_a(\omega)+f_9\overset{\circ}{g}(\omega)=0,$$

$$h_p\frac{dT_p(t,\omega)}{dt}+\overline{J}_{pa}-\overline{J}_{sp}+f_{10}\overset{\circ}{T}_s(t,\omega)+f_{11}\overset{\circ}{T}_p(t,\omega)+f_{12}\overset{\circ}{T}_a(\omega)=0,$$

where $f_k=\phi(\overline{T}_c(t),\overline{T}_s(t),\overline{T}_p(t),\overline{g},t)$ and $\overline{J}_{ca(cs,sp)}=\varphi(\overline{T}_c(t),\overline{T}_s(t),\overline{T}_p(t),\overline{g},t)$ are functions of mean temperatures $\overline{T}_c(t)$, $\overline{T}_s(t)$, $\overline{T}_p(t)$ and heat conduction of the gap $\overline{g}$.

The latter equations can be written in the form of matrix equation:

$$H\frac{dT(t,\omega)}{dt}+\overline{F}(\overline{T},t)+A(\overline{T},t)\overset{\circ}{T}(t,\omega)+D(\overline{T},t)\overset{\circ}{g}(\omega)+B(\overline{T},t)\overset{\circ}{T}_a(\omega)=P(\omega)I_1,$$

with the initial condition

$$T(0,\omega)=T_a(\omega)I_4,$$

where $T(t,\omega)=(T_c(t,\omega),T_s(t,\omega),T_p(t,\omega))'$ is a stochastic temperature vector; $\overline{T}(t)=(\overline{T}_c(t),\overline{T}_s(t),\overline{T}_p(t))'$ is a vector of mean stochastic temperatures; $\overset{\circ}{T}(t,\omega)=T(t,\omega)-\overline{T}(t)=(\overset{\circ}{T}_c(t,\omega),\overset{\circ}{T}_s(t,\omega),\overset{\circ}{T}_p(t,\omega))'$ is a vector of centered temperatures; H is a diagonal matrix with elements arranged along diagonal h_c, h_s, h_p; $\overline{F}(t)=(\overline{J}_{ca}+\overline{J}_{cs},\overline{J}_{sp}-\overline{J}_{cs},\overline{J}_{pa}+\overline{J}_{sp})'$ is a heat flow vector; $A(\overline{T},t)$ is a square matrix equal to

$$A(\overline{T},t)=\begin{pmatrix} f_1 & f_2 & 0\\ f_5 & f_6 & f_7\\ 0 & f_{10} & f_{11}\end{pmatrix};$$

$D(\overline{T},t)=(f_4,f_9,0)'$, $D(\overline{T},t)=(f_3,f_8,f_{12})'$, $I_1=(100)'$, $I_4=(111)'$ are vectors; $(\cdot)'$ denotes transpose operation.

Applying to the obtained matrix equation the expectation operator, obtain equations for finding the expectation temperatures $\overline{T}_c(t)$, $\overline{T}_s(t)$, $\overline{T}_p(t)$ ($\overline{P}$ is mathematical expectation of the IC consumption power):

$$H\frac{d\overline{T}(t)}{dt}+\overline{F}(\overline{T},t)=\overline{P}I_1,\quad \overline{T}(0)=\overline{T}_aI_4.$$

The matrix equation for the covariance matrix $K_{TT}(t)$ whose diagonal elements are equal to temperature variances $Var_{Tc}(t)$, $Var_{Ts}(t)$, $Var_{Tp}(t)$ has the form:

$$\frac{dK_{TT}(t)}{dt}+H^{-1}AK_{TT}(t)+K_{TT}(t)A'H^{-1}=-H^{-1}DK_1(t)-H^{-1}BK_2(t)+H^{-1}I_1K_3(t)$$

$$-K_1'(t)D'H^{-1}-K_2'(t)B'H^{-1}+K_3'(t)I_1'H^{-1},\quad K_{TT}(0)=Var_{Ta}I_4I_4',$$

where Var_{Ta} is the given variance of stochastic environmental temperature; $K_1(t)$, $K_2(t)$, and $K_3(t)$ are vectors being, respectively, the first, second and third rows of the covariance matrix $K_{TG}(t)=E\{\overset{\circ}{G}(\omega)\overset{\circ}{T}'(t,\omega)\}$ between stochastic vectors of the input determining factors $\overset{\circ}{G}(\omega)=(\overset{\circ}{g}(\omega),\overset{\circ}{T}_a(\omega),\overset{\circ}{P}(\omega))'$ of the thermal process in the ES, and centered temperatures $\overset{\circ}{T}(t,\omega)=(\overset{\circ}{T}_c(t,\omega),\overset{\circ}{T}_s(t,\omega),\overset{\circ}{T}_p(t,\omega))'$ of the ES elements.

The covariance matrix $K_{TG}(t) = (K_1(t), K_2(t), K_3(t))'$ is determined from the following matrix equation:

$$\frac{dK_{TG}(t)}{dt} + K_{TG}(t)A'H^{-1}$$

$$= -Var_g I_1 D'H^{-1} - Var_{Ta} I_2 B'H^{-1} + Var_P I_3 I_1' H^{-1}, \quad K_{TG}(0) = Var_{Ta} I_2 I_4',$$

where Var_g, Var_{Ta}, and Var_P, are the known variances of the input interval-stochastic parameters, namely, heat conductivity of the gap, environment temperature, and IC power, respectively; $I_2 = (010)'$, $I_3 = (001)'$.

The obtained matrix equations with the corresponding initial conditions completely define the sought-for statistical measures of stochastic temperatures $T_c(t,\omega)$, $T_s(t,\omega)$, $T_p(t,\omega)$: the mathematical expectations $\overline{T}_c(t)$, $\overline{T}_s(t)$, $\overline{T}_p(t)$, the variances $Var_{Tc}(t)$, $Var_{Ts}(t)$, $Var_{Tp}(t)$, the mean square deviations $\sigma_{Tc}(t)$, $\sigma_{Ts}(t)$, $\sigma_{Tp}(t)$, and also the covariances between different stochastic temperatures. Having found from the solution of these equations the statistical measures of stochastic temperatures, we will find the lower $T_{Bot,i}(t)$ and upper $T_{Up,i}(t)$ boundaries of intervals $[T_{Bot,i}(t), T_{Up,i}(t)]$, which will include the real values of temperatures $T_i(t,\omega)$ of the ES elements ($i = c, s, p$), i.e., $T_{Bot,i}(t) = \overline{T}_i(t) - \chi \cdot \sigma_{Ti}(t)$ and $T_{Up,i}(t) = \overline{T}_i(t) + \chi \cdot \sigma_{Ti}(t)$.

The calculations of the nonstationary statistical measures by the given equations were performed by Runge–Kutta method on PC (Core 2, CPU 3.2 GHz) and the computation time for the ES under consideration was less than several seconds. The initial data for variations of the interval-stochastic parameters were: $\sigma_P/\overline{P} = 4\%$ for the IC consumption power, $\sigma_\delta/\overline{\delta} = 29\%$ for thickness of the air gap between IC package and the PCB, and $\Delta_{Ta}/\overline{T}_a = 2.5\%$ for the environment temperature.

The calculated results that refer to the nonstationary statistical measures of temperature of the IC package, $T_c(t,\omega)$, are shown in Fig. 2 (mathematical expectation), Fig. 3 (variance), and Fig. 4 real temperature variation interval). Dynamics of the thermal process evolution in the ES under consideration (Figs. 2, 3, and 4) is shown for the first 20 s; at that the thermal process proper reaches the steady regime during about 2.5 min. The boundaries of the real temperature variation intervals (Fig. 4) were calculated with $\chi = 3$ for which the probability of detecting values of temperature $T_c(t,\omega)$ beyond the interval $[T_{Bot,c}(t), T_{Up,c}(t)]$, according to Chebyshev inequality, is less than 1/9.

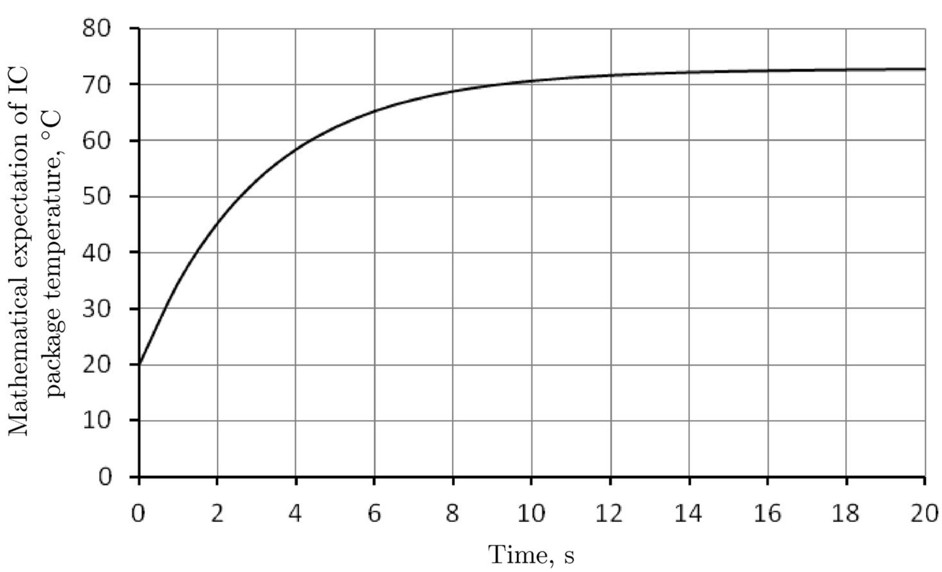

Fig. 2. Nonstationary mathematical expectation (°C) of stochastic temperature of the IC package.

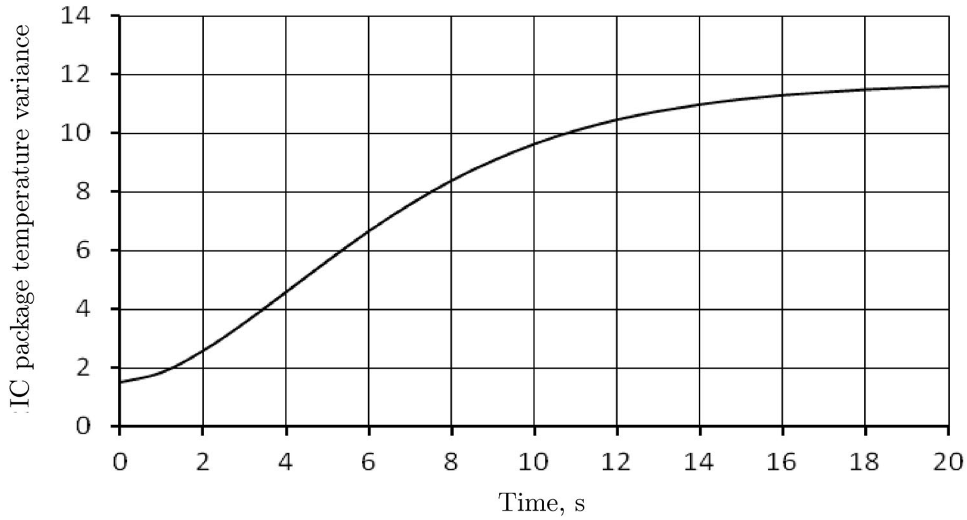

Fig. 3. Nonstationary variance (°C^2) of stochastic IC package temperature.

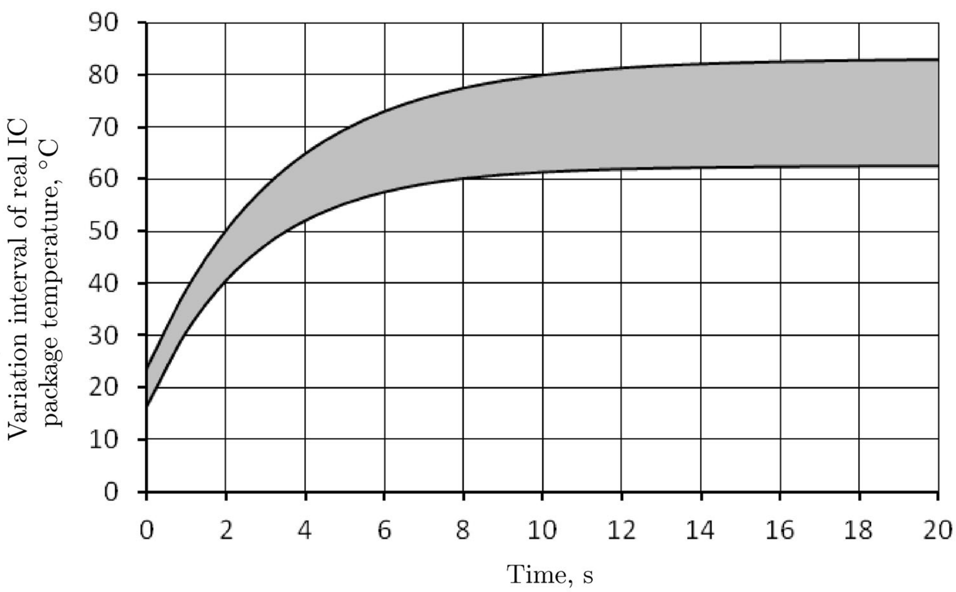

Fig. 4. Nonstationary variation interval of real IC package temperature (°C).

It follows from the obtained dependences that in the steady regime the mathematical expectation if IC package temperature will be 75.5°C, whereas the real steady temperature of the package will be within the interval [101.7°C, 125.7°C] whose width is 24°C (Fig. 4). In other words, steady temperature of the IC package $T_c(\omega)$ for functioning of the real ESs may have any value from the interval $T_c(\omega) \in [62.6°C, 83.1°\text{C}]$; at that, the probability to observe in real ESs temperature values beyond (higher or lower) this interval will be less than 0.1. The steady value of the variance of MC package temperature (Fig. 3) is $Var_{Tc} = 12.1°\text{C}^2$ and sets during about 2.5 min.

The found interval temperature values are significant for thermal ES design because they allow one to predict (at the stage of design) the range of temperature between its highest and least values that will occur in practice in during real ES functioning; moreover, they allow more reliable evaluation of electrical and reliability characteristics of the designed ES.

5. CONCLUSIONS

The method of mathematical modeling the interval-stochastic thermal processes in ES, which has been developed in the present paper and in [2], allows calculations of time variations of intervals of real temperature values of the ES elements. The method has no analogs in the foreign and domestic practice of thermal ES design. It considers the nonstationary, nonlinear and interval-stochastic character of thermal processes in ESs. For modeling the stochastic thermal processes in ESs we have developed a method that enables one to find equations for nonstationary statistical measures of stochastic temperature distributions of ES elements, namely, mathematical expectations, variances, covariances between ES element temperatures. The finite equations for statistical measures of temperatures of the ES elements are nonstationary, nonlinear differential equations in ordinary first-order derivatives, which can easily be solved on current-technology computers. The method developed in this paper is applied in practice for modeling and thermal design of electronic systems. It has proved its adequacy in creation of state-of-the-art competitive electronic systems.

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