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Integral Formulas for the Painlevé-2 Transcendent

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Abstract—In the work we use integral formulas for calculating the monodromy data for the Painlevé-2 equation. The perturbation theory for the auxiliary linear system is constructed and formulas for the variation of the monodromy data are obtained. We also derive a formula for solving the linearized Painlevé-2 equation based on the Fourier-type integral of the squared solutions of the auxiliary linear system of equations.

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1. INTRODUCTION

We consider a scheme of the isomonodromic deformation method for the Painlevé-2 equation in the following form:

$$u'' = 2u^3 + xu. \tag{1.1}$$

The approach is based on the method of isomonodromic deformations developed in [1–3] and [4]. Here we obtain integral formulas which allow us to use the method of the isomonodromic deformations to study variations in the Stokes coefficients and derive a formula for a solution of the linearized Panlevé equation:

$$v'' = 6u^2 v + xv. \tag{1.2}$$

The formulas obtained for solution of (1.2) use integrals of squared solutions to the auxiliary linear problem. Such formulas for solutions of linearized equations are used widely for perturbations of $(1+1)$-dimensional integrable equations by the inverse scattering transform method for the first corrections of perturbation theory [5, 6] and for corrections from the continuous spectrum [7]. For $(2+1)$-dimensional integrable equations the formulas for linearized equations were obtained in [8, 9]. Here we derive analogously formulas for the theory of integration of the linearized Painlevé-2 equation. One of the examples for using the Painlevé-2 equation and its perturbation can be found in [15], and another one, in [16].

The approach developed here allows us to study and to obtain formulas for the variations of the Stokes constants, which are the parameters of the Painlevé transcendent. These formulas and the formula for the solution of (1.2) open a way to study the properties of the linearized equation using the global properties of the Painlevé transcendent.

The general structure of the paper is as follows. In Section 2, we present the Stokes theory for solutions to the auxiliary system of equations and derive integral formulas for Stokes matrices. In Section 3, integral formulas for the Painlevé-2 transcendent are obtained using the integral representation of the solution to the Riemann–Hilbert problem for the auxiliary system of equations. In Section 4, formulas for the variation of the Stokes multipliers are derived. Section 5 provides a formula for solving the linearized Painlevé-2 equation.

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2. INTEGRAL FORMULAS FOR THE STOKES MULTIPLIERS

In this section the integral formulas for solving the auxiliary system of equations for the parameter λ are given according to the theory from [1, 2, 4]. These integral formulas are used to obtain integral formulas for the Stokes multipliers of the auxiliary system of equations.

Consider an auxiliary system of equations that determines the dependence of the function Ψ on the complex variable λ:

$$\frac{d\Psi}{d\lambda} = A\Psi, \quad A = -i(4\lambda^2 + x + 2u^2)\sigma_3 + 4u\lambda\sigma_1 - 2u'\sigma_2. \tag{2.1}$$

Here the notation for Pauli matrices is accepted:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

In addition to the system of equations (2.1), the function Ψ satisfies the system of differential equations for the real variable x:

$$\frac{d\Psi}{dx} = U\Psi, \quad U = -i\lambda\sigma_3 + u\sigma_1. \tag{2.3}$$

The Painlevé-2 equation provides a condition for the existence of a solution of both systems of equations, (2.1) and (2.3) [10].

The solution of the system of equations (2.1) has the singular point at $\lambda = \infty$. The asymptotics of the solution of this system for $\lambda \to \infty$ can be constructed using the WKB [11] method in the form:

$$\Psi^\infty \sim \left(I + \sum_{n=1}^{\infty} \lambda^{-n} M_n(u, u', x)\right) \exp\big(-i\Omega(\lambda)\sigma_3\big). \tag{2.4}$$

The present formula yields a formal series representation for the matrix Ψ^∞. The asymptotic formula (2.4) uniquely determines the analytic solution within each sector for a single column of the matrix Ψ. A key aspect of the Stokes phenomenon is the connection between asymptotic expansions in different sectors. Below, we will discuss the sectors and corresponding asymptotic expansions in detail.

In (2.4) $M_n(u, u', x)$ is a matrix. To obtain the coefficients of M_n, one should substitute the formula (2.4) into the system (2.1) and eliminate coefficients of power λ^{-k} for any k. An explicit form of such an asymptotic expansion was given in [1] up to order of λ^{-1}. However, we need the asymptotic expansion of third order in λ^{-1}:

$$\Psi^\infty \sim \left(I + \frac{1}{2\lambda}\begin{pmatrix} i(u^2x - (u')^2 + u^4) & -iu \\ iu & -i(u^2 - (u')^2 + u^4) \end{pmatrix} + \frac{1}{8\lambda^2}\begin{pmatrix} p_{11} & p_{21} \\ p_{21} & p_{11} \end{pmatrix} + \frac{1}{48\lambda^3}\begin{pmatrix} q_{11} & -q_{21} \\ q_{21} & q_{11} \end{pmatrix} + O(\lambda^{-4})\right) \times \exp\big(-i\Omega(\lambda)\sigma_3\big), \tag{2.5}$$

where $\Omega(\lambda) = \big(4\lambda^3/3 + \lambda x\big)$ and the coefficients p_{11}, p_{21}, q_{11} and q_{21} are derived using the computer algebra system "Maxima":

$$p_{11} = -(u^4x^2 + (2u^6 - 2u^2(u')^2)x + (u')^4 - 2u^4(u')^2 + u^8 - u^2),$$
$$p_{21} = -2(u^3x - u(u')^2 - u' + u^5),$$

$$
\begin{aligned}
q_{11} &= iu^6x^3 + (-3iu^4\,(u')^2 + 3iu^8 + 2iu^2)\,x^2 \\
&+ (3iu^2\,(u')^4 + (-6iu^6 - 2i)\,(u')^2 + 3iu^{10} - iu^4)x \\
&- i(u')^6 + 3iu^4\,(u')^4 + (3iu^2 - 3iu^8)\,(u')^2 \\
&+ 2iu\,u' + iu^{12} - 3iu^6, \\
q_{21} &= -3(iu^5\,x^2 + (-2iu^3\,(u')^2 - 2iu^2\,u' + 2iu^7 + 2iu)x \\
&+ iu(u')^4 + 2(u')^3 - 2iu^5\,(u')^2 - 2u^4\,u' + iu^9 + iu^3).
\end{aligned}
$$

The main term of this asymptotics oscillates on the lines $\Im(4\lambda^3/3 + \lambda x) = 0$. In the neighborhood of an infinity, these lines have asymptotes by the straight lines $\arg(\lambda) = \pi(k-1)/3$, $k = 1, \ldots, 6$. For each of these six lines in the neighborhood of infinity, one can define a function Ψ_k by the given asymptotic direction $\arg(\lambda) = \pi(k-1)/3$:

$$\Psi_k \sim \Psi^\infty, \quad k = 1, 2, 3, 4, 5, 6.$$

Since each of the Ψ_k matrices is a fundamental solution system for (2.1), they can be expressed in terms of each other:

$$\Psi_{k+1} = \Psi_k S_k. \tag{2.6}$$

Here S_k is a matrix consisting of parameters that depend on the solution of the Painlevé-2 equation, but do not depend on the parameter λ. These S_k matrices are called Stokes matrices. The symbols correspond to those used in the book [4].

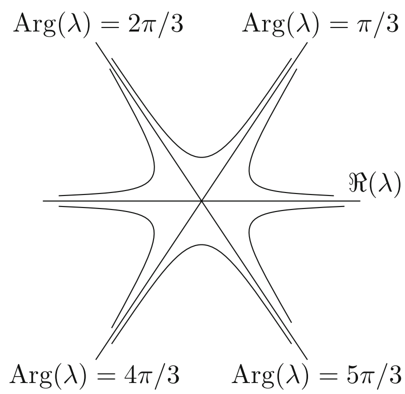

Fig. 1. The Stokes rays at the directions $\pi(k-1)/3$, $k = 1, 2, 3, 4, 5, 6$ and the curves of integration, which approach ∞ in the following directions: ∞_k, ∞_{k+1}.

To derive integral formulas for the Stokes matrix, it is convenient to use the following substitution:

$$\Psi_k = \exp\big(-i\Omega(\lambda)\sigma_3\big)\Phi_k. \tag{2.7}$$

Using the system of equations (2.1), one can derive a similar system of equations for the matrix Φ_k:

$$\frac{d}{d\lambda}\Phi_k = \big(\exp\left(i\Omega\sigma_3\right)A\exp\left(-i\Omega\sigma_3\right) + i\Omega'\sigma_3\big)\,\Phi_k. \tag{2.8}$$

For the matrix Φ_k the following condition is true:

$$\Phi_k \to I, \quad \lambda = Re^{i(k-1)\pi/3}, \quad R \to \infty. \tag{2.9}$$

It can be verified that the solution to the scattering problem defined by (2.8) and (2.9) satisfies a system of integral equations:

$$\Phi_k(\lambda) = I + \int_{\infty_k}^{\lambda} \big(\exp\left(i\Omega\sigma_3\right)A\exp\left(-i\Omega\sigma_3\right) + i\Omega'\sigma_3\big)\,\Phi_k d\mu. \tag{2.10}$$

Here the integral is considered an improper integral, where $\infty_k = R\exp\big(i\pi(k-1)/3\big)$, for $R \to \infty$.

The formulas (2.6), (2.7) and the integral equations (2.10) yield the following limit form of the connection formula (2.6):

$$\Phi_{k+1}|_{\infty_k} = \Phi_k|_{\infty_k} S_k.$$

The next important step is to use these integral equations to obtain formulas for the coefficients of the Stokes matrix (2.6). For this one should consider the integral on the right-hand side as an improper integral for both limits of integration. In this case the initial and final points of the integration are $\lambda \to \infty_k$ and $\lambda \to \infty_{k+1}$.

If we do not touch the integral convergence, we obtain the following formula:

$$I + \int_{\infty_{k+1}}^{\infty_k} \big(\exp(i\Omega\sigma_3)\, A \exp(-i\Omega\sigma_3) + i\Omega'\sigma_3\big)\, \Phi_{k+1} d\mu = S_k.$$

Now S_k can be expressed using Ψ_{k+1}. The integrand in the previous formula can be written as two terms:

$$\big(\exp(i\Omega\sigma_3)\, A \exp(-i\Omega\sigma_3) + i\Omega'\sigma_3\big)\, \Phi_{k+1}$$
$$= \exp(i\Omega\sigma_3)\, A \exp(-i\Omega\sigma_3)\, \Phi_{k+1} + i\Omega'\sigma_3\Phi_{k+1}.$$

Now we note that

$$\exp(-i\Omega\sigma_3)\Phi_{k+1} = \Psi_{k+1}.$$

Then

$$\exp(i\Omega\sigma_3)\, A \exp(-i\Omega\sigma_3)\, \Phi_{k+1} = \exp(i\Omega\sigma_3)\, A\Psi_{k+1},$$

and

$$i\Omega'\sigma_3\Phi_{k+1} = \exp(i\Omega\sigma_3) i\Omega'\sigma_3 \exp(-i\Omega\sigma_3)\Phi_{k+1} = \exp(i\Omega\sigma_3) i\Omega'\sigma_3\Psi_{k+1}.$$

As a result, we obtain

$$S_k = I + \int_{\infty_{k+1}}^{\infty_k} \exp(i\Omega\sigma_3)\big(A + i\Omega'\sigma_3\big)\, \Psi_{k+1} d\mu. \tag{2.11}$$

Due to the Stokes phenomenon, the integrands have different asymptotic behaviors as $\mu \to \infty_{k+1}$ and as $\mu \to \infty_k$. Therefore, we consider below all components in the integral formula for the matrix S_k separately and show the convergence of the improper integrals in (2.11). The following calculations take into account the asymptotic properties of the matrix Ψ_{k+1} from the formula (2.9).

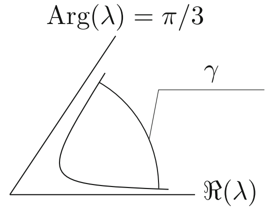

Fig. 2. The integrals over the closed curve γ in the formula (2.11) are equal to zero due to the Cauchy theorem for analytic functions.

The integrands in the diagonal elements are analytic functions with respect to λ and the integrands decrease as λ^{-2} for $\lambda \to \infty$. Therefore, we use the Cauchy theorem for such functions and consider the integrals over an arc of a large circle with radius R as $R \to \infty$ (see, Fig. 2).

Let us consider carefully calculations for the matrix S_1 as an example. The function Ψ_2 oscillates near the ray $\mathrm{Arg}(\lambda) = \pi/3$. The asymptotic behaviors of Ψ_2 coincide with the asymptotic behaviors of Ψ_∞ in the sector $0 < \mathrm{Arg}(\lambda) < 2\pi/3$. Therefore, instead of integrals over the path near the Stokes rays one can use the integrals over an arc of a large circle.

For other matrices S_k one can use the same calculations in the corresponding sectors of the complex plane of λ.

Let us consider the integrand on an arc of the large circle. Straightforward calculations yield the following formulas. The integrands in $(S_1)_{11}$ and $(S_1)_{22}$ have the order $O(\mu^{-2})$. As a result, we get

$$(S_k)_{11} = 1 + \lim_{R\to\infty} \int_{R\exp(i\pi(k+1)/3)}^{R\exp(i\pi k/3)} \mathcal{O}(R^{-2}) d\mu = 1.$$

$$(S_k)_{22} = 1 + \lim_{R\to\infty} \int_{R\exp(i\pi(k+1)/3)}^{R\exp(i\pi k/3)} \mathcal{O}(R^{-2}) d\mu = 1.$$

In the integrands for $(S_1)_{12}$ and $(S_1)_{21}$ the forms of the exponent are important. We do not show here such a simple calculation.

The elements of the matrix S_k lying on the off-diagonal have exponents in the integrand for large values of λ:

$$(S_k)_{12} = \lim_{R\to\infty} \int_{R\exp(i\pi(k+1)/3)}^{R\exp(i\pi k/3)} \big(4iu\mu + \big(2u^3x - 2uw^2 - 2w + 2u^5\big) \\ +\mathcal{O}(1/R)\big) \exp\big(2i(4\mu^3/3 + x\mu)\big) d\mu \tag{2.12}$$

$$(S_k)_{21} = \lim_{R\to\infty} \int_{R\exp(i\pi(k+1)/3)}^{R\exp(i\pi k/3)} \big(-4iu\mu + \big(2u^3x - 2u\,w^2 - 2w + 2u^5\big) \\ +\mathcal{O}(1/R)\big) \exp\big(-2i(4\mu^3/3 + x\mu)\big) d\mu. \tag{2.13}$$

The values of the integrals in the formulas (2.12) and (2.13) depend on the sign $\Re(i\mu^3)$ on the integration path. Therefore, it is convenient to evaluate the integrals for different values of k.

If $k = 1, 3, 5$, then on the arc $\pi(k-1)/3 < \arg(\mu) < \pi k/3$ we get $\Re(i\mu^3) = -R\sin\big(3\arg(\mu)\big) < 0$. In this case, it can be shown that

$$(S_k)_{12} = 0.$$

Similarly, for $k = 2, 4, 6$ on the arc $\pi(k-1)/3 < \arg(\mu) < \pi k/3$ we get $\Re(-i\mu^3) = R\sin\big(3\arg(\mu)\big) < 0$, that is,

$$(S_k)_{21} = 0.$$

As a result, we get

$$S_k = \begin{pmatrix} 1 & 0 \\ s_k & 1 \end{pmatrix}, \quad k = 1, 3, 5;$$

$$S_k = \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}, \quad k = 2, 4, 6.$$

In the terms of [1] we rewrite: $s_1 = a,\ s_2 = b,\ s_3 = c,\ s_4 = d,\ s_5 = e,\ s_6 = f$ and $s_1 = s_4$, $s_2 = s_5$, $s_3 = s_6$.

Formulas for s_k can be obtained by multiplying the matrix in the integrands in (2.11) and the formula (2.1):

$$\begin{pmatrix} e^{i\Omega} & 0 \\ 0 & e^{-i\Omega} \end{pmatrix} \begin{pmatrix} -2iu^2 & 4u\mu + 2iu' \\ 4u\mu - 2iu' & 2iu^2) \end{pmatrix} \begin{pmatrix} (\Psi_k)_{11} & (\Psi_k)_{12} \\ (\Psi_k)_{21} & (\Psi_k)_{22} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\Omega(\mu)} & 0 \\ 0 & e^{-i\Omega(\mu)} \end{pmatrix} \begin{pmatrix} -2iu^2(\Psi_k)_{11} + (4u\mu + 2iu')(\Psi_k)_{21} & -2iu^2(\Psi_k)_{12} + (4u\mu + 2iu')(\Psi_k)_{22} \\ (4u\mu - 2iu)(\Psi_k)_{11} + 2iu^2(\Psi_k)_{21} & (4u\mu - 2iu)(\Psi_k)_{12} + 2iu^2(\Psi_k)_{22} \end{pmatrix}.$$

Now one should use $\Omega = (4/3)\mu^3 + \mu x$.

The main result of Section 2 is explicit formulas for monodromy data.

Theorem 1. *The monodromy data for solutions of* (2.1) *can be obtained by the following integral formulas:*

$$s_k = 2\int_{\infty_{k+1}}^{\infty_k} \big((2u\mu - iu')(\Psi_k)_{11} + iu^2(\Psi_k)_{21}\big)\, e^{-i(4\mu^3/3+x\mu)}d\mu, \quad k = 1, 3, 5;$$

$$s_k = 2\int_{\infty_{k+1}}^{\infty_k} \big((2u\mu + iu')(\Psi_k)_{22} - iu^2(\Psi_k)_{12}\big)\, e^{i(4\mu^3/3+x\mu)}d\mu, \quad k = 2, 4, 6.$$

The formulas for the integrals over λ can be used at any regular point of the solution of the Painlevé-2 equation.

Solutions of Painlevé equations are meromorphic functions of a complex variable x. A connection of the Stokes data s_k and singularities were considered, for example, in the book [2]. In the formulas we integrate with respect to λ for defining the Stokes data s_k, the values of $u(x)$ and $u'(x)$ can be used at any regular point.

3. INTEGRAL FORMULA FOR THE PAINLEVÉ TRANSCENDENT

The analytical properties of the functions Ψ_k allow us to formulate the problem of conjugation of functions for the analytical continuation of the function Ψ_k into neighboring sectors of the complex plane of the parameter λ. To obtain the integral equations, we can conveniently use Sokhotsky's formulas [12]. Similar constructions were done in the work [1]. As a result, we obtained a system of equations for the first and second columns of analytical equations in the complex plane λ:

$$\Psi^{(1)}e^{i\Omega} = \begin{pmatrix}1\\0\end{pmatrix} - \mathrm{Res}_{\mu=0}\frac{\Psi^{(1)}e^{i\Omega}}{\mu-\lambda} + \frac{s_1}{2\pi i}\int_{C_{42}}\frac{\Psi^{(2)}e^{i\Omega}}{\mu-\lambda}d\mu + \frac{s_2}{2\pi i}\int_{C_{46}}\frac{\Psi^{(2)}e^{i\Omega}}{\mu-\lambda}d\mu + \frac{s_2 s_3}{2\pi i}\int_{C_{64}}\frac{\Psi^{(1)}e^{i\Omega}}{\mu-\lambda}d\mu,$$

$$\Psi^{(2)}e^{-i\Omega} = \begin{pmatrix}0\\1\end{pmatrix} - \mathrm{Res}_{\mu=0}\frac{\Psi^{(2)}e^{-i\Omega}}{\mu-\lambda} + \frac{s_2}{2\pi i}\int_{C_{53}}\frac{\Psi^{(1)}e^{-i\Omega}}{\mu-\lambda}d\mu + \frac{s_3}{2\pi i}\int_{C_{51}}\frac{\Psi^{(1)}e^{-i\Omega}}{\mu-\lambda}d\mu + \frac{s_1 s_2}{2\pi i}\int_{C_{53}}\frac{\Psi^{(2)}e^{-i\Omega}}{\mu-\lambda}d\mu.$$

The solution of the Painlevé-2 equation is usually represented using the asymptotics as $\lambda \to \infty$ for the off-diagonal components of the matrix Ψ [1]:

$$u(x) = \lim_{\lambda\to\infty} \lambda i \Psi_{12} e^{-i\Omega}$$

or

$$u(x) = -\lim_{\lambda\to\infty} \lambda i \Psi_{21} e^{i\Omega}.$$

If we use the integral equations for the matrix Ψ, then we can obtain an alternative expression for the second Painlevé transcendent in terms of the components of the functions Ψ_k.

For this we use the following formula:

$$\frac{1}{1-q} = 1 + \frac{q}{1-q}.$$

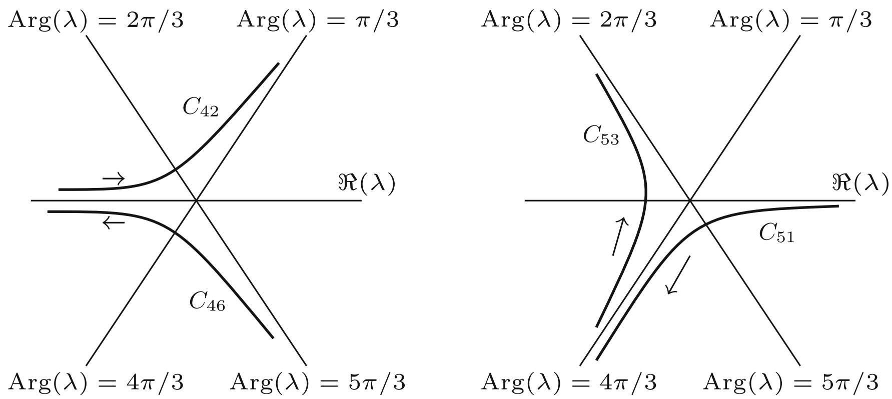

Fig. 3. Integration paths in the Riemann problem for the matrix function Ψ and for calculating the Painlevé transcendent by integral formulas.

If we denote $q = \mu/\lambda$, then

$$\lim_{\lambda\to\infty} \lambda \int_a^b \frac{f(\mu)}{\mu-\lambda} d\mu = \lim_{\lambda\to\infty} \frac{\lambda}{\lambda} \int_a^b \frac{f(\mu)}{\frac{\mu}{\lambda}-1} d\mu$$
$$= -\int_a^b f(\mu) d\mu - \lim_{\lambda\to\infty} \int_a^b f(\mu) \frac{\frac{\mu}{\lambda}}{\frac{\mu}{\lambda}-1} d\mu$$
$$= -\int_a^b f(\mu) d\mu - \lim_{\lambda\to\infty} \frac{1}{\lambda} \int_a^b \frac{f(\mu)\mu}{\frac{\mu}{\lambda}-1} d\mu.$$

If

$$\left| \int_a^b f(\mu)\mu d\mu \right| < \infty,$$

then

$$\lim_{\lambda\to\infty} \lambda \int_a^b \frac{f(\mu)}{\mu-\lambda} d\mu = -\int_a^b f(\mu) d\mu.$$

In our case the interval of the integration is unbounded. The integrals should be considered as improper. On the Stokes curves, the exponents in the integrands are oscillatory. Therefore, the leading terms of the asymptotics as $\mu \to \infty$ are bounded. Let us consider:

$$\lim_{\mu\to\infty_k} (\Psi_k)_{22} \mu \exp\big(i\Omega(\mu)\big) \sim \mu \exp\Big(2i\big((4/3)\mu^3 + \mu x\big)\Big).$$

The improper integral with such integrands exists. It yields the same formulas as on the bounded interval.

For one more integral we get

$$\lambda \int_0^{\infty_4} (\Psi_4)_{21} \exp(i\Omega) d\mu.$$

The asymptotic behavior of the integrand is

$$\lambda \frac{(\Psi_4)_{21} \exp(i\Omega)}{\mu-\lambda} \sim \frac{\lambda}{\mu-\lambda} \frac{iu}{2\mu}.$$

Then the integral can be represented in the following form:

$$\lambda \int_0^{\infty_4} \frac{(\Psi_4)_{21} \exp(i\Omega)}{\mu-\lambda} d\mu + \lambda \int_{\infty_6}^0 \frac{(\Psi_4)_{21} \exp(i\Omega)}{\mu-\lambda} d\mu$$

$$
\begin{aligned}
&= \lambda \int_{-a}^{\infty_4} \left(\frac{(\Psi_4)_{21} \exp(i\Omega)}{\mu - \lambda} - \frac{1}{\mu - \lambda} \frac{iu}{2\mu} \right) d\mu \\
&+ \lambda \int_{\infty_6}^{a \exp(5i\pi/3)} \left(\frac{(\Psi_4)_{21} \exp(i\Omega)}{\mu - \lambda} - \frac{1}{\mu - \lambda} \frac{iu}{2\mu} \right) d\mu \\
&+ \lambda \int_{-a}^{\infty_4} \left(\frac{1}{\mu - \lambda} \frac{iu}{2\mu} \right) d\mu + \lambda \int_{\infty_6}^{a \exp(5i\pi/3)} \left(\frac{1}{\mu - \lambda} \frac{iu}{2\mu} \right) d\mu
\end{aligned}
$$

As a result, we get

$$
\begin{aligned}
u(x) = &-\frac{s_1}{2\pi} \int_0^{\infty_4} (\Psi_4)_{21} e^{i\Omega} d\mu - \frac{s_1}{2\pi} \int_{\infty_2}^0 (\Psi_2)_{21} e^{i\Omega} d\mu \\
&- \frac{s_2}{2\pi} \int_0^{\infty_4} (\Psi_4)_{21} e^{i\Omega} d\mu - \frac{s_2}{2\pi} \int_{\infty_6}^0 (\Psi_6)_{21} e^{i\Omega} d\mu \\
&- \frac{s_2 s_3}{2\pi} \int_0^{\infty_4} (\Psi_4)_{21} e^{i\Omega} d\mu - \frac{s_2 s_3}{2\pi} \int_{\infty_6}^0 (\Psi_6)_{21} e^{i\Omega} d\mu. \qquad (3.1)
\end{aligned}
$$

Another expression can be obtained by using the Ψ_{12} component:

$$
\begin{aligned}
u(x) = &\frac{s_2}{2\pi} \int_0^{\infty_5} (\Psi_5)_{12} e^{-i\Omega} d\mu + \frac{s_2}{2\pi} \int_{\infty_3}^0 (\Psi_3)_{12} e^{-i\Omega} d\mu \\
&+ \frac{s_3}{2\pi} \int_0^{\infty_5} (\Psi_5)_{12} e^{-i\Omega} d\mu + \frac{s_3}{2\pi} \int_{\infty_1}^0 (\Psi_1)_{12} e^{-i\Omega} d\mu + \\
&+ \frac{s_1 s_2}{2\pi} \int_0^{\infty_5} (\Psi_5)_{12} e^{-i\Omega} d\mu + \frac{s_1 s_2}{2\pi} \int_{\infty_3}^0 (\Psi_3)_{12} e^{-i\Omega} d\mu. \qquad (3.2)
\end{aligned}
$$

The formulas in (3.1) and (3.2) are the main results of Section 3.

Note that a limit of small value $u(x)$ and related auxiliary systems were considered, for example, in the book [4].

4. VARIATION OF THE STOKES MULTIPLIERS

Consider the effect of perturbations on the Stokes multipliers associated with the scattering problem (2.1). For an infinitesimal perturbation of the coefficients of the system (2.1) $u = u + \delta u$, we can obtain a system of equations for the variation of $\delta\Psi$:

$$
\frac{d}{d\lambda} \delta\Psi = A \delta\Psi + \delta A \Psi, \quad \delta A = -i4u\delta u \sigma_3 + 4\delta u \lambda \sigma_1 - 2\delta u' \sigma_2. \qquad (4.1)
$$

Below we will use the notation

$$
\frac{d}{dx}(\delta u) \equiv \delta u'.
$$

The general solution to the system of equations (4.1) can be represented as

$$
\delta\Psi = \Psi C + \Psi \int \Psi^{-1} \delta A \Psi d\mu, \qquad (4.2)
$$

where C is a matrix composed of arbitrary constants, which are parameters of the solution of the system (4.1). This matrix will be used for constructing a special solution of (4.1).

Now consider the matrix in the integrand of (4.2) using the formula $\det \Psi \equiv 1$:

$$
\begin{aligned}
(\Psi^{-1} \delta A \Psi)_{11} = &(\Psi_{2,1} \Psi_{2,2} - \Psi_{1,1} \Psi_{1,2}) \, 4\lambda \delta u \\
&+ (\Psi_{2,1} \Psi_{2,2} + \Psi_{1,1} \Psi_{1,2}) \; 2i\delta u' \\
&- (\Psi_{1,1} \Psi_{2,2} + \Psi_{1,2} \Psi_{2,1}) \, 4iu\delta u,
\end{aligned}
$$

$$\begin{aligned}
(\Psi^{-1}\delta A\Psi)_{21} &= \left(\Psi_{1,1}^2 - \Psi_{2,1}^2\right)4\lambda\delta u - \left(\Psi_{2,1}^2 + \Psi_{1,1}^2\right)2i\delta u' \\
&\quad + 8i\Psi_{1,1}\,\Psi_{2,1}uv, \\
(\Psi^{-1}\delta A\Psi)_{12} &= \left(\Psi_{2,2}^2 - \Psi_{1,2}^2\right)4\lambda\delta u + \left(\Psi_{2,2}^2 + \Psi_{1,2}^2\right)2i\delta u' \\
&\quad - 8i\Psi_{1,2}\,\Psi_{2,2}u\delta u, \\
(\Psi^{-1}\delta A\Psi)_{22} &= \left(\Psi_{1,1}\,\Psi_{1,2} - \Psi_{2,1}\,\Psi_{2,2}\right)4\lambda\delta u \\
&\quad - \left(\Psi_{2,1}\,\Psi_{2,2} + \Psi_{1,1}\,\Psi_{1,2}\right)2i\delta u' \\
&\quad + \left(\Psi_{1,1}\,\Psi_{2,2} + \Psi_{1,2}\,\Psi_{2,1}\right)4iu\delta u.
\end{aligned}$$

The infinitesimal variation $\delta\Psi$ can be used to calculate variations of the Stokes multipliers. Specifically, we obtain

$$\delta\Psi_{k+1} = \delta\Psi_k S_k + \Psi_k \delta S_k,$$

as $\lambda \to \infty_k$ we obtain the condition

$$\delta\Psi_k = 0, \quad \lambda \to \infty_k.$$

Then

$$\delta\Psi_{k+1} \sim \exp(-i\Omega\sigma_3)\delta S_k, \quad \lambda \to \infty_k. \tag{4.3}$$

On the other side one obtains

$$\delta\Psi_{k+1} \sim \Psi_{k+1}\int_{\infty_{k+1}}^{\infty_k} \Psi_{k+1}^{-1}\delta A\Psi_{k+1}d\mu, \quad \lambda \to \infty_k.$$

Let us change

$$\Psi_{k+1} = \Psi_k S_k \sim \exp(-i\Omega\sigma_3)S_k, \quad \lambda \to \infty_k,$$

then

$$\delta\Psi_{k+1} \sim \exp(-i\Omega\sigma_3)S_k\int_{\infty_{k+1}}^{\infty_k} \Psi_{k+1}^{-1}\delta A\Psi_{k+1}d\mu, \quad \lambda \to \infty_k. \tag{4.4}$$

Equate (4.3) and (4.4), multiply the left side by $\exp(i\Omega\sigma_3)$. This yields

$$\delta S_k = S_k\int_{\infty_{k+1}}^{\infty_k} \Psi_{k+1}^{-1}\delta A\Psi_{k+1}d\mu. \tag{4.5}$$

Only one element of the matrix S_k, denoted by s_k, depends on u and u'. Therefore,

$$\delta S_k = \begin{pmatrix} 0 & 0 \\ \delta s_k & 0 \end{pmatrix}, \quad k = 1, 3, 5.$$

Denote

$$\int_{\infty_{k+1}}^{\infty_k} \Psi_{k+1}^{-1}\delta A\Psi_{k+1}d\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the formula (4.5) has the following form:

$$\begin{pmatrix} 0 & 0 \\ \delta s_k & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_k & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & 0 \\ \delta s_k & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ as_k + c & bs_k + d \end{pmatrix}.$$

It yields: $a = 0$, $b = 0$, $d = 0$ and $\delta s_k = c$, where

$$c = \int_{\infty_{k+1}}^{\infty_k} \left(\left(\Psi_{1,1}^2 - \Psi_{2,1}^2\right) 4\lambda\delta u - \left(\Psi_{2,1}^2 + \Psi_{1,1}^2\right) 2i\delta u' + 8i\Psi_{1,1}\,\Psi_{2,1} u\delta u\right) d\mu.$$

Then the integrals of the diagonal elements along the integration paths marked in Fig. 1 give zeros. It is convenient to define

$$\psi_1^+ = \Psi_{11}^2 + \Psi_{21}^2, \quad \psi_1^- = \Psi_{11}^2 - \Psi_{21}^2, \quad \psi_1 = \Psi_{11}\Psi_{21};$$
$$\psi_2^+ = \Psi_{12}^2 + \Psi_{22}^2, \quad \psi_1^- = \Psi_{22}^2 - \Psi_{12}^2, \quad \psi_2 = \Psi_{12}\Psi_{22}.$$

For reasons that mirror the calculations of the Stokes multipliers in Section 2, we obtain the following statement.

Theorem 2. *Variations of the Stokes multipliers are defined by the following integrals:*

$$\delta s_k = \int_{\infty_{k+1}}^{\infty_k} 4\mu\delta u\,(\psi_1^-)_{k+1} - 2i\delta u'\,(\psi_1^+)_{k+1} + 8iu\delta u\,(\psi_1)_{k+1} d\mu,\ k = 1, 3, 5; \tag{4.6}$$

$$\delta s_k = \int_{\infty_{k+1}}^{\infty_k} 4\mu\delta u\,(\psi_2^-)_{k+1} + 2i\delta u'\,(\psi_2^+)_{k+1} - 8iu\delta u\,(\psi_2)_{k+1} d\mu,\ k = 2, 4, 6. \tag{4.7}$$

The formulas for the integrals over λ can be used at any regular point of the solution of the Painlevé-2 equation.

5. FORMULA FOR SOLUTION OF THE LINEARIZED PAINLEVÉ-2 EQUATION

Consider the equations for quadratic expressions ψ_1^+, ψ_1^-, ψ_1:

$$\frac{d\psi_1^+}{d\lambda} = -2i(4\lambda^2 + x + 2u^2)\psi_1^- + 16\lambda u\psi_1. \tag{5.1}$$

$$\frac{d\psi_1^-}{d\lambda} = -2i(4\lambda^2 + x + 2u^2)\psi_1^+ + 8iu'\psi_1. \tag{5.2}$$

$$\frac{d\psi_1}{d\lambda} = 4\lambda u\,\psi_1^+ - 2iu'\,\psi_1^-. \tag{5.3}$$

Similar expressions are obtained for the derivatives of the same quadratic expressions with respect to x:

$$\frac{d\psi_1^+}{dx} = -2i\lambda\psi_1^- + 4u\psi_1, \tag{5.4}$$

$$\frac{d\psi_1^-}{dx} = -2i\lambda\psi_1^+, \tag{5.5}$$

$$\frac{d\psi_1}{dx} = u\psi_1^+. \tag{5.6}$$

We will assume that the variation δu represents a solution to the linearized Painlevé-2 equation:

$$\delta u'' = (6u^2 + x)\delta u. \tag{5.7}$$

Differentiating Eq. (4.6) with respect to x, using Eqs. (5.4), (5.5), and the linearized equation (5.7) gives (with s_1 used for concreteness)

$$\frac{d}{dx}\delta s_1 = \int_{\infty_6}^{\infty_1} \left((-2i(4\mu^2 + x + 2u^2)\psi_1^+ + 8iu'\psi_1)\delta u - 2if\psi_1^+\right) d\mu$$
$$= \delta u \int_{\infty_6}^{\infty_1} \frac{d\psi_1^-}{d\mu} d\mu.$$

To calculate the integral involving the derivative of λ, we will consider the representation of ψ_1^- in terms of the squares of the first column of the Ψ-function. For $\lambda \to \infty_6$ and $r = |\lambda|$, $\alpha = \mathrm{Arg}(\lambda)$ we obtain

$$\Psi_{11}^2 \sim \exp\left(-2i\left(\frac{4}{3}r^3 e^{3i\alpha} + xre^{i\alpha}\right)\right),$$
$$\Psi_{21}^2 \sim \frac{-u^2 e^{-2i\alpha}}{4r^2}\exp\left(-2i\left(\frac{4}{3}r^3 e^{3i\alpha} + xre^{i\alpha}\right)\right), \quad \lambda \to \infty_6,$$

As $\lambda \to \infty_1$ and $r = |\lambda|$, $\beta = \mathrm{Arg}(\lambda) - \pi/3$ we get:

$$\Psi_{11}^2 \sim \exp\left(2i\left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right)(1 + O(r^{-1})) + O(1/r)$$
$$+ s_1^2 \exp\left(-2i\left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right)\left(\frac{u^2 e^{-2i\pi/3-2i\beta}}{4r^2} + O(r^{-3})\right),$$
$$\Psi_{21}^2 \sim s_1^2 \exp\left(-2i\left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right)(1 + O(r^{-1})) + O(1/r)$$
$$+ \exp\left(2i\left(\frac{4}{3}r^3 e^{3i\beta} - xre^{i\pi/3}e^{i\beta}\right)\right)\left(\frac{e^{2i\pi/3+2i\beta}}{r^2} + O(r^{-3})\right).$$

The integral involving the derivative of ψ_1^- with respect to λ can be written as the sum of integrals:

$$\int_{\infty_6}^{\infty_1} \frac{d\psi_1^-}{d\mu} d\mu = \int_{\mathcal{L}_{11}} \frac{d\Psi_{11}^2}{d\mu} d\mu + \int_{\mathcal{L}_{21}} \frac{d\Psi_{21}^2}{d\mu} d\mu.$$

For each of the integrals, we deform the contour so that at its ends the functions Ψ_{11}^2 and Ψ_{21}^2, respectively, vanish. The choice of an integration path for representing solutions to the auxiliary linear equations connecting the Painlevé-2 equation was considered in [13] and [14].

Let us consider the asymptotics of Ψ_{11}^2 near the ray to ∞_1, as $\mathrm{Arg}(\lambda) = \pi/3 + \beta$, where $\beta \ll 1$ and $r \to \infty$. The curve, where the real part of the exponent is equal to zero:

$$\Re\left(2i\left(\frac{4}{3}\lambda^3 + x\lambda\right)\right) = 0,$$

as $\lambda = re^{i\pi/3}e^{i\beta}$, has the form

$$\frac{8}{3}r^2 \sin(3\beta) - x\sin(\beta) - \sqrt{3}x\cos(\beta) = 0.$$

As $r \to \infty$ this curve has the asymptotics

$$\beta \sim \frac{x\sqrt{3}}{8r^2} + O(r^{-4})$$

The asymptotics of Ψ_{11} is

$$\Psi_{11}^2 \sim \exp\left(-\frac{8}{3}r^3\beta + rx\beta + \sqrt{3}xr\right)O(1) + \frac{\exp\left(\frac{8}{3}r^3\beta - xr\beta - \sqrt{3}xr\right)}{r^2}O(1).$$

The first term of this formula decreases when $\beta > x\sqrt{3}/(8r^2)$, and the second term decreases when

$$\left(\frac{8}{3}r^3\beta - xr\beta - \sqrt{3}xr\right) - 2\ln(r) < 0,$$

or as $r \to \infty$:

$$\beta < \frac{\sqrt{3}}{8r^2}x + \frac{1}{4r^3}\ln(r).$$

Then $\Psi_{11}^2 \to 0$ as $r \to \infty$ and:

$$\frac{\pi}{3} + \frac{\sqrt{3}}{8r^2}x < \mathrm{Arg}(\lambda) < \frac{\pi}{3} + \frac{\sqrt{3}}{8r^2}x + \frac{1}{4r^3}\ln(r).$$

In the same way one can get the asymptotics of Ψ_{21}^2 near the ray $(0, \infty_1)$ for $\mathrm{Arg}(\lambda) = \pi/3 + \beta$, where $\beta \ll 1$ and $r \to \infty$

$$\Psi_{21}^2 \sim \frac{\exp\left(-8r^3\beta + xr\beta + \sqrt{3}xr\right)}{r^2}O(1) + \exp\left(8r^3\beta - xr\beta - \sqrt{3}xr\right)O(1).$$

For this function, the exponents have the opposite sign compared to the other asymptotics of Ψ_{11}^2. Therefore, similar considerations give the condition: $\Psi_{21}^2 \to 0$ as $r \to \infty$, leading to

$$\pi/3 - \frac{x\sqrt{3}}{8r^2} > \mathrm{Arg}(\lambda) > \pi/3 - \frac{x\sqrt{3}}{8r^2} - \frac{1}{4r^3}\ln(r).$$

The integral in the formula for the derivative of Ψ_{11}^2 has the integration path $\mathcal{L}_{11}$, as shown in Fig. 4. The path begins at the point $\mathcal{L}_{11}^-$ in the sector $-\Delta < \mathrm{Arg}(\lambda) < 0$, where $\Delta > 0$, and finishes at the point $\mathcal{L}_{11}^+$, where $\pi/3 + \frac{x\sqrt{3}}{8r^2} + \frac{1}{4r^3}\ln(r) > \mathrm{Arg}(\mathcal{L}_{11}^+) > \pi/3 + \frac{x\sqrt{3}}{8r^2}$ and $r = |\lambda|$.

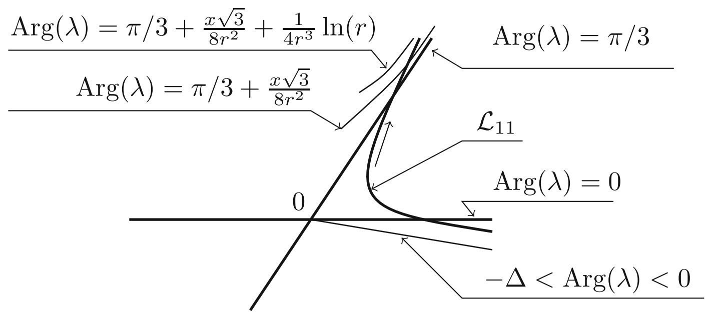

Fig. 4. The ends of the path of integration $\mathcal{L}_{11}$ lie in the sectors $\pi/3 + \frac{x\sqrt{3}}{8r^2} < \mathrm{Arg}(\lambda) < \pi/3 + \frac{x\sqrt{3}}{8r^2} + \frac{1}{4r^3}\ln(r)$ and $-\Delta < \mathrm{Arg}(\lambda) < 0$, for $\forall\Delta > 0$.

The path $\mathcal{L}_{21}$ is obtained using the same considerations. This path is shown in Fig. 5.

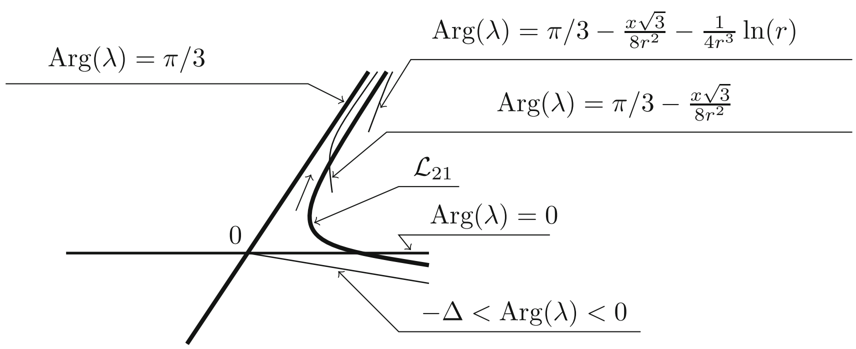

Fig. 5. The ends of the path of integration $\mathcal{L}_{21}$ lie in the sectors $\pi/3 - \frac{x\sqrt{3}}{8r^2} - \frac{1}{4r^3}\ln(r) < \mathrm{Arg}(\lambda) < \pi/3 - \frac{x\sqrt{3}}{8r^2}$ and $-\Delta < \mathrm{Arg}(\lambda) < 0$ for $\forall\Delta > 0$.

Then the following formula is valid:

$$\frac{d}{dx}\delta s_1 = 0. \tag{5.8}$$

The formulas for the squares of Ψ allow us to represent the solution of the linearized Painlevé-2 equation in terms of quadratic expressions from Ψ. Indeed, we differentiate with respect to x Eq. (5.4) by virtue of Eqs. (5.5) and (5.6):

$$\frac{d^2\psi_1^+}{dx^2} = 4(-\lambda^2 + u^2)\psi_1^+ + 4u'\psi_1.$$

In this equation, the last term on the right-hand side can be replaced using Eq. (5.2). This gives

$$\frac{d^2\psi_1^+}{dx^2} = (x + 6u^2)\psi_1^+ - i\frac{1}{2}\frac{d\psi_1^-}{d\lambda}.$$

The same calculations for ψ_1^- give

$$\frac{d^2\psi_1^-}{dx^2} = 4(-\lambda^2)\psi_1^- - 8i\lambda u\psi_1.$$

and, using replacement, this yields

$$\frac{d^2\psi_1^-}{dx^2} = (x + 2u^2)\psi_1^- - i\frac{1}{2}\frac{d\psi_1^+}{d\lambda}.$$

The formulas obtained above are useful for deriving an integral representation of the solution to the linearized Painlevé-2 equation, which is one of the main goals of this work.

Consider the integral

$$v(x) = \int_{\infty_1}^{\infty_6} \psi_1^+(\lambda, x)d\lambda.$$

The second derivative of this integral is

$$\frac{d^2}{dx^2}\int_{\infty_6}^{\infty_1} \psi_1^+(\lambda, x)d\lambda = (6u^2 + x)\int_{\infty_6}^{\infty_1} \psi_1^+(\lambda, x)d\lambda$$
$$- i\frac{1}{2}\int_{\mathcal{L}_{11}} \frac{d}{d\lambda}\psi_{11}^2 d\lambda + i\frac{1}{2}\int_{\mathcal{L}_{21}} \frac{d}{d\lambda}\Psi_{21}^2 d\lambda.$$

Then, by following similar reasoning used in the derivation of formula (5.8), we find that the solution to the linearized Painlevé-2 equation

$$v'' = (6u^2 + x)v$$

can be represented as

$$v(x) = \int_{\infty_6}^{\infty_1} \psi_1^+(\lambda, x)d\lambda. \tag{5.9}$$

In the same way we obtain the formula

$$y = \int_{\infty_6}^{\infty_1} \psi_1^-(\lambda, x)d\lambda,$$

which is a solution for

$$y'' = (x + 2u^2)y.$$

Theorem 3. *A solution of the linearized Painlevé-2 equation has the following integral representation:*

$$v(x) = \int_{\infty_1}^{\infty_6} \psi_1^+(\lambda, x) d\lambda.$$

The formula for the integral over λ can be used at any regular point of the solution of the Painlevé-2 equation.

Let $v(x)$ and $v_1(x)$ be linearly independent solutions of the linearized Painlevé-2 equation. Since their Wronskian is a constant, say 1, one has

$$vv_1' - v'v_1 = 1.$$

Then $v_1(x)$ can be found due to the Wronskian formula and one gets

$$v_1(x) = v(x) \int \frac{dx}{v^2(x)}.$$

If $v(\xi) = 0$, then $v'(\xi) \neq 0$, and the integral should be considered in the regularized sense:

$$v_1(\xi) = \frac{1}{v'(\xi)}.$$

Corollary 1. *A general solution for the linearized Painlevé-2 equation can be represented in the form*

$$y(x) = Cv(x) + C_1 v_1(x), \quad C, C_2 \in \mathbb{R}.$$

6. CONCLUSION

In this work we obtain integral formulas for the monodromy data for connection of auxiliary linear equations with the Painlevé-2 equation. The formulas allow us to derive the perturbation theory for the auxiliary linear system (2.1) and to obtain formulas for infinitesimal variations of the monodromy data. We also derive an integral formula for the solution to the linearized Painlevé-2 equation. This formula utilizes the squares of the solutions to the auxiliary system of equations in (2.1).

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. Flaschka, H. and Newell, A. C., Monodromy- and Spectrum-Preserving Deformations: 1, *Comm. Math. Phys.,* 1980, vol. 76, no. 1, pp. 65–116.
2. Its, A. R. and Novokshenov, V. Yu., *The Isomonodromic Deformation Method in the Theory of Painlevé Equations,* Lecture Notes in Math., vol. 1191, Berlin: Springer, 1986.
3. Kapaev, A. A., Asymptotic Behavior of the Solutions of the Painlevé Equation of the First Kind, *Differ. Equ.,* 1988, vol. 24, no. 10, pp. 1107–1115; see also: *Differ. Uravn.,* 1988, vol. 24, no. 10, pp. 1684–1695.
4. Fokas, A. S., Its, A. R., Kapaev, A. A., and Novokshenov, V. Yu., *Painlevé Transcendents: The Riemann – Hilbert Approach,* Math. Surveys Monogr., vol. 128, Providence, R.I.: AMS, 2006.
5. Kaup, D. J., A Perturbation Expansion for the Zakharov – Shabat Inverse Scattering Transform, *SIAM J. Appl. Math.,* 1976, vol. 31, no. 1, pp. 121–133.
6. Karpman, V. I. and Maslov, E. M., Perturbation Theory for Solitons, *Sov. Phys. JETP,* 1977, vol. 46, no. 2, pp. 537–559; see also: *Zh. Èksper. Teoret. Fiz.,* 1977, vol. 46, no. 2, pp. 281–291.
7. Kiselev, O. M., Kink Asymptotics of the Perturbed Sine-Gordon Equation, *Theoret. and Math. Phys.,* 1992, vol. 93, no. 1, pp. 1106–1111; see also: *Teoret. Mat. Fiz.,* 1992, vol. 93, no. 1, pp. 39–48.
8. Kiselev, O. M., Perturbation Theory for the Dirac Equation in Two-Dimensional Space, *J. Math. Phys.,* 1998, vol. 39, no. 4, pp. 2333–2345.
9. Kiselev, O. M., Basis Functions Associated with the Two-Dimensional Dirac System, *Funct. Anal. Appl.,* 1998, vol. 32, no. 1, pp. 56–59; see also: *Funktsional. Anal. i Prilozhen.,* 1998, vol. 32, no. 1, pp. 72–76.
10. Garnier, R., Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Ann. Sci. Éc. Norm. Supér.,* 1912, vol. 29, pp. 1–126.
11. Wasow, W. R., *Asymptotic Expansion for Ordinary Differential Equations,* Pure Appl. Math., vol. 14, New York: Interscience, 1965.
12. Sochocki, Yu. V., On Definite Integrals and Functions Used in Series Expansions, *PhD Dissertation,* St. Petersburg, St. Petersburg Univ., 1873, 131 pp. (Russian).
13. Suleĭmanov, B. I., “Quantizations” of the Second Painlevé Equation and the Problem of the Equivalence of Its *L-A* Pairs, *Theoret. and Math. Phys.,* 2008, vol. 156, no. 3, pp. 1280–1291; see also: *Teoret. Mat. Fiz.,* 2008, vol. 156, no. 3, pp. 364–377.
14. Joshi, N., Kitaev, A. V., and Treharne, P. A., On the Linearization of the First and Second Painlevé Equations, *J. Phys. A,* 2009, vol. 42, no. 5, 055208, 18 pp.
15. Gallo, C. and Pelinovsky, D., On the Thomas – Fermi Ground State in a Harmonic Potential, *Asymptot. Anal.,* 2011, vol. 73, nos. 1–2, pp. 53–96.
16. Kiselev, O. M., An Asymptotic Structure of the Bifurcation Boundary of the Perturbed Painlevé-2 Equation, *Chaos Solitons Fractals,* 2021, vol. 151, Paper No. 111299, 7 pp.

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