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Twisted States in a System of Nonlinearly Coupled Phase Oscillators

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Abstract—We study the dynamics of the ring of identical phase oscillators with nonlinear nonlocal coupling. Using the Ott – Antonsen approach, the problem is formulated as a system of partial derivative equations for the local complex order parameter. In this framework, we investigate the existence and stability of twisted states. Both fully coherent and partially coherent stable twisted states were found (the latter ones for the first time for identical oscillators). We show that twisted states can be stable starting from a certain critical value of the medium length, or on a length segment. The analytical results are confirmed with direct numerical simulations in finite ensembles.

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INTRODUCTION

The study of the systems of coupled oscillators is a topic of much current interest in theoretical and experimental nonlinear science. The reason is in a fundamental importance of such models in various fields of modern science and technology. They allow for an adequate description of mechanical objects (for example, coupled pendulums [1–3], metronomes mounted on a common base [4, 5]), of processes in electrical networks [6, 7], of solid state structures [8, 9] and molecular chains [10, 11], etc. Further references to specific experimental and theoretical studies can be found in the book [1] and the recent review [12].

A large number of key fundamental phenomena in nonlinear oscillatory media of different nature can be studied in the framework of the phase approximation [1, 13]. This, in particular, includes synchronization in its manifold representations [1, 2]. A transformation of an accurate and specific mathematical formulation of a problem in terms of differential equations into a universal description in terms of dynamic equations for the phase variables allows for identification of common principles and general patterns in the behavior of physical, chemical, biological and social oscillatory systems [1]. A seminal Kuramoto model and its modifications [12, 14, 15] are spectacular examples of the power of phase reduction in studies of populations of oscillators interacting through a common

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field. An important feature of this coupling topology is the lack of information about the elements positions in space. Therefore, in this case, despite the fact that the cluster formation process is possible, one cannot formulate the problem of the formation of spatial patterns of synchrony. The latter task is, however, possible for spatially ordered media with nonglobal coupling, local or nonlocal. Here, because of the absence of permutation symmetry, positions of oscillators are important. In several recent studies [16–18], stationary spatial patterns in phase oscillators media with nonlocal coupling have been described as states with a spatially inhomogeneous profile of a local (in space) complex order parameter, which measures the level of synchronization of the neighboring elements.

In this paper we study twisted states [19–21] in a set of identical nonlocally coupled phase oscillators, uniformly distributed on a ring. A fully coherent twisted state in such a configuration occurs when the phase differences between neighboring oscillators is a constant, i.e., the phase profile has a constant gradient. In the case of a partially coherent twisted state, the phases of units deviate from the fully ordered gradient phase distribution, however, the phase of the local order parameter has a constant gradient. The fully coherent twisted state was observed in systems of identical phase oscillators [19, 20]. Remarkably, in the work [22] the authors essentially considered such twisted states in a model of a coupled map lattice with a complex state variable (although the term "twisted states" itself was not used there). The partially coherent twisted state has been reported for a population with nonidentical natural frequencies [21]. In this article, we describe both the fully coherent and the partially coherent twisted states in a system of *identical* phase elements with *nonlinear* nonlocal coupling.

The paper is organized as follows. In Section 1 we formulate the model as a set of phase equations, and introduce the local order parameter using the Ott – Antonsen ansatz. In Section 2 the twisted states (fully coherent and partially coherent) are found as solutions of the Ott – Antonsen equations. Furthermore, their stability is determined and numerical simulations are presented. We summarize our findings in Section 3.

1. BASIC MODEL

We consider a continuous medium of nonlocally coupled identical phase oscillators with natural frequency ω, continuously distributed over the interval $[0; L)$ with periodic boundary conditions. Such a configuration is equivalent to the situation where the elements of a one-dimensional oscillatory medium are located on a ring of length L. We describe this system using the dynamical variable $\phi(x,t)$ defined at each point $x \in [0; L)$ and obeying the integro-differential equation:

$$\partial_t \phi(x,t) = \omega + \mathrm{Im}\left[H(x,t)e^{-i\phi(x,t)-i\alpha(H(x,t)))}\right]. \tag{1.1}$$

Here $H(x,t)$ is the force acting on the oscillator at site x, it is defined as a convolution over the states of all other units

$$H(x,t) = \int_0^L G(x-\widetilde{x})e^{i\phi(\widetilde{x},t)}d\widetilde{x}. \tag{1.2}$$

Note that the integral in (1.2) is understood in the Lebesgue sense, so that no spatial smoothness or continuity of function $e^{i\phi(x,t)}$ is needed. We stress once again that our basic model is a continuous medium of oscillators, not a lattice system. The kernel $G(y)$ characterizes the interaction strength within the medium, and satisfies the unit normalization condition $\int_{-L/2}^{L/2} G(y)dy = 1$. For $G(y)$ we choose a function:

$$G(y) = \frac{\cosh(|y| - L/2)}{2\sinh(L/2)}, \tag{1.3}$$

which describes exponentially decaying interactions (with proper boundary conditions). This function naturally appears as a Green function for a Laplace equation (see Eq. (1.7) below) on a ring [17].

Parameter α is a phase shift in the coupling. Usually, the value α is considered to be a constant α_0. However, nonlinear effects in coupling can have a significant influence on the dynamics of the system. In order to take them into account, we set the value of phase shift as $\alpha(H) = \alpha_0 + \alpha_1|H|^2$ [18, 23, 24]. In the study of this model we are primarily interested in the dynamics of the system, depending on the medium length L.

Equations (1.1)–(1.3) determine the dynamics of the investigated continuous medium. Because coupling in (1.2) is nonlocal, there are no terms supporting smoothness and continuity of the phase profile, thus $\phi(x,t)$ is generally a nonsmooth function of x: neighboring phases can differ by as much as π. Hence, the original equation is inconvenient for the formulation of the pattern formation problem and its further analysis. On the other hand, this problem does not appear in numerical simulations below, where we perform a spatial discretization with 1000 points per unit length along the x axis. The key step allowing for formulating a pattern formation problem is in using the procedure of averaging (coarse-graining) over a small δ-neighborhood of a point x. Following this approach, we introduce a local order parameter

$$Z(x,t) = \frac{1}{2\delta}\int_{-\delta}^{\delta} e^{i\phi(x+\tilde{x},t)} d\tilde{x}, \tag{1.4}$$

which is a continuous complex function of the coordinate x and time t, satisfying the inequality $|Z(x,t)| \leqslant 1$. If $|Z(x,t)| = 1$, the behavior of neighboring elements of the medium is synchronous. If $|Z(x,t)| < 1$, then the phase oscillators are partially synchronous. We employ the Ott – Antonsen reduction [25], allowing for formulation of a closed dynamical equations for $Z(x,t)$:

$$\partial_t Z = i\omega Z + (e^{-i\alpha(H)}H - e^{i\alpha(H)}H^* Z^2)/2. \tag{1.5}$$

According to definition (1.2) and to the definition of the local order parameter, the field $H(x,t)$ can be expressed via $Z(x,t)$:

$$H(x,t) = \int_0^L G(x-\widetilde{x})Z(\widetilde{x},t)d\widetilde{x}. \tag{1.6}$$

It is noteworthy that the Lebegues integral over a nonsmooth phase profile in (1.2) is transformed, by virtue of coarse-graining, to the Cauchy integral over the smooth order parameter in (1.6). Further, using (1.3), it is not difficult to pass from relation (1.6) to the equivalent differential equation:

$$\partial^2_{xx}H - H = -Z \tag{1.7}$$

with periodic boundary conditions:

$$H(0,t) = H(L,t), \quad \partial_x H(0,t) = \partial_x H(L,t). \tag{1.8}$$

Thus, the dynamics of the oscillatory medium under study can be described within the Ott – Antonsen approximation as a set of partial derivative equations (1.5), (1.7) for the complex field of local order parameter $Z(x,t)$, with boundary conditions (1.8).

2. TWISTED STATES

Let us define a class of stationary twisted states. A fully coherent twisted state in system (1.1) is realized if the phase profile $\phi(x,t)$ is a linear function of x, satisfying the boundary condition: $\phi(x,t) = \phi(0,t) + 2\pi mx/L$ (here $m = \pm 1, \pm 2, \ldots$ is the number of phase rotations). This is clearly a gradient profile. For a partially coherent twisted state, the phases of the individual elements can deviate from the gradient distribution, however, the locally spatially averaged phase, i. e., the argument of the local order parameter, has a gradient profile. A characteristic feature of these regimes is that the absolute value of the order parameter $|Z(x,t)|$ at all points of the medium is a constant value, which corresponds to the same degree of local coherence of the oscillators, however, the phase of the $Z(x,t)$ makes an integer number of rotations along the ring. The global order parameter $\int_0^L e^{i\phi(x,t)}dx$ vanishes in the twisted states.

2.1. Twisted States as Solutions of the Ott – Antonsen Equations

For uniformly rotating twisted states, the solutions $Z(x,t)$ and $H(x,t)$ are defined by the following expressions:

$$Z(x,t) = z(x)e^{i(\omega+\Omega)t}, \quad H(x,t) = h(x)e^{i(\omega+\Omega)t}, \tag{2.1}$$

$$z(x) = z_0 e^{-iqx}, h(x) = h_0 e^{-iqx}, \tag{2.2}$$

where $z_0 = \text{const}$, $h_0 = \text{const}$, $q = 2\pi m/L$, $m = \pm 1, \pm 2, \ldots$, Ω is the rotation frequency. We restrict ourselves to the case $m = 1$, and the values of the parameters satisfying the condition: $0 \leqslant \alpha_0, \alpha_1 \leqslant \pi/2$.

Substituting (2.1) and (2.2) into (1.7), we obtain expressions relating the values of h_0 and z_0:

$$h_0 = \frac{z_0}{1+q^2}. \tag{2.3}$$

To find the parameters z_0 and h_0, we substitute (2.1), (2.2), (2.3) into (1.5), from which we obtain the following system:

$$\begin{aligned} \sin\alpha(H)\left(-\frac{1+|z_0|^2}{1+q^2}\right) &= 2\Omega, \\ \cos\alpha(H)\left(\frac{1-|z_0|^2}{1+q^2}\right) &= 0. \end{aligned} \tag{2.4}$$

The system (2.4) has two solutions. The first solution

$$z_0 = z_s = 1, \quad h_0 = h_s = \frac{1}{1+q^2} \tag{2.5}$$

corresponds to a fully coherent twisted state (FCTS) (Fig. 1a). It exists for all values of α_0, α_1. The second solution

$$z_0 = z_{ps} = (1+q^2)\sqrt{\frac{\pi - 2\alpha_0}{2\alpha_1}}, \quad h_0 = h_{ps} = \sqrt{\frac{\pi - 2\alpha_0}{2\alpha_1}} \tag{2.6}$$

corresponds to a partially coherent twisted state (PCTS) (Fig. 1b). The region of existence of this state is defined by the inequality $\alpha_1 > \pi/2 - \alpha_0$.

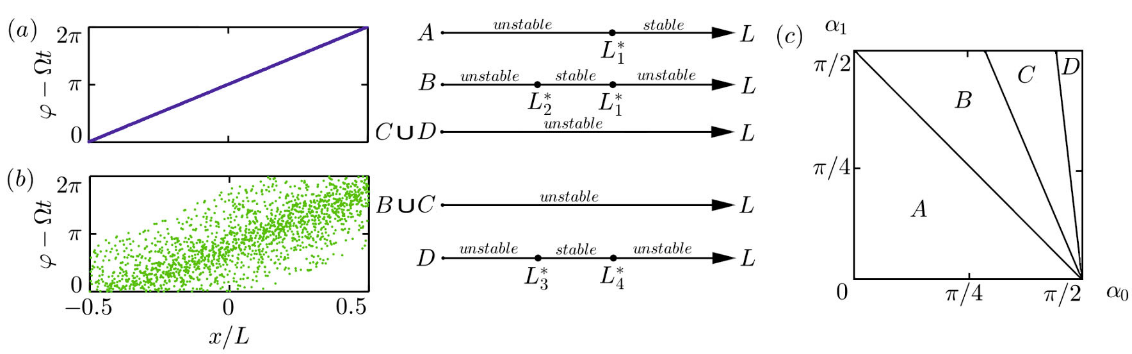

Fig. 1. Panels (a) and (b) illustrate twisted states via snapshots of the phases on the ring. (a) Fully coherent twisted state for $\alpha_1 = 0.7853$, $\alpha_0 = 0.7853$, $L = 12$, $z_{ps} = 1$, $h_{ps} = 0.785$. (b) Partially coherent twisted state for $\alpha_1 = 1.57$, $\alpha_0 = 1.468$, $L = 7.85$, $z_s = 0.42$, $h_s = 0.256$. Panel (c) shows regions of existence and stability of different twisted states on a parameter plane (α_0, α_1). Region A: $0 < \alpha_1 < \pi/2 - \alpha_0$, FCTS is stable at $L > L_1^*$, PCTS does not exist. Region B: $\pi/2 - \alpha_0 < \alpha_1 < 9(\pi/2 - \alpha_0)/4$, FCTS is stable at $L_2^* < L < L_1^*$, PCTS is unstable for any L. Region C: FCTS and PCTS are unstable for any L. Region D: FCTS is unstable for any L, PCTS is stable for $L_3^* < L < L_4^*$. The middle panel (arrows) illustrates dependences of stability of different states on the length L.

2.2. Stability Analysis

The stability of a stationary twisted state (2.1), (2.2) can be analyzed via linearization of equation (1.5). For this we substitute the ansatz

$$Z(x,t) = \big(z_0 + \hat{Z}(x,t)\big)e^{-iqx+i(\omega+\Omega)t}, \tag{2.7}$$

where $\hat{Z}(x,t)$ is a periodic in x small perturbation, into the Ott–Antonsen equation (1.5). Similarly, we represent $H(x,t)$ as

$$H(x,t) = \big(h_0 + \hat{H}(x,t)\big)e^{-iqx+i(\omega+\Omega)t}. \tag{2.8}$$

As a result of linearization, we obtain an equation that determines the dynamics of $\hat{Z}$:

$$\partial_t\hat{Z} = -\left(i\Omega + e^{i\alpha}\frac{r^2}{1+q^2}\right)\hat{Z} + \frac{1}{2}\left(e^{-i\alpha}\hat{H} - e^{i\alpha}r^2\hat{H}^*\right) - \frac{i\alpha_1 r^2}{2(1+q^2)^2}\left(\hat{H}^* + \hat{H}\right)\left(e^{-i\alpha} + r^2 e^{i\alpha}\right), \tag{2.9}$$

where

$$\alpha = \alpha_0 + \alpha_1\frac{r^2}{(1+q^2)^2}, \tag{2.10}$$

$$\hat{H}(x,t) = \int\limits_0^L G(x-\widetilde{x})e^{iq(x-\widetilde{x})}\hat{Z}(\widetilde{x},t)d\widetilde{x}. \tag{2.11}$$

We use the Bogolyubov method and represent the perturbation $\hat{Z}(x,t)$ in the form

$$\hat{Z}(x,t) = ae^{\lambda t - ik_n x} + b^* e^{\lambda^* t + ik_n x}, \tag{2.12}$$

where $k_n = 2\pi n/L$, n is number of the harmonic of the perturbation $\hat{Z}$. Then $\hat{H}(x,t)$ is given by the expression

$$\hat{H}(x,t) = \frac{a^* e^{\lambda t - ik_n x}}{1+(q+k_n)^2} + \frac{b^* e^{\lambda^* t + ik_n x}}{1+(q-k_n)^2}. \tag{2.13}$$

Substituting (2.12), (2.13) into (2.9), we get the following eigenvalue problem:

$$\lambda(n)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.14}$$

where

$$M_{11} = -\left(i\Omega + e^{i\alpha}\frac{r^2}{1+q^2}\right) + \frac{e^{-i\alpha}}{2\big(1+(q+k_n)^2\big)} - \frac{i\alpha_1 r^2\left(e^{-i\alpha} + r^2 e^{i\alpha}\right)}{2(1+q^2)^2\big(1+(q+k_n)^2\big)}, \tag{2.15}$$

$$M_{12} = -\frac{e^{i\alpha}r^2}{2\big(1+(q-k_n)^2\big)} - \frac{i\alpha_1 r^2\left(e^{-i\alpha} + r^2 e^{i\alpha}\right)}{2(1+q^2)^2\big(1+(q-k_n)^2\big)}, \tag{2.16}$$

$$M_{21} = -\frac{e^{-i\alpha}r^2}{2\big(1+(q+k_n)^2\big)} + \frac{i\alpha_1 r^2\left(e^{i\alpha} + r^2 e^{-i\alpha}\right)}{2(1+q^2)^2\big(1+(q+k_n)^2\big)}, \tag{2.17}$$

$$M_{22} = -\left(-i\Omega + e^{-i\alpha}\frac{r^2}{1+q^2}\right) + \frac{e^{i\alpha}}{2\big(1+(q-k_n)^2\big)} + \frac{i\alpha_1 r^2\left(e^{i\alpha} + r^2 e^{-i\alpha}\right)}{2(1+q^2)^2\big(1+(q-k_n)^2\big)}. \tag{2.18}$$

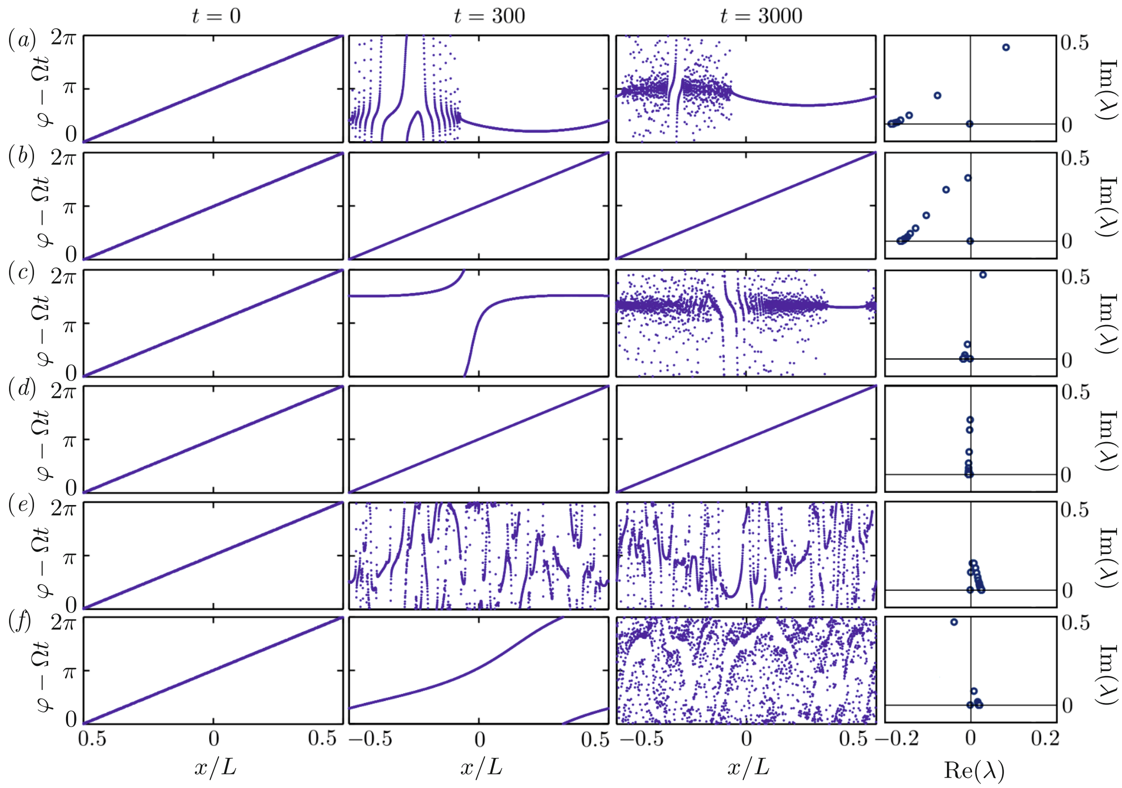

Fig. 2. Fully coherent twisted states. Left three columns: results of direct numerical simulations of the set of $N = 2048$ oscillators, at different points in time. The right column shows the spectrum of linear perturbations $\lambda_{1,2}(n)$ for $n = 0, 1, 2, \dots, 100$. (a) Region A. Parameters: $\alpha_1 = 0.7853$, $\alpha_0 = 0.7853$, $L = 5.0$, $z_s = 1$, $h_s = 0.388$. The unstable twisted state evolves to an inhomogeneous state. (b) Region A. Parameters: $\alpha_1 = 0.7853$, $\alpha_0 = 0.7853$, $L = 10.0$, $z_s = 1$, $h_s = 0.717$. Stable twisted state. (c) Region B. Parameters: $\alpha_1 = 0.15$, $\alpha_0 = 1.457$, $L = 3.0$, $z_s = 1$, $h_s = 0.185$. The unstable twisted state evolves to an inhomogeneous state. (d) Region B. Parameters: $\alpha_1 = 0.15$, $\alpha_0 = 1.457$, $L = 10.0$, $z_s = 1$, $h_s = 0.717$. Stable twisted state. (e) Region B. Parameters: $\alpha_1 = 0.15$, $\alpha_0 = 1.457$, $L = 25.0$, $z_s = 1$, $h_s = 0.94$. The unstable twisted state, evolving to a turbulent regime. (f) Region $C \cup D$. Parameters: $\alpha_1 = 1.2$, $\alpha_0 = 1.457$, $L = 3.0$, $z_s = 1$, $h_s = 0.185$. The unstable twisted state, evolving to a turbulent regime.

2.3. Stability of a Fully Coherent Twisted State

First, we consider fully coherent twisted states, when $h_0 = h_s$, $z_0 = z_s$. For them, the real parts of the eigenvalues have the form

$$\operatorname{Re}\lambda_1 = -\frac{L^2\cos(\alpha)}{L^2 + 4\pi^2}, \tag{2.19}$$

$$\operatorname{Re}\lambda_2(n) = -\frac{4L^2n^2\pi^2(L^2 - 12\pi^2 + 4\pi^2n^2)\cos(\alpha)}{(L^2 + 4\pi^2)(L^2 + 4\pi^2 - 8\pi^2 n + 4\pi^2n^2)(L^2 + 4\pi^2 + 8\pi^2 n + 4\pi^2 n^2)}. \tag{2.20}$$

It is easy to demonstrate that in this case stability is determined by eigenvalues with $n = 1$, i. e., by the first harmonic

$$\operatorname{Re}\lambda_2(1) = -\frac{4\pi^2(8\pi^2 - L^2)\cos(\alpha)}{(L^2 + 4\pi^2)(L^2 + 16\pi^2)}. \tag{2.21}$$

Analysis of expressions (2.19) and (2.20) shows that there are three characteristic regions of (α_0, α_1) parameter values. In the first region $0 < \alpha_1 < \pi/2 - \alpha_0$ (Fig. 1c, region A) the fully coherent twisted state is stable at $L > L_1^*$, where $L_1^* = 2\sqrt{2}\pi$. In the second region $\pi/2 - \alpha_0 < \alpha_1 < 9(\pi/2 - \alpha_0)/4$ (Fig. 1c, region B) this state is stable when L belongs to the interval $L_2^* < L < L_1^*$, where $L_2^* = \left(\sqrt{2\alpha_1/(\pi - 2\alpha_0)} - 1\right)^{-1/2}$. In the third region $9(\pi/2 - \alpha_0)/4 < \alpha_1 < \pi/2$ (Fig. 1c, region C∪D) this regime is unstable for any L.

We illustrate the analytic stability considerations above with direct numerical simulations in Fig. 2. Here phase snapshots at different moments in time and the eigenvalue spectrum of fully coherent twisted states are shown. In the case of instability, these states evolve to various inhomogeneous or turbulent regimes.

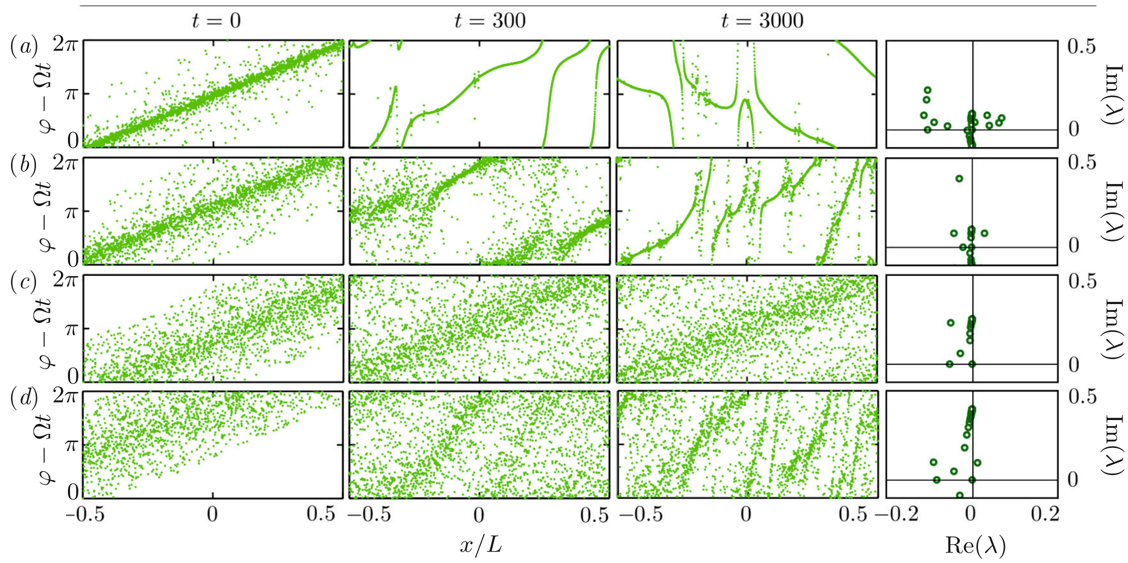

Fig. 3. Partially coherent twisted states. Panels show the same quantities as those in Fig. 2. (a) Region $B \cup C$. Parameters: $\alpha_1 = 1.2$, $\alpha_0 = 1.0$, $L = 12.0$, $z_{ps} = 0.879$, $h_{ps} = 0.716$. The unstable twisted state evolves to the turbulent state. (b) Region D. Parameters: $\alpha_1 = 1.57$, $\alpha_0 = 1.468$, $L = 4.85$, $z_{ps} = 0.685$, $h_{ps} = 0.275$. The unstable twisted state evolves to the turbulent state. (c) Region D. Parameters: $\alpha_1 = 1.57$, $\alpha_0 = 1.468$, $L = 7.85$, $z_{ps} = 0.42$, $h_{ps} = 0.288$. Stable twisted state. (e) Region D. Parameters: $\alpha_1 = 1.57$, $\alpha_0 = 1.468$, $L = 15.7$, $z_{ps} = 0.297$, $h_{ps} = 0.294$. The unstable twisted state evolves to the turbulent state.

2.4. *Partially Coherent Twisted State*

Analyzing the eigenvalues $\lambda(n)$ of the problem (2.14) numerically for $h_0 = h_{ps}$, $z_0 = z_{ps}$, we find that a partially coherent twisted state is unstable for any L in the region $B \cup C$ (Fig. 1c) and stable for $L_3^* < L < L_4^*$ in the region D (Fig. 1c). It is noteworthy that in this case stability of a twisted state can also be affected by the harmonics of perturbations $\hat{Z}(x, t)$ with $n > 1$. Time evolution of stable and unstable partially synchronized twisted states is illustrated in Fig. 3, where phase snapshots for different points in time and the eigenvalue spectrum are shown.

Thus, we have shown that a partially coherent twisted state can be realized in a system of identical nonlocally coupled oscillators. Moreover, there is no region of system parameters α_0, α_1, L, where this state could coexist with a fully coherent twisted state, i. e., no bistability coherent-partially coherent twisted state has been found.

3. CONCLUSION

In summary, we investigated twisted states in a system of identical nonlocally coupled phase oscillators with a nonlinear phase shift. The system was reformulated in terms of a local complex order parameter as a system of partial differential equations, by virtue of the Ott–Antonsen reduction. Twisted states were found as exact solutions of these equations, and the regions of existence and stability of fully coherent and partially coherent states are described. Remarkably, twisted states can be stable, starting from a certain critical value of the medium length, or on a length segment. We stress that the existence of a partially coherent state is caused not by the heterogeneity of the oscillator frequencies [21], but by the presence of a nonlinear dependence of the phase shift on the field H, i. e., it is a purely dynamical phenomenon in an ordered medium of identical oscillators. The analytically obtained results are confirmed by direct numerical simulation within the framework of the phase model.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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