Periodic Billiards Within Conics in the Minkowski Plane and Akhiezer Polynomials

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Abstract—We derive necessary and sufficient conditions for periodic and for elliptic periodic trajectories of billiards within an ellipse in the Minkowski plane in terms of an underlining elliptic curve. We provide several examples of periodic and elliptic periodic trajectories with small periods. We observe a relationship between Cayley-type conditions and discriminantly separable and factorizable polynomials. Equivalent conditions for periodicity and elliptic periodicity are derived in terms of polynomial-functional equations as well. The corresponding polynomials are related to the classical extremal polynomials. In particular, the light-like periodic trajectories are related to the classical Chebyshev polynomials. Similarities and differences with respect to the previously studied Euclidean case are highlighted.

1. INTRODUCTION

Billiards within quadrics in pseudo-Euclidean spaces were studied in [13, 14, 18]. In [15, 16], the relationship between the billiards within quadrics in the Euclidean spaces and extremal polynomials has been studied. The aim of this paper is to develop the connection between extremal polynomials and billiards in the Minkowski plane. The paper is devoted to 150th anniversary of S. A. Chaplygin and the conference proceedings contribution is [4].

This paper is organized as follows. In Section 2, we recall the basic notions connected with the Minkowski plane, confocal families of conics, relativistic ellipses and hyperbolas, and billiards. In Section 3, we give a complete description of the periodic billiard trajectories in algebro-geometric terms. In Section 4, we use the conditions obtained in the previous section to study examples of periodic trajectories with small periods.

We also emphasize an intriguing connection between the Cayley-type conditions and discriminantly separable polynomials. The notion of relativistic ellipses and hyperbolas enables definition of Jacobi-type elliptic coordinates in the Minkowski setting. Since the correspondence between Cartesian and elliptic coordinates is not one-to-one, there is a notion of elliptic periodicity which refers to a weaker assumption that a trajectory is periodic in elliptic coordinates. In Section 5, we provide an algebro-geometric characterization of trajectories to be n-elliptic periodic without being n-periodic. Section 6 provides examples and connections with discriminantly separable polynomials. In Section 7, we derive a characterization of elliptic periodic trajectories using polynomial equations.

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In the last section (Section 8), we establish the connection between characteristics of periodic billiard trajectories and extremal polynomials: the Zolotarev polynomials, the Akhiezer polynomials on symmetric intervals, and the general Akhiezer polynomials on two intervals. We conclude our study of the relationship of billiards in the Minkowski plane with the extremal polynomials by relating the case of light-like trajectories to the classical Chebyshev polynomials, see Section 9.

Apart from similarities with previously studied Euclidean spaces, see [15, 16], there are also significant differences: for example, among the obtained extremal polynomials are such with winding numbers $(3, 1)$, which was never the case in the Euclidean setting.

2. CONFOCAL FAMILIES OF CONICS AND BILLIARDS

The Minkowski plane is $\mathbf{R}^2$ with *the Minkowski scalar product*: $\langle X, Y \rangle = X_1 Y_1 - X_2 Y_2$.

The Minkowski distance between points X, Y is $\text{dist}(X, Y) = \sqrt{\langle X - Y, X - Y \rangle}$. Since the scalar product can be negative, notice that the Minkowski distance can have imaginary values as well. In that case, we choose the value of the square root with the positive imaginary part.

Let ℓ be a line in the Minkowski plane, and v its vector. ℓ is called *space-like* if $\langle v, v \rangle > 0$, *time-like* if $\langle v, v \rangle < 0$, and *light-like* if $\langle v, v \rangle = 0$. Two vectors x, y are *orthogonal* in the Minkowski plane if $\langle x, y \rangle = 0$. Note that a light-like vector is orthogonal to itself.
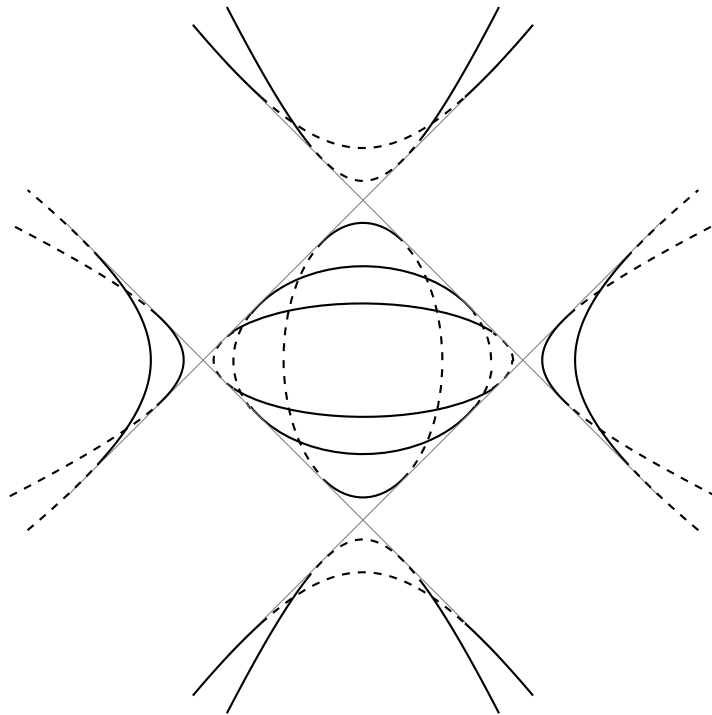
Confocal families. Denote by

$$\mathcal{E} \ : \ \frac{x^2}{a} + \frac{y^2}{b} = 1 \tag{2.1}$$

an ellipse in the plane, with a, b being fixed positive numbers.

The associated family of confocal conics is

$$\mathcal{C}_\lambda \ : \ \frac{x^2}{a - \lambda} + \frac{y^2}{b + \lambda} = 1, \quad \lambda \in \mathbf{R}. \tag{2.2}$$

The family is shown in Fig. 1.

Fig. 1. Family of confocal conics in the Minkowski plane. The solid lines represent relativistic ellipses, and the dashed ones relativistic hyperbolas.

We may distinguish the following three subfamilies in the family (2.2): for $\lambda \in (-b, a)$, conic $\mathcal{C}_\lambda$ is an ellipse; for $\lambda < -b$, conic $\mathcal{C}_\lambda$ is a hyperbola with x-axis as the major one; for $\lambda > a$, it is a hyperbola again, but now its major axis is the y-axis. In addition, there are three degenerate quadrics: $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_\infty$ corresponding to the y-axis, the x-axis, and the line at infinity, respectively.

The confocal family has three pairs of foci: $F_1(\sqrt{a+b}, 0)$, $F_2(-\sqrt{a+b}, 0)$; $G_1(0, \sqrt{a+b})$, $G_2(0, -\sqrt{a+b})$; and $H_1(1 : -1 : 0)$, $H_2(1 : 1 : 0)$ on the line at infinity.

We notice four distinguished lines:

$$\mathsf{x} + \mathsf{y} = \sqrt{a+b}, \quad \mathsf{x} + \mathsf{y} = -\sqrt{a+b},$$
$$\mathsf{x} - \mathsf{y} = \sqrt{a+b}, \quad \mathsf{x} - \mathsf{y} = -\sqrt{a+b}.$$

These lines are common tangents to all conics from the family.

Conics in the Minkowski plane have geometric properties analogous to the conics in the Euclidean plane. Namely, for each point on conic $\mathcal{C}_\lambda$, either sum or difference of its Minkowski distances from the foci F_1 and F_2 is equal to $2\sqrt{a - \lambda}$; either sum or difference of the distances from the other pair of foci G_1, G_2 is equal to $2\sqrt{-b - \lambda}$ [13].

In the Minkowkski plane, it is natural to consider relativistic conics, which are suggested in [7]. In this section, we give a brief account of the related analysis.

Consider points $F_1(\sqrt{a+b}, 0)$ and $F_2(-\sqrt{a+b}, 0)$.

For a given constant $c \in \mathbf{R}^+ \cup i\mathbf{R}^+$, a *relativistic ellipse* is the set of points X satisfying $\mathrm{dist}(F_1, X) + \mathrm{dist}(F_2, X) = 2c$, while a *relativistic hyperbola* is the union of the sets given by the following equations:

$$\mathrm{dist}(F_1, X) - \mathrm{dist}(F_2, X) = 2c,$$
$$\mathrm{dist}(F_2, X) - \mathrm{dist}(F_1, X) = 2c.$$

Relativistic conics can be described as follows.

$\mathbf{0 < c < \sqrt{a+b}.}$ The corresponding relativistic conics lie on ellipse $\mathcal{C}_{a-c^2}$ from family (2.2). The ellipse $\mathcal{C}_{a-c^2}$ is split into four arcs by touching points with the four common tangent lines; thus, the relativistic ellipse is the union of the two arcs intersecting the y-axis, while the relativistic hyperbola is the union of the other two arcs.

$\mathbf{c > \sqrt{a+b}.}$ The relativistic conics lie on $\mathcal{C}_{a-c^2}$ – a hyperbola with x-axis as the major one. Each branch of the hyperbola is split into three arcs by touching points with the common tangents; thus, the relativistic ellipse is the union of the two finite arcs, while the relativistic hyperbola is the union of the four infinite ones.

$\mathbf{c}$ **is imaginary.** The relativistic conics lie on hyperbola $\mathcal{C}_{a-c^2}$ – a hyperbola with y-axis as the major one. As in the previous case, the branches are split into six arcs in total by common points with the four tangents. The relativistic ellipse is the union of the four infinite arcs, while the relativistic hyperbola is the union of the two finite ones.

Notice that all relativistic ellipses are disjoint with each other, as well as all relativistic hyperbolas, see Fig. 1. Moreover, at the intersection point of a relativistic ellipse which is a part of the geometric conic $\mathcal{C}_{\lambda_1}$ from the confocal family (2.2) and a relativistic hyperbola belonging to $\mathcal{C}_{\lambda_2}$, it is always $\lambda_1 < \lambda_2$.

Elliptic coordinates. Each point inside ellipse $\mathcal{E}$ has elliptic coordinates (λ_1, λ_2), such that $-b < \lambda_1 < 0 < \lambda_2 < a$.

The differential equation of the lines touching a given conic $\mathcal{C}_\gamma$ is

$$\frac{d\lambda_1}{\sqrt{(a - \lambda_1)(b + \lambda_1)(\gamma - \lambda_1)}} + \frac{d\lambda_2}{\sqrt{(a - \lambda_2)(b + \lambda_2)(\gamma - \lambda_2)}} = 0. \qquad (2.3)$$

Billiards. Let v be a vector and p a line in the Minkowski plane. Decompose vector v into the sum $v = a + n_p$ of a vector n_p orthogonal to p and a belonging to p. Then the vector $v' = a - n_p$ is *the billiard reflection* of v on p. It is easy to see that v is also the billiard reflection of v' with respect to p. Moreover, since $\langle v, v \rangle = \langle v', v' \rangle$, vectors v, v' are of the same type.

Note that $v = v'$ if v is contained in p and $v' = -v$ if it is orthogonal to p. If n_p is light-like, which means that it belongs to p, then the reflection is not defined.

Line ℓ' is *the billiard reflection* of ℓ off ellipse $\mathcal{E}$ if their intersection point $\ell \cap \ell'$ belongs to $\mathcal{E}$ and the vectors of ℓ, ℓ' are reflections of each other with respect to the tangent line of $\mathcal{E}$ at this point.

The lines containing segments of a given billiard trajectory within $\mathcal{E}$ are all of the same type: they are all either space-like, time-like, or light-like. For a detailed explanation, see [18].

Billiard trajectories within ellipses in the Minkowski plane have caustic properties: each segment of a given trajectory will be tangent to the same conic confocal with the boundary, see [13]. For more on the Minkowski plane and related integrable systems, see [7, 17, 21].

3. PERIODIC TRAJECTORIES

Sections 3–8 deal with the trajectories with nondegenerate caustic $\mathcal{C}_\gamma$, which will mean that $\gamma \in \mathbf{R} \setminus \{-b, a\}$. Such trajectories are either space-like or time-like. The case of light-like trajectories which correspond to the degenerate caustic $\mathcal{C}_\infty$ is considered separately, in Section 9.

The periodic trajectories of elliptical billiards in the Minkowski plane can be characterized in algebro-geometric terms using the underlying elliptic curve:

Theorem 1. *The billiard trajectories within $\mathcal{E}$ with nondegenerate caustic $\mathcal{C}_\gamma$ are n-periodic if and only if $nQ_0 \sim nQ_\gamma$ on the elliptic curve:*

$$\mathscr{C} \ : \ y^2 = \varepsilon(a - x)(b + x)(\gamma - x), \tag{3.1}$$

with Q_0 being a point of $\mathscr{C}$ corresponding to $x = 0$, Q_γ to $x = \gamma$, and $\varepsilon = \operatorname{sign} \gamma$.

Proof. Along a billiard trajectory within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$, the elliptic coordinate λ_1 traces the segment $[\alpha_1, 0]$, and λ_2 the segment $[0, \beta_1]$, where α_1 is the largest negative and β_1 the smallest positive member of the set $\{a, -b, \gamma\}$.

Case 1. If $\mathcal{C}_\gamma$ is an ellipse and $\gamma < 0$, then $\alpha_1 = \gamma$, $\beta_1 = a$. The coordinate λ_1 takes the value $\lambda_1 = \gamma$ at the touching points with the caustic and the value $\lambda_1 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is time-like. On the other hand, λ_2 takes the value $\lambda_2 = a$ at the intersections with y-axis, and $\lambda_2 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is space-like.

Case 2. If $\mathcal{C}_\gamma$ is an ellipse and $\gamma > 0$, then $\alpha_1 = -b$, $\beta_1 = \gamma$. The coordinate λ_1 takes the value $\lambda_1 = -b$ at the intersections with x-axis and value $\lambda_1 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is time-like. On the other hand, λ_2 takes the value $\lambda_2 = \gamma$ at the touching points with the caustic, and $\lambda_2 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is space-like.

Case 3. If $\mathcal{C}_\gamma$ is a hyperbola, then $\alpha_1 = -b$, $\beta_1 = a$. The coordinate λ_1 takes the value $\lambda_1 = -b$ at the intersections with x-axis and value $\lambda_1 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is time-like. On the other hand, λ_2 takes the value $\lambda_2 = a$ at the intersections with y-axis, and $\lambda_2 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is space-like.

In each case, the elliptic coordinates change monotonously between their extreme values.

Consider an n-periodic billiard trajectory and denote by n_1 the number of reflections off time-like arcs, i.e., off relativistic ellipses, and by n_2 the number of reflections off space-like arcs, i.e., relativistic hyperbolas. Obviously, $n_1 + n_2 = n$. Integrating (2.3) along the trajectory, we get

$$n_1 \int_{\alpha_1}^{0} \frac{d\lambda_1}{\sqrt{\varepsilon(a - \lambda_1)(b + \lambda_1)(\gamma - \lambda_1)}} + n_2 \int_{\beta_1}^{0} \frac{d\lambda_2}{\sqrt{\varepsilon(a - \lambda_2)(b + \lambda_2)(\gamma - \lambda_2)}} = 0, \tag{3.2}$$

i.e.,

$$n_1(Q_0 - Q_{\alpha_1}) + n_2(Q_0 - Q_{\beta_1}) \sim 0.$$

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In Case 1, this is equivalent to

$$n_1(Q_0 - Q_\gamma) + n_2(Q_0 - Q_a) \sim n(Q_0 - Q_\gamma),$$

since a closed trajectory crosses the y-axis an even number of times, i.e, n_2 must be even, and $2Q_a \sim 2Q_\gamma$.

Similarly, in Case 2, it follows since n_1 is even, and in Case 3 both n_1 and n_2 need to be even. □

From the proof of Theorem 1, we have:

Corollary 1. *The period of a closed trajectory with hyperbola as caustic is even.*

Theorem 2. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic if and only if:*

$$C_2 = 0, \quad \begin{vmatrix} C_2 & C_3 \\ C_3 & C_4 \end{vmatrix} = 0, \quad \begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0, \dots \quad for \quad n = 3, 5, 7, \dots$$

$$B_3 = 0, \quad \begin{vmatrix} B_3 & B_4 \\ B_4 & B_5 \end{vmatrix} = 0, \quad \begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} = 0, \dots \quad for \quad n = 4, 6, 8, \dots.$$

Here, we have denoted:

$$\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)} = B_0 + B_1 x + B_2 x^2 + \dots,$$

$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma - x} = C_0 + C_1 x + C_2 x^2 + \dots,$$

the Taylor expansions around $x = 0$.

Proof. Denote by Q_∞ the point of $\mathscr{C}$ (3.1) corresponding to $x = \infty$ and notice that

$$2Q_\gamma \sim 2Q_\infty. \tag{3.3}$$

Consider first n even. Because of (3.3), the condition $nQ_0 \sim nQ_\gamma$ is equivalent to $nQ_0 \sim nQ_\infty$, which is equivalent to the existence of a meromorphic function of $\mathscr{C}$ with the unique pole at Q_∞ and unique zero at Q_0, such that the pole and the zero are both of multiplicity n. The basis of $\mathscr{L}(nQ_\infty)$ is

$$1, x, x^2, \dots, x^{n/2}, y, xy, x^{n/2-2}y, \tag{3.4}$$

thus, a nontrivial linear combination of those functions with a zero of order n at $x = 0$ exists if and only if

$$\begin{vmatrix} B_{n/2+1} & B_{n/2} & \dots & B_3 \\ B_{n/2+2} & B_{n/2+1} & \dots & B_4 \\ \dots & & & \\ B_{n-1} & B_n & \dots & B_{n/2+1} \end{vmatrix} = 0.$$

Now suppose n is odd. Because of (3.3), the condition $nQ_0 \sim nQ_\gamma$ is equivalent to $nQ_0 \sim (n-1)Q_\infty + Q_\gamma$, which is equivalent to the existence of a meromorphic function of $\mathscr{C}$ with only two poles: of order $n-1$ at Q_∞ and a simple pole at Q_γ, and unique zero at Q_0. The basis $\mathscr{L}\big((n-1)Q_\infty + Q_\gamma\big)$ is

$$1, x, x^2, \dots, x^{(n-1)/2}, \frac{y}{\gamma - x}, \frac{xy}{\gamma - x}, \dots, \frac{x^{(n-1)/2-1}y}{\gamma - x}, \tag{3.5}$$

thus, a nontrivial linear combination of those functions with a zero of order n at $x = 0$ exists if and only if

$$
\begin{vmatrix}
C_{(n-1)/2+1} & C_{(n-1)/2} & \cdots & C_2 \\
C_{(n-1)/2+2} & C_{(n-1)/2+1} & \cdots & C_3 \\
\cdots & & & \\
C_{n-1} & C_n & \cdots & C_{(n-1)/2+1}
\end{vmatrix} = 0.
$$

$\square$

4. TRAJECTORIES WITH SMALL PERIODS AND DISCRIMINANTLY SEPARABLE POLYNOMIALS

4.1. Examples of Periodic Trajectories: $3 \leqslant n \leqslant 8$

3-periodic trajectories

There is a 3-periodic trajectory of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ in the Minkowski plane if and only if, according to Theorem 2, the caustic is an ellipse, i. e., $\gamma \in (-b, a)$ and $C_2 = 0$.
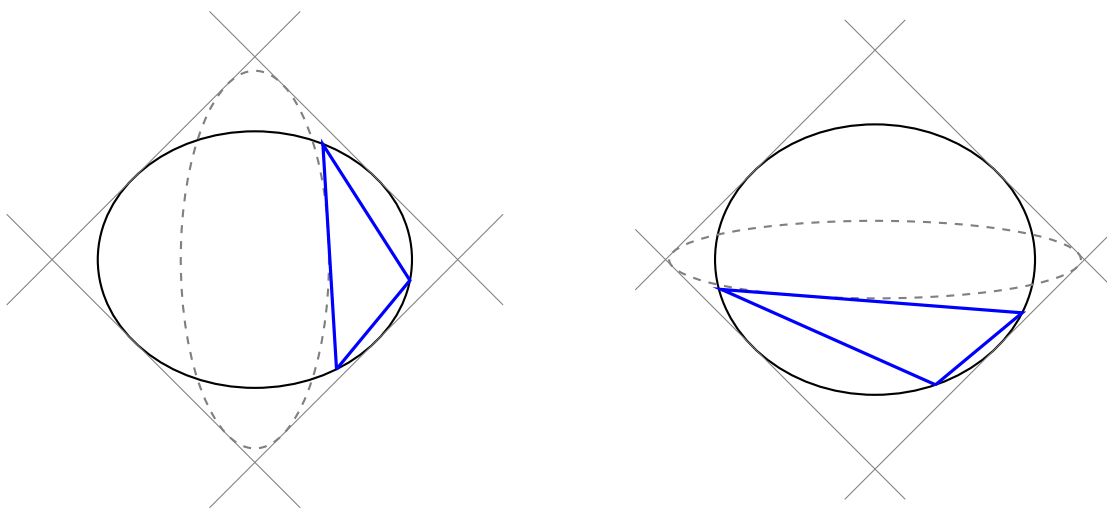
We solve the equation

$$
C_2 = \frac{3a^2b^2 + 2ab(a-b)\gamma - (a+b)^2\gamma^2}{8(ab)^{3/2}\gamma^{5/2}} = 0, \tag{4.1}
$$

which yields the following two solutions for the parameter γ for the caustic:

$$
\gamma_1 = \frac{ab}{(a+b)^2}(a - b + 2\sqrt{a^2 + ab + b^2}), \quad \gamma_2 = -\frac{ab}{(a+b)^2}(-a + b + 2\sqrt{a^2 + ab + b^2}). \tag{4.2}
$$

Notice that both caustics $\mathcal{C}_{\gamma_2}$ and $\mathcal{C}_{\gamma_1}$ are ellipses since $-b < \gamma_2 < 0 < \gamma_1 < a$.

Two examples of a 3-periodic trajectories are shown in Fig. 2.

Fig. 2. A 3-periodic trajectory with an ellipse along the y-axis as caustic ($a = 3$, $b = 2$, $\gamma \approx 2.332$) is shown on the left, while another trajectory with an ellipse along the x-axis as caustic ($a = 7$, $b = 5$, $\gamma \approx -4.589$) is on the right.

4-periodic trajectories

There is a 4-periodic trajectory of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ in the Minkowski plane if and only if $B_3 = 0$. We solve the equation
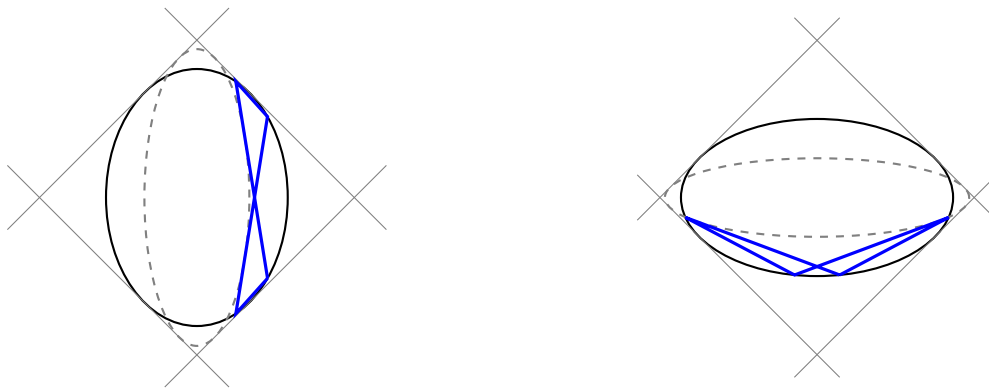
$$B_3 = -\frac{(ab + a\gamma + b\gamma)(ab + a\gamma - b\gamma)(ab - a\gamma - b\gamma)}{16(ab\gamma)^{5/2}} = 0, \tag{4.3}$$

which yields the following solutions for the parameter γ for the caustic:

$$\gamma_1 = -\frac{ab}{a+b}, \qquad \gamma_2 = -\frac{ab}{a-b}, \qquad \gamma_3 = \frac{ab}{a+b}. \tag{4.4}$$

Since $\gamma_1 \in (-b, 0)$, $\gamma_3 \in (0, a)$ and $\gamma_2 \notin (-b, a)$, therefore conic $\mathcal{C}_{\gamma_2}$ is a hyperbola, whereas conics $\mathcal{C}_{\gamma_1}$ and $\mathcal{C}_{\gamma_3}$ are ellipses.

In Figs. 3 and 4, examples of 4-periodic trajectories with each type of caustic are shown.

Fig. 3. A 4-periodic trajectory with an ellipse along the y-axis as caustic ($a = 2$, $b = 4$, $\gamma = 4/3$) is shown on the left, while another trajectory with an ellipse along the x-axis as caustic ($a = 9$, $b = 3$, $\gamma = -9/4$) is on the right.
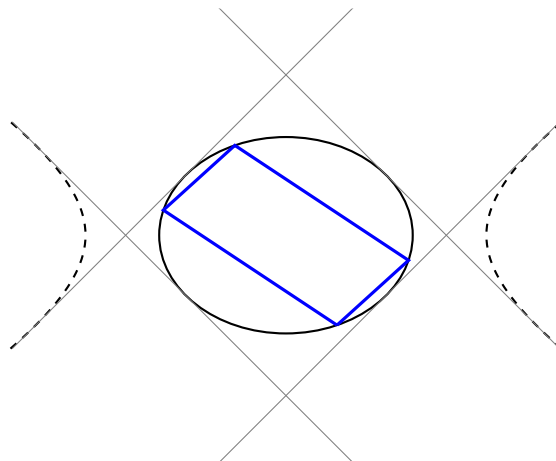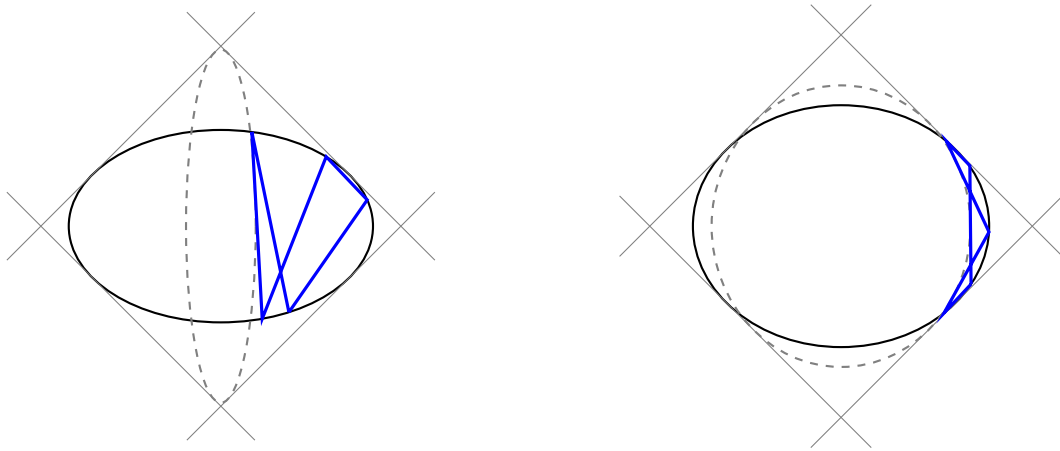

Fig. 4. A 4-periodic trajectory with a hyperbola along the x-axis as caustic ($a = 5$, $b = 3$, $\gamma = -15/2$).
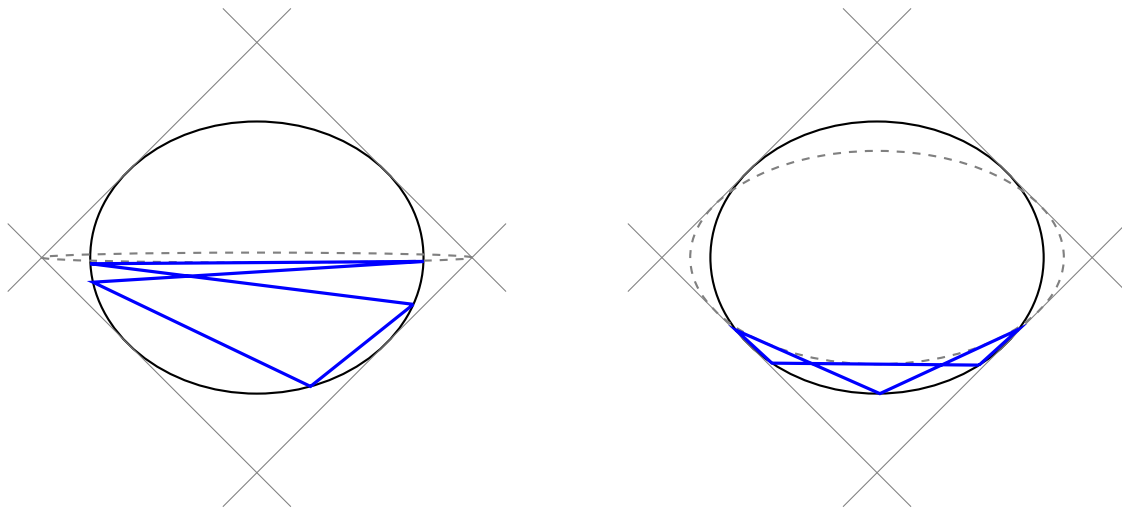
5-periodic trajectories

There is a 5-periodic trajectory of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if, according to Theorem 2, the caustic is an ellipse, i.e., $\gamma \in (-b, a)$, and $C_2 C_4 - C_3^2 = 0$, which is equivalent to

$$0 = (a+b)^6 \gamma^6 - 2ab(a-b)(a-3b)(3a-b)(a+b)^2 \gamma^5 - a^2 b^2 \left(29a^2 - 54ab + 29b^2\right)(a+b)^2 \gamma^4$$

$$- 36a^3 b^3 (a-b)(a+b)^2 \gamma^3 - a^4 b^4 \left(9a^2 + 34ab + 9b^2\right)\gamma^2 + 10a^5 b^5 (a-b)\gamma + 5a^6 b^6. \tag{4.5}$$

Examples of 5-periodic billiard trajectories are shown in Figs. 5 and 6.

Fig. 5. 5-periodic trajectories with an ellipse along the y-axis as caustic. On the left, the particle is bouncing 4 times off the relativistic ellipse and once off relativistic hyperbola ($a = 5$, $b = 2$, $\gamma \approx 4.7375$), while on the right the billiard particle is reflected twice off relativistic ellipse and 3 times off relativistic hyperbola ($a = 6$, $b = 4$, $\gamma \approx 1.4205$).

Fig. 6. 5-periodic trajectories with an ellipse along the x-axis as caustic. On the left, the particle is bouncing once off the relativistic ellipse and 4 times off the relativistic hyperbola ($a = 6$, $b = 4$, $\gamma \approx -3.9947$), while on the right the billiard particle is reflected twice off the relativistic hyperbola and 3 times off the relativistic ellipse ($a = 6$, $b = 4$, $\gamma \approx -1.5413$).

6-periodic trajectories

There is a 6-periodic trajectory of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if $B_3 B_5 - B_4^2 = 0$, which is equivalent to

$$
\begin{aligned}
0 = &\Big(-(a+b)^2\gamma^2 + 2ab(a-b)\gamma + 3a^2b^2\Big)\Big((a+b)(a-3b)\gamma^2 + 2ab(a+b)\gamma + a^2b^2\Big) \\
&\times \Big((a+b)^2\gamma^2 + 2ab(a-b)\gamma + a^2b^2\Big)\Big(-(a+b)(3a-b)\gamma^2 - 2ab(a+b)\gamma + a^2b^2\Big).
\end{aligned}
\tag{4.6}
$$

The first factor, $-(a+b)^2\gamma^2 + 2ab(a-b)\gamma + 3a^2b^2$, is a constant multiple of C_2 (see Eq. (4.1)), thus it produces 3-periodic trajectories, which have already been studied.

The discriminant of the third factor $(a+b)^2\gamma^2 + 2ab(a-b)\gamma + a^2b^2$ is $-16a^3b^3$, which is negative, therefore the expression has no real roots in γ.

Next, we consider the second factor: $(a+b)(a-3b)\gamma^2 + 2ab(a+b)\gamma + a^2b^2 = 0$, which has two real solutions:
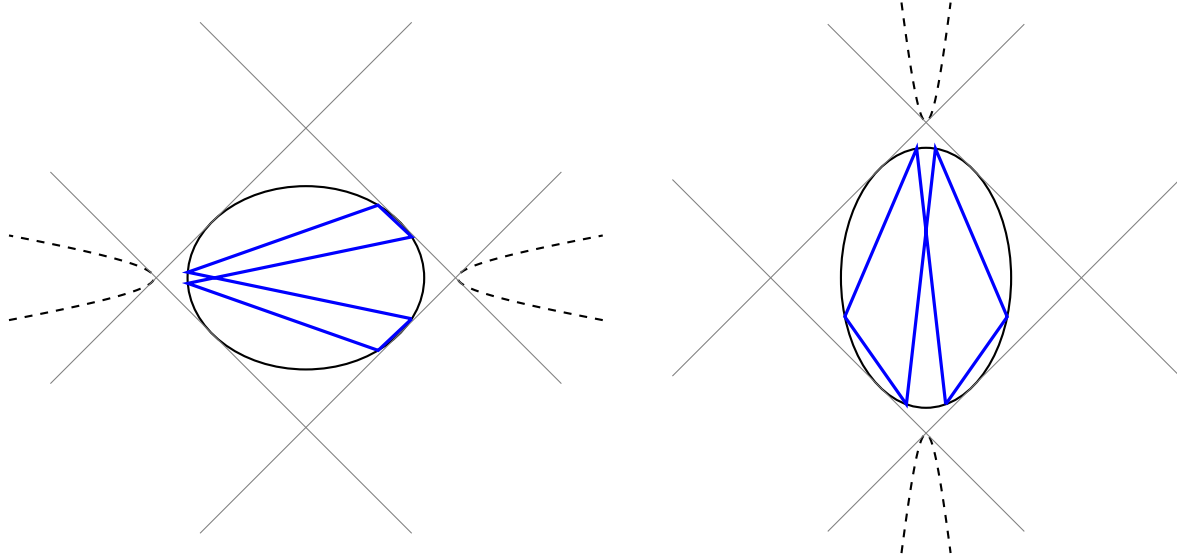
$$
\gamma = \frac{ab}{(a+b)(a-3b)}\Big(-a-b \pm 2\sqrt{ab+b^2}\Big).
$$

Finally, we consider the fourth factor: $-(a+b)(3a-b)\gamma^2 - 2ab(a+b)\gamma + a^2b^2 = 0$, which yields two real solutions:

$$\gamma = \frac{ab}{(a+b)(3a-b)}\left(-a-b \pm 2\sqrt{ab+a^2}\right).$$

Examples of 6-periodic trajectories with hyperbolas as caustics are shown in Fig. 7.

Fig. 7. A 6-periodic trajectory with a hyperbola along the x-axis as caustic ($a = 5$, $b = 3$, $\gamma \approx -3.2264$ is shown on the left, while another trajectory with a hyperbola along the y-axis as caustic ($a = 3$, $b = 7$ and $\gamma \approx 3.1189$) is on the right. On the left, the particle bounces off the relativistic ellipse twice and 4 times the relativistic hyperbola, while on the right the particle bounces off the relativistic ellipse 4 times and the relativistic hyperbola twice.
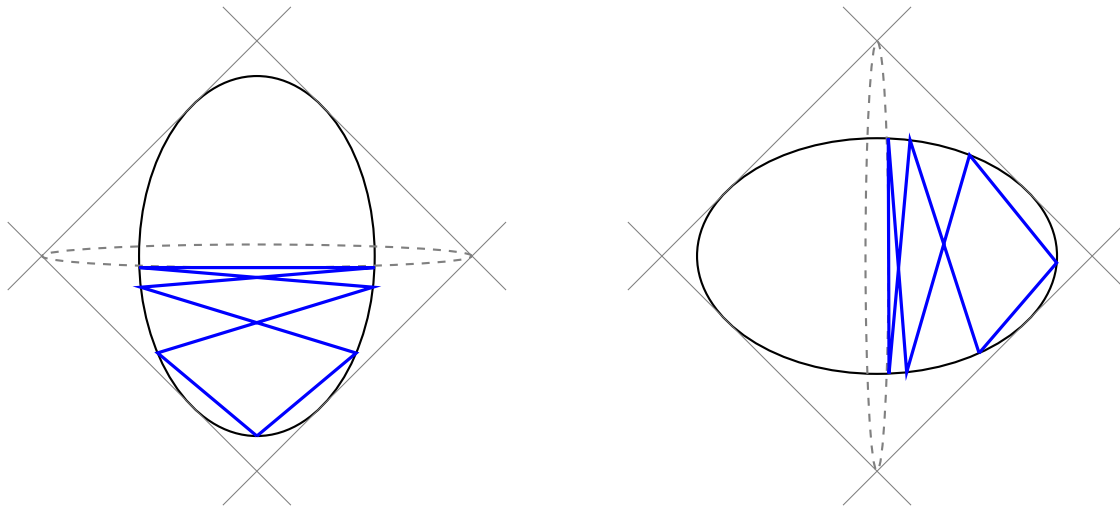
7-periodic trajectories

According to Theorem 2, there is a 7-periodic trajectory of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if the caustic is an ellipse, i.e., $\gamma \in (-b, a)$, and

$$\begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0,$$

which is equivalent to

$$\begin{aligned}
0 = & -(a+b)^{12}\gamma^{12} + 4ab(a-b)(a-3b)(3a-b)(a^2-6ab+b^2)(a+b)^6\gamma^{11} \\
& + 2a^2b^2(59a^4 - 332a^3b + 626a^2b^2 - 332ab^3 + 59b^4)(a+b)^6\gamma^{10} \\
& + 28a^3b^3(a-b)(13a^2 - 38ab + 13b^2)(a+b)^6\gamma^9 \\
& + a^4b^4(7a^2 + 30ab + 7b^2)(63a^4 - 84a^3b - 38a^2b^2 - 84ab^3 + 63b^4)(a+b)^2\gamma^8 \\
& - 8a^5b^5(a-b)(21a^4 - 420a^3b - 50a^2b^2 - 420ab^3 + 21b^4)(a+b)^2\gamma^7 \qquad (4.7)\\
& - 12a^6b^6(105a^4 - 420a^3b + 422a^2b^2 - 420ab^3 + 105b^4)(a+b)^2\gamma^6 \\
& - 24a^7b^7(a-b)(75a^2 - 106ab + 75b^2)(a+b)^2\gamma^5 \\
& - 3a^8b^8(437a^2 - 726ab + 437b^2)(a+b)^2\gamma^4 - 4a^9b^9(a-b)(121a^2 + 250ab + 121b^2)\gamma^3 \\
& - 14a^{10}b^{10}(3a^2 + 14ab + 3b^2)\gamma^2 + 28a^{11}b^{11}(a-b)\gamma + 7a^{12}b^{12}.
\end{aligned}$$

Examples of 7-periodic trajectories are shown in Fig. 8.

Fig. 8. A 7-periodic trajectory with an ellipse along the x-axis as caustic ($a = 3$, $b = 7$, $\gamma \approx -6.9712$) is shown on the left, while another trajectory with an ellipse along the y-axis as caustic ($a = 7$, $b = 3$ and $\gamma \approx 6.9712$) is on the right. On the left, the particle bounces once off the relativistic ellipse and 6 times off the relativistic hyperbola, while on the right the particle bounces 6 times off the relativistic ellipse and once off the relativistic hyperbola.

8-periodic trajectories

There is an 8-periodic trajectory of the billiard within ellipse (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if

$$\begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} = 0,$$

which is equivalent to

$$\begin{aligned}
0 = {}& (ab - a\gamma - b\gamma)(ab + a\gamma + b\gamma)(ab + a\gamma - b\gamma) \\
& \left((a+b)^4\gamma^4 - 4ab(a+b)(-b+a)^2\gamma^3 - 2a^2b^2(a+b)(5a - 3b)\gamma^2 - 4a^3b^3(a+b)\gamma \right. \\
& \left. + a^4b^4 \right) \left((a+b)^4\gamma^4 + 4ab(a+b)(-b+a)^2\gamma^3 + 2a^2b^2(a+b)(3a - 5b)\gamma^2 \right. \\
& \left. + 4a^3b^3(a+b)\gamma + a^4b^4 \right) \left((a^2 - 6ab + b^2)(a+b)^2\lambda^4 + 4ab(-b+a)(a+b)^2\gamma^3 \right. \\
& \left. + 2a^2b^2(3a^2 + 2ab + 3b^2)\gamma^2 + 4a^3b^3(-b+a)\gamma + a^4b^4 \right).
\end{aligned} \tag{4.8}$$

In Figs. 9 and 10, three examples of 8-periodic trajectories are shown.

Summary of numbers of touching points with relativistic ellipses and hyperbolas

In the table below, we summarize the examples given in this section. Here, n_1 and n_2 represent the numbers of bouncing points off the relativistic ellipses and the relativistic hyperbolas, respectively.
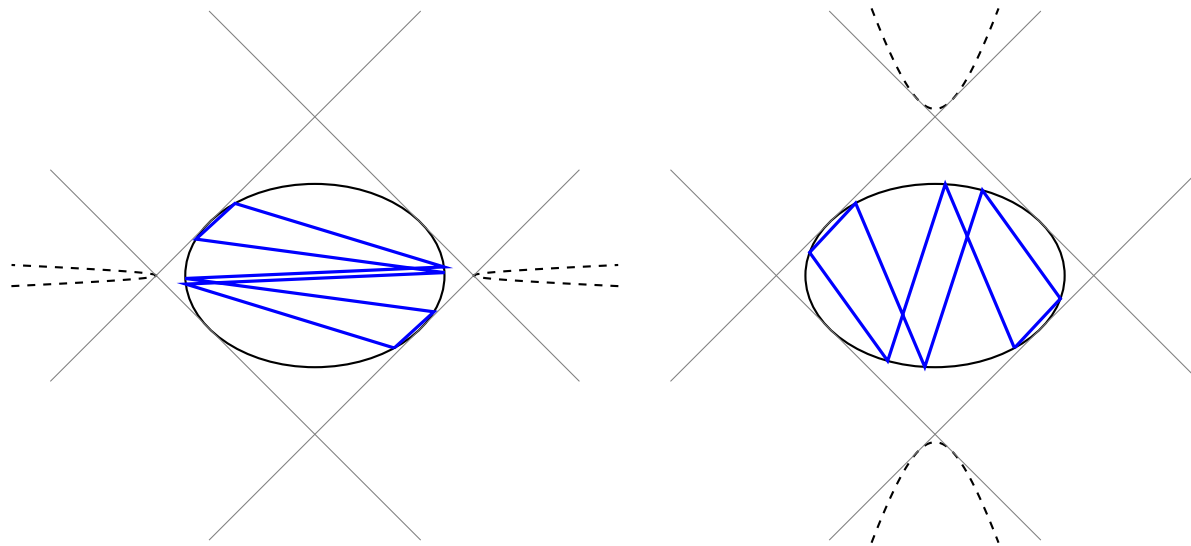
Fig. 9. On the left, an 8-periodic trajectory with a hyperbola along x-axis as caustic ($a = 6$, $b = 3$, $\gamma \approx -3.0151$), with 2 vertices on relativistic ellipses and 6 on relativistic hyperbolas. On the right, an 8-periodic trajectory with a hyperbola along y-axis as caustic ($a = 6$, $b = 3$, $\gamma \approx 6.9168$), with 6 vertices on relativistic ellipses and 2 on relativistic hyperbolas.
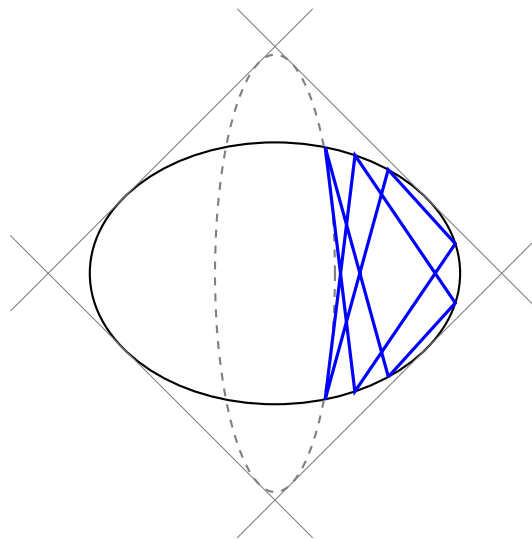
Fig. 10. An 8-periodic trajectory with an ellipse along y-axis as caustic ($a = 6$, $b = 3$, $\gamma \approx 5.3707$). There are 2 reflections off the relativistic hyperbola and 6 off the relativistic ellipses.

4.2. Cayley-type Conditions and Discriminantly Separable Polynomials

Similarly to the case of Euclidean plane [16], the Cayley-type conditions obtained above have a very interesting algebraic structure. Namely, the numerators of the corresponding expressions are polynomials in 3 variables. As examples below show, those polynomials have factorizable discriminants which, after a change of variables, lead to discriminantly separable polynomials in the sense of the following definition (see Remark 2 for more details).

Definition 1 ([8]). A polynomial $F(x_1, \ldots, x_n)$ is *discriminantly separable* if there exist polynomials $f_1(x_1), \ldots, f_n(x_n)$ such that the discriminant $\mathscr{D}_{x_i} F$ of F with respect to x_i satisfies

$$\mathscr{D}_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j),$$

for each $i = 1, \ldots, n$.

Period $n_1 + n_2$	Caustic	n_1	n_2
$n = 3$	Ellipse along y-axis	2	1
	Ellipse along x-axis	1	2
$n = 4$	Ellipse along x-axis	2	2
	Ellipse along y-axis	2	2
	Hyperbola along x-axis	2	2
$n = 5$	Ellipse along y-axis	2	3
	Ellipse along x-axis	3	2
	Ellipse along y-axis	4	1
	Ellipse along x-axis	1	4
$n = 6$	Hyperbola along x-axis	2	4
	Ellipse along y-axis	4	2
$n = 7$	Ellipse along x-axis	1	6
	Ellipse along y-axis	6	1
$n = 8$	Hyperbola along x-axis	2	6
	Hyperbola along y-axis	6	2
	Ellipse along x-axis	6	2

Discriminantly factorizable polynomials were introduced in [9] in connection with n-valued groups. Various applications of discriminantly separable polynomials in continuous and discrete integrable systems were presented in [10–12]. The connection between Cayley-type conditions in the Euclidean setting and discriminantly factorizable and separable polynomials has been observed in [16]. As examples below show, the Cayley conditions in the Minkowski plane provide examples of discriminantly factorizable polynomials which, after a change of variables, have separable discriminants. It would be interesting to establish this relationship as a general statement.

Example 1. The expression (4.1) is

$$\mathsf{G}_2(\gamma, a, b) = -\left(a + b\right)^2 \gamma^2 + 2\, ab\, (a - b)\, \gamma + 3\, a^2 b^2,$$

and its discriminant with respect to γ is

$$\mathscr{D}_\gamma \mathsf{G}_2 = 2^4 \left(a^2 + ab + b^2\right) a^2 b^2,$$

which is obviously factorizable.

Example 2. The expression (4.3) is

$$\mathsf{G}_3(\gamma, a, b) = -(ab + a\gamma + b\gamma)(ab + a\gamma - b\gamma)(ab - a\gamma - b\gamma)$$

and its discriminant with respect to γ is factored as

$$\mathscr{D}_\gamma \mathsf{G}_3 = 2^6 a^8 b^8 \left(a + b\right)^2.$$

Example 3. The expression (4.5) is

$$\begin{aligned}
\mathsf{G}_6(\gamma, a, b) = &\ (a + b)^6 \gamma^6 - 2ab\, (a - b)\, (a - 3b)\, (3a - b)\, (a + b)^2 \gamma^5 \\
&- a^2 b^2 \left(29a^2 - 54ab + 29b^2\right)(a + b)^2 \gamma^4 - 36a^3 b^3 (a - b)\, (a + b)^2 \gamma^3 \\
&- a^4 b^4 \left(9a^2 + 34ab + 9b^2\right) \gamma^2 + 10a^5 b^5 (a - b)\, \gamma + 5a^6 b^6.
\end{aligned}$$

is discriminantly factorizable since its discriminant with respect to γ is

$$\mathscr{D}_\gamma \mathsf{G}_6 = -5 \cdot 2^{44}(27a^6 + 81a^5 b + 322\, a^4 b^2 + 509a^3 b^3 + 322a^2 b^4 + 81ab^5 + 27b^6)\, (a + b)^8\, b^{38} a^{38}.$$

Example 4. Let us denote the expression (4.6) as

$$\mathsf{G}_8(\gamma, a, b) = (3a - b)(a - 3b)(a + b)^6\gamma^8 + 8ab(a - b)(a + b)^6\gamma^7$$
$$- 4\, a^2 b^2 (3\, a^4 - 24\, a^3 b + 10\, a^2 b^2 - 24\, ab^3 + 3\, b^4)\, (a + b)^2\, \gamma^6$$
$$- 8\, a^3 b^3\, (a - b)\, \left(9\, a^2 - 14\, ab + 9\, b^2\right)\, (a + b)^2\, \gamma^5$$
$$- 10\, a^4 b^4\, \left(11\, a^2 - 18\, ab + 11\, b^2\right)\, (a + b)^2\, \gamma^4 - 72\, a^5 b^5\, (a - b)\, (a + b)^2\, \gamma^3$$
$$- 4\, a^6 b^6\, (a + 3\, b)\, (3\, a + b)\, \gamma^2 + 8\, a^7 b^7 (a - b)\gamma + 3\, a^8 b^8.$$

We find that the discriminant of G_8 with respect to γ factors is

$$\mathscr{D}_\gamma \mathsf{G}_8 = -2^{88}\, \left(a^2 + ab + b^2\right) (a + b)^{18}\, b^{74} a^{74}.$$

Example 5. The discriminant $\mathscr{D}_\gamma \mathsf{G}_{12}$ of the expression in (4.7) is

$$\mathscr{D}_\gamma \mathsf{G}_{12} = -\, 2^{184} \cdot 7^2\, (a + b)^{40}\, (ab)^{172} \times$$
$$\times\, (84375 a^{12} + 506250 a^{11} b + 4266243\, a^{10} b^2 + 16690590\, a^9 b^3 + 34989622\, a^8 b^4$$
$$+\, 45383698\, a^7 b^5 + 46564971\, a^6 b^6 + 45383698\, a^5 b^7 + 34989622\, a^4 b^8 + 16690590\, a^3 b^9$$
$$+\, 4266243\, a^2 b^{10} + 506250\, ab^{11} + 84375\, b^{12}),$$

thus, G_{12} is a discriminantly factorizable polynomial.

Example 6. The discriminant $\mathscr{D}_\gamma \mathsf{G}_{15}$ of the expression (4.8) is

$$\mathscr{D}_\gamma \mathsf{G}_{15} = -\, 2^{246}(ab)^{278}\, \left(27\, a^2 + 46\, ab + 27\, b^2\right) (a + b)^8\, \times$$
$$\times\, \left(a^5 + 5\, a^4 b + 10\, a^3 b^2 + 10\, a^2 b^3 + 5\, ab^4 + b^5\right) \times$$
$$\times\, (a^7 + 7\, a^6 b + 21\, a^5 b^2 + 35\, a^4 b^3 + 35\, a^3 b^4 + 21\, a^2 b^5 + 7\, ab^6 + b^7) \times$$
$$\times\, (8\, a^{26} + 27\, b^{26} + 200\, a^{25} b + 2427\, a^{24} b^2 + 19048\, a^{23} b^3 + 108652\, a^{22} b^4 + 479688\, a^{21} b^5$$
$$+\, 1703702\, a^{20} b^6 + 4993208\, a^{19} b^7 + 12286692\, a^{18} b^8 + 25688608\, a^{17} b^9 + 46007797\, a^{16} b^{10}$$
$$+\, 70961808\, a^{15} b^{11} + 94556312\, a^{14} b^{12} + 108998288\, a^{13} b^{13} + 108671412\, a^{12} b^{14}$$
$$+\, 93545968\, a^{11} b^{15} + 69297712\, a^{10} b^{16} + 43955208\, a^9 b^{17} + 23703317\, a^8 b^{18} + 10761608\, a^7 b^{19}$$
$$+\, 4059132\, a^6 b^{20} + 1248808\, a^5 b^{21} + 305302\, a^4 b^{22} + 57048\, a^3 b^{23} + 7652\, a^2 b^{24} + 656\, ab^{25})$$
$$\times\, (27\, a^{26} + 8\, b^{26} + 656\, a^{25} b + 7652\, a^{24} b^2 + 57048\, a^{23} b^3 + 305302\, a^{22} b^4 + 1248808\, a^{21} b^5$$
$$+\, 4059132\, a^{20} b^6 + 10761608\, a^{19} b^7 + 23703317\, a^{18} b^8 + 43955208\, a^{17} b^9 + 69297712\, a^{16} b^{10}$$
$$+\, 93545968\, a^{15} b^{11} + 108671412\, a^{14} b^{12} + 108998288\, a^{13} b^{13} + 94556312\, a^{12} b^{14}$$
$$+\, 70961808\, a^{11} b^{15} + 46007797\, a^{10} b^{16} + 25688608\, a^9 b^{17} + 12286692\, a^8 b^{18} + 4993208\, a^7 b^{19}$$
$$+\, 1703702\, a^6 b^{20} + 479688\, a^5 b^{21} + 108652\, a^4 b^{22} + 19048\, a^3 b^{23} + 2427\, a^2 b^{24} + 200\, ab^{25}),$$

so G_{15} is a discriminantly factorizable polynomial.

Remark 1. Since the determinants obtained in Theorem 2 are symmetric in a, $-b$, and γ, the discriminants with respect to a and b of the polynomials in Examples 1–6 will be also factorizable.

Remark 2. We observed in Examples 1–6 that all polynomials are discriminantly factorizable. However, it is important to note that their factors are homogeneous, thus, by a change of variables, say, $(a, b) \mapsto (a, \hat{b})$, with $\hat{b} = \dfrac{b}{a}$, transforms the polynomials into discriminantly separable polynomials in new variables $(a, \hat{b})$:

$$\mathscr{D}_\gamma \mathsf{G}_2 = 2^4\, a^8 \hat{b}^2 \left(1 + \hat{b} + \hat{b}^2\right),$$
$$\mathscr{D}_\gamma \mathsf{G}_3 = 2^6\, a^{18} \hat{b}^8 \left(1 + \hat{b}\right)^2,$$

$$\mathscr{D}_\gamma \mathsf{G}_6 = -5.2^{44} a^{90} \hat{b}^{38} \left(27 + 81\hat{b} + 322\hat{b}^2 + 509\hat{b}^3 + 322\hat{b}^4 + 81\hat{b}^5 + 27\hat{b}^6 \right) \left(1 + \hat{b} \right)^8,$$

$$\mathscr{D}_\gamma \mathsf{G}_8 = -2^{88} a^{168} \hat{b}^{74} \left(1 + \hat{b} + \hat{b}^2 \right) \left(1 + \hat{b} \right)^{18}.$$

5. ELLIPTIC PERIODIC TRAJECTORIES

Points of the plane which are symmetric with respect to the coordinate axes share the same elliptic coordinates, and hence there is no bijection between the elliptic and the Cartesian coordinates. Thus, we introduce a separate notion of periodicity in elliptic coordinates.

Definition 2. A billiard trajectory is *n-elliptic periodic* if it is n-periodic in elliptic coordinates joined to the confocal family (2.2).

Now we will derive algebro-geometric conditions for elliptic periodic trajectories.

Theorem 3. *A billiard trajectory within $\mathcal{E}$ with the caustic $\mathcal{C}_\gamma$ is n-elliptic periodic without being n-periodic if and only if one of the following conditions is satisfied on $\mathscr{C}$:*

(a) $\mathcal{C}_\gamma$ is an ellipse, $0 < \gamma < a$, and $nQ_0 - (n-1)Q_\gamma - Q_{-b} \sim 0$;

(b) $\mathcal{C}_\gamma$ is an ellipse, $-b < \gamma < 0$, and $nQ_0 - (n-1)Q_\gamma - Q_a \sim 0$;

(c) $\mathcal{C}_\gamma$ is a hyperbola, n is even and $nQ_0 - (n-2)Q_\gamma - Q_{-b} - Q_a \sim 0$;

(d) $\mathcal{C}_\gamma$ is a hyperbola, n is odd, and $nQ_0 - (n-1)Q_\gamma - Q_a \sim 0$;

(e) $\mathcal{C}_\gamma$ is a hyperbola, n is odd, and $nQ_0 - (n-1)Q_\gamma - Q_{-b} \sim 0$.

Moreover, such trajectories are always symmetric with respect to the origin in Case (c). They are symmetric with respect to the x-axis in Cases (b) and (d), and with respect to the y-axis in Cases (a) and (e).

Proof. Let M_0 be the initial point of a given n-elliptic periodic trajectory, and M_1 the next point on the trajectory with the same elliptic coordinates. Then, integrating (2.3) M_0 to M_1 along the trajectory, we get

$$n_1(Q_0 - Q_{\alpha_1}) + n_2(Q_0 - Q_{\beta_1}) \sim 0,$$

where $n = n_1 + n_2$, and n_1 is the number of times that the particle hits the arcs of $\mathcal{E}$ with time-like metrics, and n_2 the number of times it hits the arcs with space-like metrics. We denoted by α_1 the largest negative member of the set $\{a, -b, \gamma\}$, and by β_1 its smallest positive member.

The trajectory is not n-periodic if and only if at least one of n_1, n_2 is odd, which then leads to the stated conclusions. $\square$

The explicit Cayley-type conditions for elliptic periodic trajectories are:

Theorem 4. *A billiard trajectory within $\mathcal{E}$ with the caustic $\mathcal{Q}_\gamma$ is n-elliptic periodic without being n-periodic if and only if one of the following conditions is satisfied:*

(a) $\mathcal{C}_\gamma$ is an ellipse, $0 < \gamma < a$, and

$$D_1 = 0, \quad \begin{vmatrix} D_1 & D_2 \\ D_2 & D_3 \end{vmatrix} = 0, \quad \begin{vmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \end{vmatrix} = 0, \dots \quad for \quad n = 2, 4, 6, \dots$$

$$E_2 = 0, \quad \begin{vmatrix} E_2 & E_3 \\ E_3 & E_4 \end{vmatrix} = 0, \quad \begin{vmatrix} E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \\ E_4 & E_5 & E_6 \end{vmatrix} = 0, \dots \quad for \quad n = 3, 5, 7, \dots;$$

(b) $\mathcal{C}_\gamma$ is an ellipse, $-b < \gamma < 0$, and

$$E_1 = 0, \quad \begin{vmatrix} E_1 & E_2 \\ E_2 & E_3 \end{vmatrix} = 0, \quad \begin{vmatrix} E_1 & E_2 & E_3 \\ E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \end{vmatrix} = 0, \ldots \quad for \quad n = 2, 4, 6, \ldots$$

$$D_2 = 0, \quad \begin{vmatrix} D_2 & D_3 \\ D_3 & D_4 \end{vmatrix} = 0, \quad \begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} = 0, \ldots \quad for \quad n = 3, 5, 7, \ldots;$$

(c) $\mathcal{Q}_\gamma$ is a hyperbola, n even and

$$C_1 = 0, \quad \begin{vmatrix} C_1 & C_2 \\ C_2 & C_3 \end{vmatrix} = 0, \quad \begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{vmatrix} = 0, \ldots \quad for \quad n = 2, 4, 6, \ldots$$

(d) $\mathcal{Q}_\gamma$ is a hyperbola, n is odd, and

$$D_2 = 0, \quad \begin{vmatrix} D_2 & D_3 \\ D_3 & D_4 \end{vmatrix} = 0, \quad \begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} = 0, \ldots \quad for \quad n = 3, 5, 7, \ldots.$$

(e) $\mathcal{Q}_\gamma$ is a hyperbola, n is odd, and

$$E_2 = 0, \quad \begin{vmatrix} E_2 & E_3 \\ E_3 & E_4 \end{vmatrix} = 0, \quad \begin{vmatrix} E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \\ E_4 & E_5 & E_6 \end{vmatrix} = 0, \ldots \quad for \quad n = 3, 5, 7, \ldots.$$

Here, we have denoted:

$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{a-x} = D_0 + D_1 x + D_2 x^2 + \ldots,$$

$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{b+x} = E_0 + E_1 x + E_2 x^2 + \ldots,$$

the Taylor expansion around $x = 0$, while Bs and Cs are as in Theorem 2.

Proof. (a) Take first n even. Using Theorem 3, we have

$$nQ_0 \sim (n-1)Q_\gamma + Q_{-b} \sim (n-2)Q_\infty + Q_{-b} + Q_\gamma \sim (n-2)Q_\infty + Q_\infty + Q_a \sim (n-1)Q_\infty + Q_a.$$

The basis of $\mathscr{L}\big((n-1)Q_\infty + Q_a\big)$ is

$$1, x, x^2, \ldots, x^{n/2-1}, \frac{y}{x-a}, \frac{xy}{x-a}, \frac{x^{n/2-1}y}{x-a},$$

thus, a nontrivial linear combination of these functions with a zero of order n at $x = 0$ exists if and only if

$$\begin{vmatrix} D_{n/2} & D_{n/2-1} & \ldots & D_1 \\ D_{n/2+1} & D_{n/2} & \ldots & D_2 \\ \ldots & & & \\ D_{n-1} & D_{n-2} & \ldots & D_{n/2} \end{vmatrix} = 0.$$

For odd n, we have

$$nQ_0 \sim (n-1)Q_\gamma + Q_{-b} \sim (n-1)Q_\infty + Q_{-b}.$$

The basis of $\mathscr{L}\big((n-1)Q_\infty + Q_{-b}\big)$ is

$$1, x, x^2, \ldots, x^{(n-1)/2}, \frac{y}{x+b}, \frac{xy}{x+b}, \frac{x^{(n-1)/2-1}y}{x+b},$$

thus, a nontrivial linear combination of these functions with a zero of order n at $x = 0$ exists if and only if

$$\begin{vmatrix} E_{(n-1)/2+1} & E_{(n-1)/2} & \cdots & E_2 \\ E_{(n-1)/2+2} & E_{(n-1)/2+1} & \cdots & E_3 \\ \cdots & & & \\ E_{n-1} & E_{n-2} & \cdots & E_{(n-1)/2+1} \end{vmatrix} = 0.$$

Case (b) is done similarly as (a).

(c) We have $nQ_0 \sim (n-2)Q_\gamma + Q_{-b} + Q_a \sim (n-1)Q_\infty + Q_\gamma$.

(d) We have $nQ_0 \sim (n-1)Q_\gamma + Q_a \sim (n-1)Q_\infty + Q_a$.

(e) We have $nQ_0 \sim (n-1)Q_\gamma + Q_{-b} \sim (n-1)Q_\infty + Q_{-b}$. $\qquad\square$

6. EXAMPLES OF ELLIPTIC PERIODIC TRAJECTORIES: $2 \leqslant n \leqslant 5$

2-elliptic periodic trajectories

There is a 2-elliptic periodic trajectory without being 2-periodic of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if, according to Theorem 4 one of the following is satisfied:

- the caustic is an ellipse, with $\gamma \in (0, a)$ and $D_1 = 0$;

- the caustic is an ellipse, with $\gamma \in (-b, 0)$ and $E_1 = 0$;

- the caustic is a hyperbola, n is even, and $C_1 = 0$.
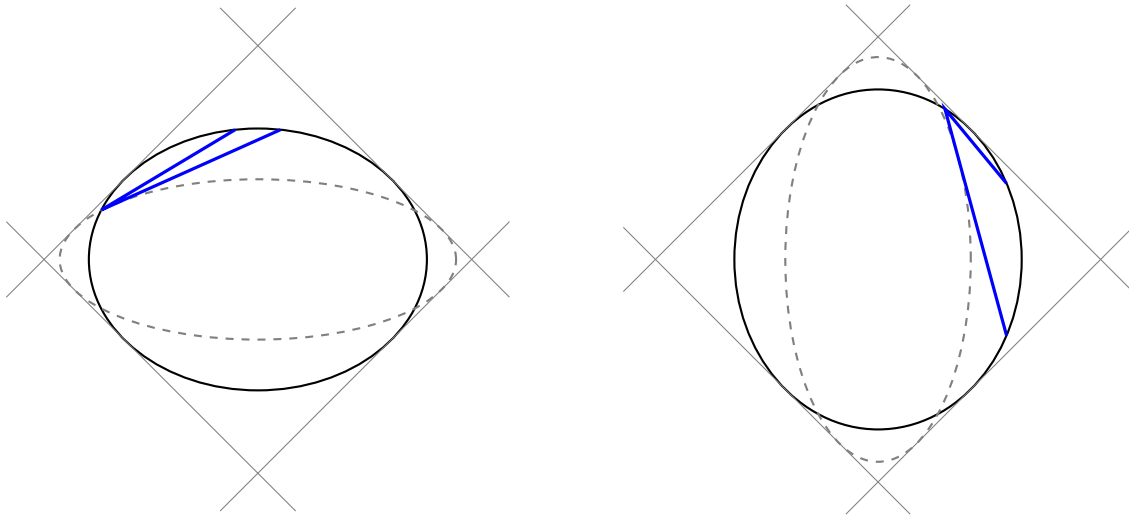
We consider the following equations:

$$D_1 = \frac{(a+b)\gamma - ab}{2\sqrt{a^3 b\gamma}} = 0, \quad E_1 = -\frac{(a+b)\gamma + ab}{2\sqrt{ab^3\gamma}} = 0, \quad C_1 = \frac{(a-b)\gamma + ab}{2\sqrt{ab\gamma^3}} = 0,$$

which, respectively, yield the solutions for the parameter γ of the caustic:

$$\gamma = \frac{ab}{a+b}, \quad \gamma = -\frac{ab}{a+b}, \quad \gamma = -\frac{ab}{a-b}.$$

Some examples of 2-elliptic periodic trajectories without being 2-periodic are shown in Figs. 11 and 12.

Fig. 11. A 2-elliptic periodic trajectories with ellipses as caustics. On the left, the caustic is an ellipse along x-axis ($a = 5$, $b = 3$, $\gamma = -15/8$), and on the right an ellipse along y-axis ($a = 5$, $b = 7$ and $\gamma = 35/12$).
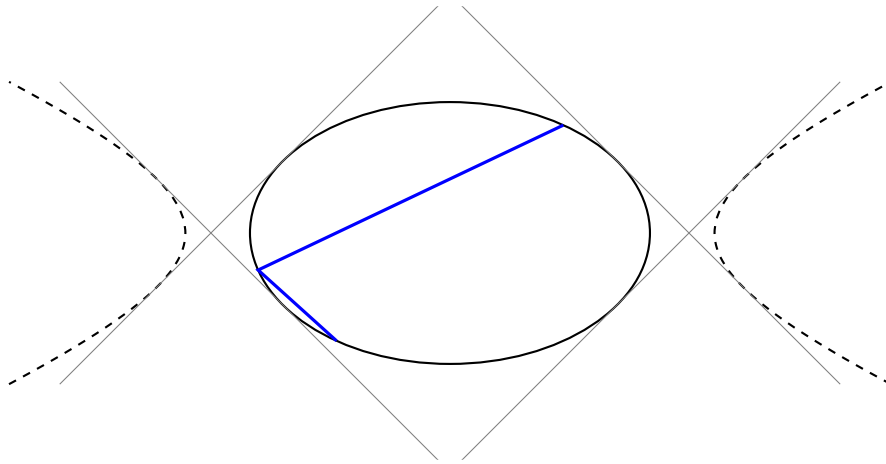
Fig. 12. A 2-elliptic periodic trajectory with a hyperbola as caustic ($a = 7$, $b = 3$, $\gamma = -5.25$).

3-elliptic periodic trajectories

There is a 3-elliptic periodic trajectory without being 3-periodic of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if one of the following is satisfied:

- $E_2 = 0$ and either the caustic is an ellipse with $\gamma \in (0, a)$ or the caustic is a hyperbola with n even;

- $D_2 = 0$ and either the caustic is an ellipse with $\gamma \in (-b, 0)$ or the caustic is a hyperbola.

The equations $E_2 = 0$ and $D_2 = 0$ are, respectively, equivalent to
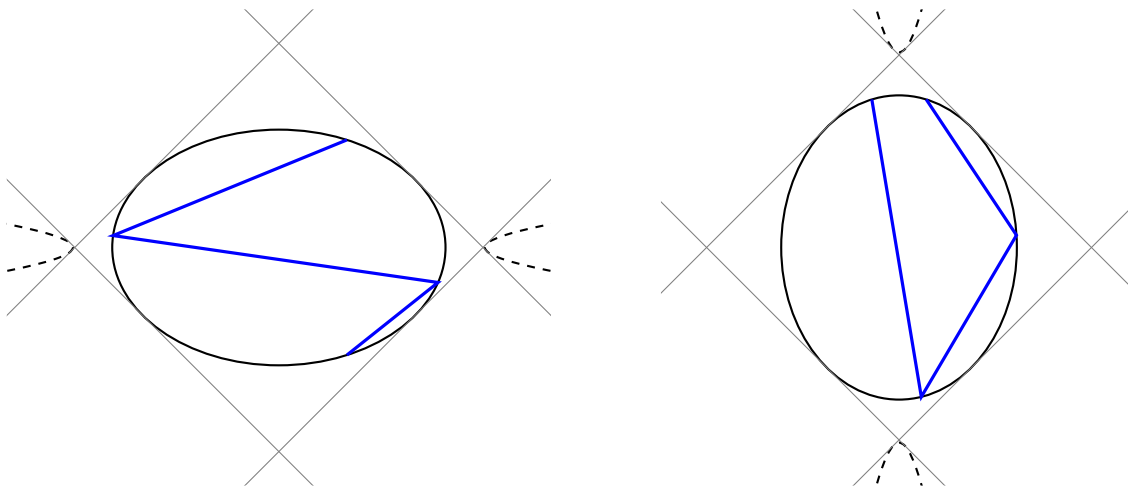
$$-(a+b)(3a-b)\gamma^2 - 2ab(a+b)\gamma + a^2b^2 = 0, \tag{6.1}$$

$$(a+b)(a-3b)\gamma^2 + 2ab(a+b)\gamma + a^2b^2 = 0, \tag{6.2}$$

which, respectively, yield the pairs of solutions for the parameter γ of the caustic:

$$\gamma = \frac{(-a-b \pm 2\sqrt{a^2+ab})ba}{(a+b)(3a-b)}, \quad \gamma = \frac{(-a-b \pm 2\sqrt{b^2+ab})ba}{(a+b)(a-3b)}.$$
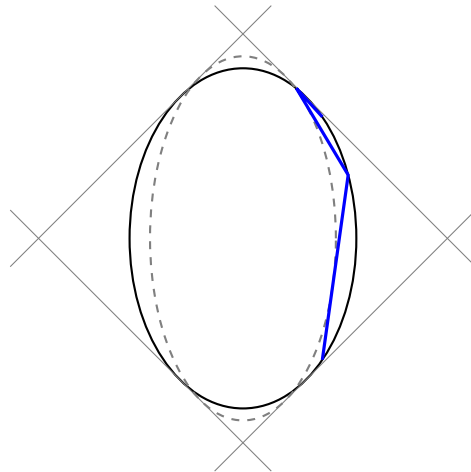
Examples of 3-elliptic periodic trajectories which are not 3-periodic are shown in Figs. 13–15.

Fig. 13. 3-elliptic periodic trajectries with hyperbolas as caustics. On the left, the caustic is orientied along the x-axis ($a = 6$, $b = 3$, $\gamma \approx -3.1595918$), and on the right along y-axis ($a = 3$, $b = 5$, $\gamma \approx 3.2264236$).
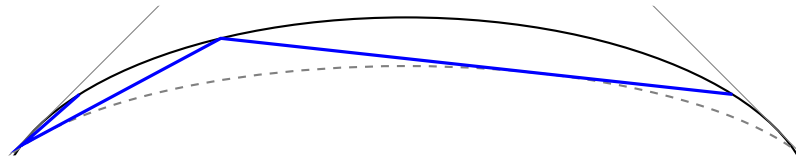
Fig. 14. A 3-elliptic periodic trajectory with an ellipse along x-axis as caustic ($a = 9$, $b = 2$, $\gamma \approx -0.8831827$).

Fig. 15. A 3-elliptic periodic trajectory with an ellipse along y-axis as caustic ($a = 4$, $b = 9$, $\gamma \approx 1.312805$).

4-elliptic periodic trajectories

There is a 4-elliptic periodic trajectory without being 4-periodic of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if, according to Theorem 4, one of the following is satisfied:

- the caustic is an ellipse, with $\gamma \in (0, a)$ and $D_3 D_1 - D_2^2 = 0$, i.e.,

$$(a + b)^4 \gamma^4 - 4ab(a + b)(a - b)^2 \gamma^3 - 2a^2 b^2 (a + b)(5a - 3b)\gamma^2 - 4a^3 b^3 (a + b)\gamma + a^4 b^4 = 0;$$

- the caustic is an ellipse, with $\gamma \in (-b, 0)$ and $E_3 E_1 - E_2^2 = 0$, i.e.,

$$(a + b)^4 \gamma^4 + 4ab(a + b)(a - b)^2 \gamma^3 + 2a^2 b^2 (a + b)(3a - 5b)\gamma^2 + 4a^3 b^3 (a + b)\gamma + a^4 b^4 = 0;$$

- the caustic is a hyperbola and $C_3C_1 - C_2^2 = 0$, i.e.,

$$(a^2 - 6ab + b^2)(a+b)^2\gamma^4 + 4ab(a-b)(a+b)^2\gamma^3 + 2a^2b^2(3a^2 + 2ab + 3b^2)\gamma^2$$
$$+ 4a^3b^3(a-b)\gamma + a^4b^4 = 0.$$

Each real solution γ for the above equations for some fixed values of a and b will produce a 4-elliptic periodic trajectory which is not 4-periodic. Some examples are shown in Fig. 16.
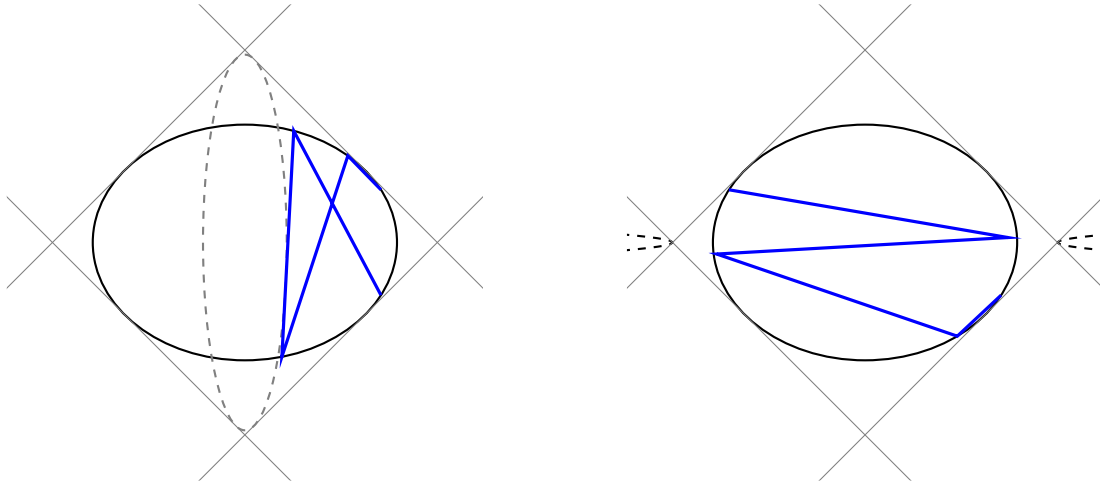
Fig. 16. 4-elliptic periodic trajectories. On the left, the caustic is an ellipse ($a = 5$, $b = 3$, $\gamma \approx 4.6216$), and it is a hyperbola on the right ($a = 5$, $b = 3$, $\gamma \approx -3.0243$).

5-elliptic periodic trajectories

According to Theorem 4, there is a 5-elliptic periodic trajectory without being 5-periodic of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma \in (0, a)$ or a hyperbola and $E_2E_4 - E_3^2 = 0$, i.e.,

$$(5a^2 - 10ab + b^2)(a+b)^4\gamma^6 + 2ab(5a - 3b)(a+b)^4\gamma^5$$
$$- a^2b^2(a+b)(9a^3 - 45a^2b - 5ab^2 - 15b^3)\gamma^4 - 4a^3b^3(a+b)(9a^2 - 10ab + 5b^2)\gamma^3$$
$$- a^4b^4(a+b)(29a - 15b)\gamma^2 - 6a^5b^5(a+b)\gamma + a^6b^6 = 0;$$

- the caustic is an ellipse, with $\gamma \in (-b, 0)$ or a hyperbola and $D_2D_4 - D_3^2 = 0$, i.e.,

$$(a^2 - 10ab + 5b^2)(a+b)^4\gamma^6 + 2ab(3a - 5b)(a+b)^4\gamma^5$$
$$+ a^2b^2(a+b)(15a^3 + 5a^2b + 45ab^2 - 9b^3)\gamma^4 + 4a^3b^3(a+b)(5a^2 - 10ab + 9b^2)\gamma^3$$
$$+ a^4b^4(a+b)(15a - 29b)\gamma^2 + 6a^5b^5(a+b)\gamma + a^6b^6 = 0.$$

Each real solution γ for the above equations for some fixed values of a and b will produce a 5-elliptic periodic trajectory which is not 5-periodic. Some examples are shown in Fig. 17.
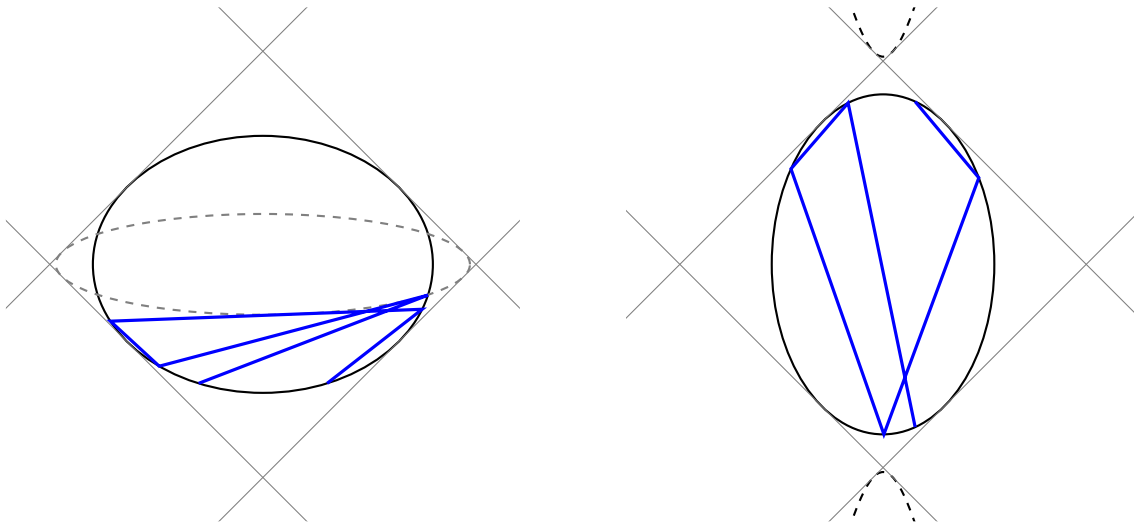
Discriminantly separable polynomials and elliptic periodicity

Since the case $n = 2$ is trivial, we start with the case $n = 3$.

From (6.1) and (6.2), we have

$$\mathsf{G}_1(a, b, \gamma) = -(a+b)(3a - b)\gamma^2 - 2ab(a+b)\gamma + a^2b^2,$$
$$\mathsf{G}_2(a, b, \gamma) = (a+b)(a - 3b)\gamma^2 + 2ab(a+b)\gamma + a^2b^2,$$

Fig. 17. 5-elliptic periodic trajectories. On the left, the caustic is an ellipse ($a = 7$, $b = 4$, $\gamma \approx -3.3848$) and a hyperbola on the right ($a = 3$, $b = 7$, $\gamma \approx 3.4462$).

and we calculate the discriminants, which factorize as follows:

$$\mathscr{D}_\gamma \mathsf{G}_1 = 16a^3b^2(a+b), \quad \mathscr{D}_\gamma \mathsf{G}_2 = 16b^3a^2(a+b).$$

Similarly, for $n = 4$, we have

$$\mathsf{G}_3(a,b,\gamma) = (a+b)^4\gamma^4 - 4ab(a+b)(a-b)^2\gamma^3 - 2a^2b^2(a+b)(5a-3b)\gamma^2 - 4a^3b^3(a+b)\gamma + a^4b^4,$$

$$\mathsf{G}_4(a,b,\gamma) = (a+b)^4\gamma^4 + 4ab(a+b)(a-b)^2\gamma^3 + 2a^2b^2(a+b)(3a-5b)\gamma^2 + 4a^3b^3(a+b)\gamma + a^4b^4,$$

$$\mathsf{G}_5(a,b,\gamma) = (a^2 - 6ab + b^2)(a+b)^2\gamma^4 + 4ab(a-b)(a+b)^2\gamma^3 + 2a^2b^2(3a^2 + 2ab + 3b^2)\gamma^2$$
$$+ 4a^3b^3(a-b)\gamma + a^4b^4.$$

The discriminants of these polynomials factorize as follows:

$$\mathscr{D}_\gamma \mathsf{G}_3 = -2^{16}a^{16}b^{14}(8a^2 + 8ab + 27b^2)(a+b)^4,$$

$$\mathscr{D}_\gamma \mathsf{G}_4 = -2^{16}a^{14}b^{16}(27a^2 + 8ab + 8b^2)(a+b)^4,$$

$$\mathscr{D}_\gamma \mathsf{G}_5 = 2^{12}(32a^6 - 491a^5b - 439a^4b^2 + 194a^3b^3 - 62a^2b^4 - 39ab^5 + 5b^6)(a+b)^3b^{15}a^{12}.$$

Using the transformation $(a,b) \mapsto (a,\hat{b})$, where $\hat{b} = \dfrac{b}{a}$, we get

$$\mathscr{D}_\gamma \mathsf{G}_1 = 16a^6\hat{b}^2(1+\hat{b}),$$

$$\mathscr{D}_\gamma \mathsf{G}_2 = 16a^5\hat{b}^3(1+\hat{b}),$$

$$\mathscr{D}_\gamma \mathsf{G}_3 = -2^{16}a^{36}\hat{b}^{14}(8 + 8\hat{b} + 27\hat{b}^2)(1+\hat{b})^4,$$

$$\mathscr{D}_\gamma \mathsf{G}_4 = -2^{16}a^{36}\hat{b}^{16}(27 + 8\hat{b} + 8\hat{b}^2)(1+\hat{b})^4,$$

$$\mathscr{D}_\gamma \mathsf{G}_5 = 2^{12}a^{36}\hat{b}^{15}(32 - 491\hat{b} - 439\hat{b}^2 + 194\hat{b}^3 - 62\hat{b}^4 - 39\hat{b}^5 + 5\hat{b}^6)(1+\hat{b})^3.$$

7. POLYNOMIAL EQUATIONS

Now we want to express the periodicity conditions for billiard trajectories in the Minkowski plane in terms of polynomial functions equations.

Theorem 5. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2, respectively, and satisfying the following:*

(a) *if* $n = 2m$ *is even, then* $d_1 = m$, $d_2 = m - 2$, *and*

$$p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-2}^2(s) = 1;$$

(b) *if* $n = 2m + 1$ *is odd, then* $d_1 = m$, $d_2 = m - 1$, *and*

$$\left(s - \frac{1}{\gamma}\right)p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)q_{m-1}^2(s) = -\operatorname{sign}\gamma.$$

Proof. We note first that the proof of Theorem 2 implies that there is a nontrivial linear combination of the bases (3.4) for n even, or (3.5) for n odd, with the zero of order n at $x = 0$.

(a) For $n = 2m$, from there we get that there are real polynomials $p_m^*(x)$ and $q_{m-2}^*(x)$ of degrees m and $m - 2$, respectively, such that the expression

$$p_m^*(x) - q_{m-2}^*(x)\sqrt{\varepsilon(a - x)(b + x)(\gamma - x)}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$p_m^*(x) + q_{m-2}^*(x)\sqrt{\varepsilon(a - x)(b + x)(\gamma - x)},$$

we get that the polynomial $\left(p_m^*(x)\right)^2 - \varepsilon(a - x)(b + x)(\gamma - x)\left(q_{m-2}^*(x)\right)^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, it follows that

$$\left(p_m^*(x)\right)^2 - \varepsilon(a - x)(b + x)(\gamma - x)\left(q_{m-2}^*(x)\right)^2 = cx^{2m},$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p_m^*. Dividing the last relation by cx^{2m} and introducing $s = 1/x$, we get the desired relation.

(b) On the other hand, for $n = 2m + 1$, we find that there are real polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m - 1$, respectively, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a - x)(b + x)(\gamma - x)}}{\gamma - x}$$

has a zero of order $2m + 1$ at $x = 0$. Multiplying that expression by

$$(\gamma - x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a - x)(b + x)(\gamma - x)}}{\gamma - x}\right),$$

we get that the polynomial $(\gamma - x)\left(p_m^*(x)\right)^2 - \varepsilon(a - x)(b + x)\left(q_{m-1}^*(x)\right)^2$ has a zero of order $2m + 1$ at $x = 0$. Since the degree of that polynomial is $2m + 1$, it follows that

$$(\gamma - x)\left(p_m^*(x)\right)^2 - \varepsilon(a - x)(b + x)\left(q_{m-1}^*(x)\right)^2 = cx^{2m+1},$$

for some constant c. Notice that c is negative, since it equals the opposite of the square of the leading coefficient of p_m^*. Dividing the last relation by $-\varepsilon cx^{2m+1}$ and introducing $s = 1/x$, we get the desired relation. $\square$

Corollary 2. *If the billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic, then there exist real polynomials $\hat{p}_n$ and $\hat{q}_{n-2}$ of degrees n and $n - 2$, respectively, which satisfy the Pell equation:*

$$\hat{p}_n^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)\hat{q}_{n-2}^2(s) = 1. \tag{7.1}$$

Proof. For $n = 2m$, take $\hat{p}_n = 2p_m^2 - 1$ and $\hat{q}_{n-2} = 2p_m q_{m-2}$. For $n = 2m + 1$, we set $\hat{p}_n = 2(\gamma s - 1)p_m^2 + \operatorname{sign}\gamma$ and $\hat{q}_{n-2} = 2p_m q_{m-1}$. $\square$

Theorem 6. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are elliptic n-periodic without being n-periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2, respectively, and satisfying the following:*

(a) $\mathcal{C}_\gamma$ *is an ellipse,* $0 < \gamma < a$, *and*

— $n = 2m$ *is even,* $d_1 = d_2 = m - 1$,

$$s\left(s - \frac{1}{a}\right)p_{m-1}^2(s) - \left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1;$$

— $n = 2m + 1$ *is odd,* $d_1 = m$, $d_2 = m - 1$,

$$\left(s + \frac{1}{b}\right)p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1;$$

(b) $\mathcal{C}_\gamma$ *is an ellipse,* $-b < \gamma < 0$, *and*

— $n = 2m$ *is even,* $d_1 = d_2 = m - 1$,

$$s\left(s + \frac{1}{b}\right)p_{m-1}^2(s) - \left(s - \frac{1}{a}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1;$$

— $n = 2m + 1$ *is odd,* $d_1 = m$, $d_2 = m - 1$,

$$\left(s - \frac{1}{a}\right)p_m^2(s) - s\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = -1;$$

(c) $\mathcal{C}_\gamma$ *is a hyperbola and* $n = 2m$ *is even,* $d_1 = d_2 = m - 1$,

$$\left(s - \frac{1}{\gamma}\right)p_{m-1}^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)q_{m-1}^2(s) = -\operatorname{sign}\gamma;$$

(d) $\mathcal{C}_\gamma$ *is a hyperbola,* $n = 2m + 1$ *is odd,* $d_1 = m$, $d_2 = m - 1$,

$$\left(s - \frac{1}{a}\right)p_m^2(s) - s\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = -1;$$

(e) $\mathcal{C}_\gamma$ *is a hyperbola,* $n = 2m + 1$ *is odd,* $d_1 = m$, $d_2 = m - 1$,

$$\left(s + \frac{1}{b}\right)p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1.$$

Proof. (a) For $n = 2m$, the proof of Theorem 4 implies that there are polynomials $p_{m-1}^*(x)$ and $q_{m-1}^*(x)$ of degrees $m - 1$, such that the expression

$$p_{m-1}^*(x) - q_{m-1}^*(x)\frac{\sqrt{(a - x)(b + x)(\gamma - x)}}{a - x}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$(a - x)\left(p_{m-1}^*(x) + q_{m-1}^*(x)\frac{\sqrt{(a - x)(b + x)(\gamma - x)}}{a - x}\right),$$

we get that the polynomial $(a - x)\left(p_{m-1}^*(x)\right)^2 - (a - x)(b + x)\left(q_{m-1}^*(x)\right)^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, it follows that

$$(a - x)\left(p_{m-1}^*(x)\right)^2 - (b + x)(\gamma - x)\left(q_{m-1}^*(x)\right)^2 = cx^{2m},$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of q^*_{m-1}. Dividing the last relation by cx^{2m} and introducing $s = 1/x$, we get the desired relation.

For $n = 2m + 1$, the proof of Theorem 4 implies that there are polynomials $p^*_m(x)$ and $q^*_{m-1}(x)$ of degrees m and $m - 1$, such that the expression

$$p^*_m(x) - q^*_{m-1}(x)\frac{\sqrt{(a-x)(b+x)(\gamma-x)}}{b+x}$$

has a zero of order $2m + 1$ at $x = 0$. Multiplying that expression by

$$(b+x)\left(p^*_m(x) + q^*_{m-1}(x)\frac{\sqrt{(a-x)(b+x)(\gamma-x)}}{b+x}\right),$$

we get that the polynomial $(b+x)\big(p^*_m(x)\big)^2 - (a-x)(\gamma-x)\big(q^*_{m-1}(x)\big)^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m + 1$, it follows that

$$(b+x)\big(p^*_m(x)\big)^2 - (a-x)(\gamma-x)\big(q^*_{m-1}(x)\big)^2 = cx^{2m+1}$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p^*_m. Dividing the last relation by cx^{2m+1} and introducing $s = 1/x$, we get the desired relation.

(b) For $n = 2m$, the proof of Theorem 4 implies that there are real polynomials $p^*_{m-1}(x)$ and $q^*_{m-1}(x)$ of degrees $m - 1$, such that the expression

$$p^*_{m-1}(x) - q^*_{m-1}(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{b+x}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$(b+x)\left(p^*_{m-1}(x) + q^*_{m-1}(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{b+x}\right),$$

we get that the polynomial $(b+x)\big(p^*_{m-1}(x)\big)^2 + (a-x)(\gamma-x)\big(q^*_{m-1}(x)\big)^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, it follows that

$$(b+x)\big(p^*_{m-1}(x)\big)^2 + (a-x)(\gamma-x)\big(q^*_{m-1}(x)\big)^2 = cx^{2m},$$

for some constant c. Notice that c is positive, since it equals to the square of the leading coefficient of q^*_{m-1}. Dividing the last relation by cx^{2m} and introducing $s = 1/x$, we get the desired relation.

For $n = 2m + 1$, the proof of Theorem 4 implies that there are polynomials $p^*_m(x)$ and $q^*_{m-1}(x)$ of degrees m and $m - 1$, such that the expression

$$p^*_m(x) - q^*_{m-1}(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{a-x}$$

has a zero of order $2m + 1$ at $x = 0$. Multiplying that expression by

$$(a-x)\left(p^*_m(x) + q^*_{m-1}(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{a-x}\right),$$

we get that the polynomial $(a-x)\big(p^*_m(x)\big)^2 + (b+x)(\gamma-x)\big(q^*_{m-1}(x)\big)^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m + 1$, it follows that

$$(a-x)\big(p^*_m(x)\big)^2 + (b+x)(\gamma-x)\big(q^*_{m-1}(x)\big)^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p^*_m. Dividing the last relation by $-cx^{2m+1}$ and introducing $s = 1/x$, we get the desired relation.

For (c), the proof of Theorem 4 implies that there are polynomials real $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma-x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(\gamma-x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma-x}\right),$$

we get that the polynomial $(\gamma-x)\left(p_m^*(x)\right)^2 - \varepsilon(a-x)(b+x)\left(q_{m-1}^*(x)\right)^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, it follows that

$$(\gamma-x)\left(p_m^*(x)\right)^2 - \varepsilon(a-x)(b+x)\left(q_{m-1}^*(x)\right)^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p_m^*. Dividing the last relation by $-\varepsilon c x^{2m+1}$ and introducing $s = 1/x$, we get the desired relation.

(d) The proof of Theorem 4 implies that there are real polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{a-x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(a-x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{a-x}\right),$$

we get that the polynomial $(a-x)\left(p_m^*(x)\right)^2 - \varepsilon(b+x)(\gamma-x)\left(q_{m-1}^*(x)\right)^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, it follows that

$$(a-x)\left(p_m^*(x)\right)^2 - \varepsilon(b+x)(\gamma-x)\left(q_{m-1}^*(x)\right)^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p_m^*. Dividing the last relation by $-cx^{2m+1}$ and introducing $s = 1/x$, we get the desired relation.

(e) For $n = 2m+1$, the proof of Theorem 4 implies that there are real polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{b+x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(b+x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{b+x}\right),$$

we get that the polynomial $(b+x)\left(p_m^*(x)\right)^2 - \varepsilon(a-x)(\gamma-x)\left(q_{m-1}^*(x)\right)^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, it follows that

$$(b+x)\left(p_m^*(x)\right)^2 - \varepsilon(a-x)(\gamma-x)\left(q_{m-1}^*(x)\right)^2 = cx^{2m+1}$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p_m^*. Dividing the last relation by cx^{2m+1} and introducing $s = 1/x$, we get the desired relation. $\square$

After Corollary 2 and relation (7.1), we see that the Pell equations arise as the functional polynomial conditions for periodicity. Let us recall some important properties of the solutions of the Pell equations.

8. CLASSICAL EXTREMAL POLYNOMIALS AND CAUSTICS

8.1. Fundamental Properties of Extremal Polynomials

From the previous section we know that the Pell equation plays a key role in functional-polynomial formulation of periodicity conditions in the Minkowski plane. The solutions of the Pell equation are the so-called extremal polynomials. Denote $\{c_1, c_2, c_3, c_4\} = \{0, 1/a, -1/b, 1/\gamma\}$ with the ordering $c_1 < c_2 < c_3 < c_4$. The polynomials $\hat{p}_n$ are the so-called *generalized Chebyshev polynomials* on two intervals $[c_1, c_2] \cup [c_3, c_4]$, with an appropriate normalization. Namely, one can consider the question of finding the monic polynomial of certain degree n which minimizes the maximum norm on the union of two intervals. Denote such a polynomial as $\hat{P}_n$ and its norm L_n. The fact that polynomial $\hat{p}_n$ is a solution of the Pell equation on the union of intervals $[c_1, c_2] \cup [c_3, c_4]$ is equivalent to the following conditions:

(i) $\hat{p}_n = \hat{P}_n / \pm L_n$

(ii) the set $[c_1, c_2] \cup [c_3, c_4]$ is the maximal subset of $\mathbf{R}$ for which $\hat{P}_n$ is the minimal polynomial in the sense above.

Chebyshev was the first who considered a similar problem on one interval, and this was how celebrated Chebyshev polynomials emerged in the 19th century. Let us recall a fundamental result about generalized Chebyshev polynomials [5, 6].

Theorem 7 (A corollary of the Krein−Levin−Nudelman Theorem [19]). *A polynomial P_n of degree n satisfies a Pell equation on the union of intervals $[c_1, c_2] \cup [c_3, c_4]$ if and only if there exists an integer n_1 such that the equation holds:*

$$n_1 \int_{c_2}^{c_3} \hat{f}(s)ds = n \int_{c_4}^{\infty} \hat{f}(s)ds, \quad \hat{f}(s) = \frac{C}{\sqrt{\prod_{i=1}^{4}(s - c_i)}}. \tag{8.1}$$

(Here C is a nonessential constant.) The modulus of the polynomial reaches its maximal values L_n at the points $c_i : |P_n(c_i)| = L_n$.
In addition, there are exactly $\tau_1 = n - n_1 - 1$ internal extremal points of the interval $[c_3, c_4]$ where $|P_n|$ reaches the value L_n, and there are $\tau_2 = n_1 - 1$ internal extremal points of $[c_1, c_2]$ with the same property.

Definition 3 ([15, 16, 20]). We call the pair (n, n_1) the partition and (τ_1, τ_2) the signature of the generalized Chebyshev polynomial P_n.

Now we are going to formulate and prove the main result of this section, which relates n_1, n_2 the numbers of reflections off relativistic ellipses and off relativistic hyperbolas, respectively, with the partition and the signature of the related solution of a Pell equation.

Theorem 8. *Given a periodic billiard trajectory with period $n = n_1 + n_2$, where n_1 is the number of reflections off relativistic ellipses, n_2 the number of reflections off the relativistic hyperbolas, then the partition corresponding to this trajectory is (n, n_1). The corresponding extremal polynomial $\hat{p}_n$ of degree n has $n_1 - 1$ internal extremal points in the first interval and $n - n_1 - 1 = n_2 - 1$ internal extremal points in the second interval.*

Proof. Recall that $c_1 < c_2 < c_3 < c_4$. From equation (3.2), one has

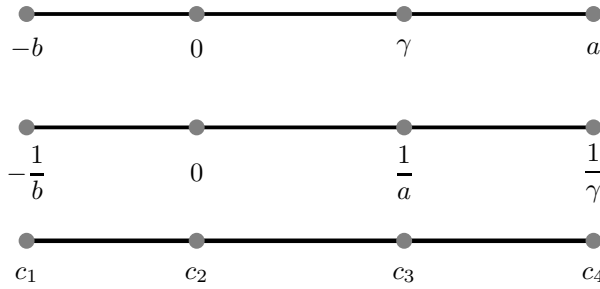$$n_1 \int_{\alpha_1}^{0} f(x)dx + n_2 \int_{\beta_1}^{0} f(x)dx = 0, \tag{8.2}$$

where α_1 is the largest negative value in $\{a, -b, \gamma\}$ and β_1 the smallest positive value in $\{a, -b, \gamma\}$.
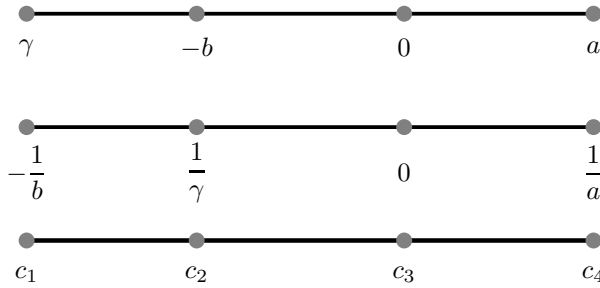The proof decomposes to the following cases:

- Case 1: $\mathcal{C}_\gamma$ is an ellipse and $\gamma < 0$, shown in Fig. 18;
- Case 2: $\mathcal{C}_\gamma$ is an ellipse and $\gamma > 0$, shown in Fig. 19;
- Case 3(i): $\mathcal{C}_\gamma$ is a hyperbola and $\gamma < -b$, shown in Fig. 20;
- Case 3(ii): $\mathcal{C}_\gamma$ is a hyperbola and $\gamma > a$, shown in Fig. 21.
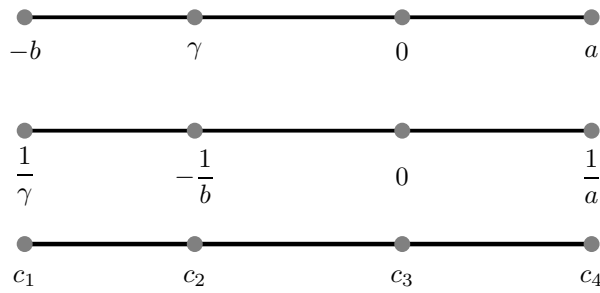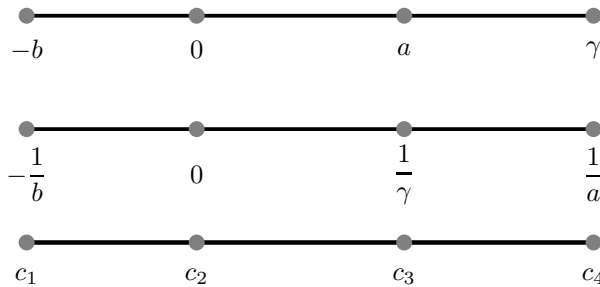
Fig. 18. Case 1: $\alpha_1 = \gamma$, $\beta_1 = a$.

Fig. 19. Case 2: $\beta_1 = \gamma$, $\alpha_1 = -b$.

Fig. 20. Case 3(i): $\beta_1 = a$, $\alpha_1 = -b$.

Fig. 21. Case 3(ii): $\beta_1 = a$, $\alpha_1 = -b$.

We provide a proof in Case 1. The proofs for other cases are analogous.
Equation (8.2) is equivalent to

$$0 = n_1 \int_\gamma^0 f(x)dx + n_1 \int_0^a f(x)dx + n_2 \int_a^0 f(x)dx - n_1 \int_0^a f(x)dx$$

$$= n_1 \int_\gamma^a f(x)dx + (n_1 + n_2) \int_a^0 f(x)dx,$$

thus

$$n_1 \int_\gamma^a f(x)dx = (n_1 + n_2) \int_0^a f(x)dx.$$

Since the cycles around the cuts on the elliptic curve are homologous:

$$\int_\gamma^a f(x)dx = \int_{-\infty}^{-b} f(x)dx,$$

(8.2) is equivalent to

$$n_1 \int_{-\infty}^{-b} f(x)dx = (n_1 + n_2) \int_0^a f(x)dx.$$

Substituting: $s = \dfrac{1}{x}$, $c_1 = \dfrac{1}{\gamma}$, $c_2 = -\dfrac{1}{b}$, $c_3 = 0$, $c_4 = \dfrac{1}{a}$ (see Fig. 18), we get that

$$n_1 \int_{-1/b}^0 \tilde{f}(s)ds = (n_1 + n_2) \int_{1/a}^\infty \tilde{f}(s)ds$$

is equivalent to

$$n_1 \int_{c_2}^{c_3} \tilde{f}(s)ds = (n_1 + n_2) \int_{c_4}^\infty \tilde{f}(s)ds, \tag{8.3}$$

where $\tilde{f}(s)ds$ is obtained from $f(x)dx$ by the substitution.   □

In particular, for $n = 3$, if the caustic $\mathcal{C}_\gamma$ is an ellipse with $\gamma < 0$, then $n_1 = 1$. Such polynomials and corresponding partitions $(3,1)$ do not arise in the study of Euclidean billiard trajectories. On the other hand, if the caustic $\mathcal{C}_\gamma$ is an ellipse with $\gamma > 0$, we have $n_1 = 2$. Such polynomials for $\gamma > 0$ can be explicitly expressed in terms of the Zolatarev polynomials, see Proposition 1. Since their partition is $(3,2)$, they appeared before in the Euclidean case (see [16]). The corresponding extremal polynomials $\hat{p}_3$ in both cases $\gamma < 0$ and $\gamma > 0$ are shown in Fig. 22. We will provide in Proposition 6 explicit formulae for such polynomials in terms of the general Akhiezer polynomials.

Let us recall that the Chebyshev polynomials $T_n(x), n = 0, 1, 2, \ldots$ defined by the recursion:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \tag{8.4}$$

for $n = 1, 2 \ldots$ can be parameterized as

$$T_n(x) = \cos n\phi, \quad x = \cos \phi, \tag{8.5}$$

or, alternatively,

$$T_n(x) = \frac{1}{2}\left(v^n + \frac{1}{v^n}\right), \quad x = \frac{1}{2}\left(v + \frac{1}{v}\right). \tag{8.6}$$

Denote $L_0 = 1$ and $L_n = 2^{1-n}, n = 1, 2, \ldots$. Then the Chebyshev theorem states that the polynomials $L_n T_n(x)$ are characterized as the solutions of the following minmax problem:

find the polynomial of degree n with the leading coefficient equal to 1 which minimizes the uniform norm on the interval $[-1, 1]$.
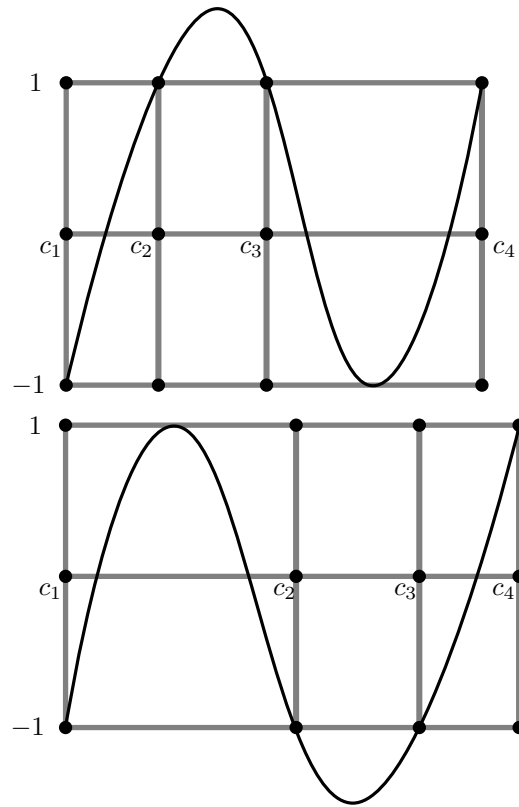
8.2. Zolotarev Polynomials

Following the ideas of Chebyshev, his student Zolatarev posed and solved a handful of problems, including the following ([5, 16]):

For the given real parameter σ and all polynomials of degree n of the form

$$p(x) = x^n - n\sigma x^{n-1} + p_2 x^{n-2} + \ldots p_n, \tag{8.7}$$

find the one with the minimal uniform norm on the interval $[-1, 1]$.

Fig. 22. On the left: the polynomial $\hat{p}_3$ corresponding to $n = 3$, $n_1 = 1$, $n_2 = 2$, $\gamma < 0$. On the right: the polynomial $\hat{p}_3$ corresponding to $n = 3$, $n_1 = 2$, $n_2 = 1$, $\gamma > 0$.

Denote this minimal uniform norm as $L_n = L(\sigma, n)$.

For $\sigma > \tan^2(\pi/2n)$, the solution z_n has the following property ([5], p. 298, Fig. 9):

$\Pi 1$ – *The equation $z_n(x) = L_n$ has $n - 2$ double solutions in the open interval $(-1, 1)$ and simple solutions at $-1, 1, \alpha, \beta$, where $1 < \alpha < \beta$, while in the union of the intervals $[-1, 1] \cup [\alpha, \beta]$ the inequality $z_n^2 \leqslant L_n$ is satisfied and $z_n > L_n$ in the complement.*

The polynomials z_n are given by the following explicit formulae:

$$z_n = \ell_n \left(v(u)^n + \frac{1}{v(u)^n} \right), \quad x = \frac{\mathrm{sn}^2 u + \mathrm{sn}^2 \frac{K}{n}}{\mathrm{sn}^2 u - \mathrm{sn}^2 \frac{K}{n}}, \tag{8.8}$$

where

$$\ell_n = \frac{1}{2^n} \left(\frac{\sqrt{\kappa} \theta_3^2(0)}{\theta_2 \left(\frac{K}{n} \right) \theta_3 \left(\frac{K}{n} \right)} \right)^{2n}, \quad v(u) = \frac{\theta_1 \left(\frac{K}{n} - u \right)}{\theta_1 \left(\frac{K}{n} + u \right)}$$

and

$$\sigma = \frac{2 \, \mathrm{sn} \frac{K}{n}}{\mathrm{cn} \frac{K}{n} \, \mathrm{dn} \frac{K}{n}} \left(\frac{1}{\mathrm{sn} \frac{2K}{n}} - \frac{\theta_0' \left(\frac{K}{n} \right)}{\theta_0 \left(\frac{K}{n} \right)} \right) - 1.$$

Here θ_i, $i = 0, 1, 2, 3$ are the classical Riemannian theta functions and

$$K = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - \kappa t^2)}}.$$

The formulae for the endpoints of the second interval are

$$\alpha = \frac{1 + \kappa^2 \, \mathrm{sn}^2 \frac{K}{n}}{\mathrm{dn}^2 \frac{K}{n}}, \quad \beta = \frac{1 + \mathrm{sn}^2 \frac{K}{n}}{\mathrm{cn}^2 \frac{K}{n}}, \tag{8.9}$$

with

$$\kappa^2 = \frac{(\alpha - 1)(\beta + 1)}{(\alpha + 1)(\beta - 1)}.$$

According to Cayley's condition, for $n = 3$ and $\gamma \in (0, a)$ we have

$$\gamma = \frac{ab(a - b) + 2ab\sqrt{a^2 + ab + b^2}}{(a + b)^2}.$$

In order to derive the formulas for $\hat{p}_3$ in terms of z_3, let us construct an affine transformation:

$$h : [-1, 1] \cup [\alpha, \beta] \rightarrow [-b^{-1}, 0] \cup [a^{-1}, \gamma^{-1}], \quad h(x) = \hat{a}x + \hat{b}.$$

We immediately get

$$\hat{a} = -\hat{b}, \quad \hat{a} = \frac{1}{2b}$$

and

$$\alpha = 2t + 1, \tag{8.10}$$

$$\gamma = \frac{2b}{\beta - 1}, \tag{8.11}$$

where $t = b/a$.

Now we get the following

Proposition 1. *The polynomial $\hat{p}_3$ can be expressed in terms of the Zolotarev polynomial z_3 up to a nonessential constant factor:*

$$\hat{p}_3(s) \sim z_3(2bs + 1).$$

To verify the proposition, we need to make sure that the definition of α and β from (8.9) for $n = 3$ and relations (8.10), (8.11) are compatible with the formula for γ we have obtained before as Cayley's condition, see (4.2).

In order to do that, we will use well-known identities for the Jacobi elliptic functions:

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \tag{8.12}$$

$$\kappa^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, \tag{8.13}$$

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{8.14}$$

$$\operatorname{sn}(K - u) = \frac{\operatorname{cn} u}{\operatorname{dn} u}. \tag{8.15}$$

In particular, we get

$$\operatorname{sn}\left(\frac{2K}{3}\right) = \frac{2 \operatorname{sn} \frac{K}{3} \operatorname{cn} \frac{K}{3} \operatorname{dn} \frac{K}{3}}{1 - \kappa^2 \operatorname{sn}^4 \frac{K}{3}}, \tag{8.16}$$

$$\operatorname{sn}\left(\frac{2}{3}K\right) = \operatorname{sn}\left(K - \frac{K}{3}\right) = \frac{\operatorname{cn} \frac{K}{3}}{\operatorname{dn} \frac{K}{3}}. \tag{8.17}$$

Let us denote

$$Y = \operatorname{sn}\left(\frac{K}{3}\right),$$

then from the previous two relations we get as in [16]:

$$1 - 2Y + 2\kappa^2 Y^3 - \kappa^2 Y^4 = 0.$$

We can express κ in terms of Y and get

$$\kappa^2 = \frac{2Y - 1}{Y^3(2 - Y)}. \tag{8.18}$$

By plugging the last relation into (8.9) for $n = 3$, we get

$$\alpha = \frac{Y^2 - 4Y + 1}{Y^2 - 1}.$$

Since at the same time from the Cayley condition we have $\alpha = 2t + 1$, with $t = b/a$, we can express Y in terms of t:

$$tY^2 + 2Y - (t + 1) = 0,$$

and

$$Y = \frac{-1 \pm \sqrt{1 + t + t^2}}{t}. \tag{8.19}$$

We plug the last relation into the formula for β from (8.9) for $n = 3$

$$\beta = \frac{1 + Y^2}{1 - Y^2},$$

and we get another formula for β in terms of t:

$$\beta = \frac{2t^2 + t + 2 - \pm 2\sqrt{t^2 + t + 1}}{-t - 2 \pm 2\sqrt{t^2 + t + 1}}. \tag{8.20}$$

We see that the last formula with the choice of the $+$ sign corresponds to a formula for β from (8.11). This formula relates β and γ from the Caley condition (4.2). From (8.20), taking the positive sign in β yields

$$\beta = \frac{2t^2 + t + 2 - 2\sqrt{t^2 + t + 1}}{-t - 2 + 2\sqrt{t^2 + t + 1}}. \tag{8.21}$$

Substituting (8.21) into (8.11) produces

$$\gamma = \frac{2b}{\beta - 1} = b\frac{-2 - t + 2\sqrt{1 + t + t^2}}{2 + t + t^2 - 2\sqrt{1 + t + t^2}}. \tag{8.22}$$

On the other hand, from the Cayley formula (4.2) we have

$$\gamma = \frac{ab}{(a + b)^2}(a - b + 2\sqrt{a^2 + ab + b^2}).$$

Knowing that $t = \dfrac{b}{a}$, the equation is equivalent to

$$\gamma = b\frac{1 - t + 2\sqrt{1 + t + t^2}}{(1 + t)^2}. \tag{8.23}$$

In order to show that the two expressions in (8.22) and (8.23) are identical, we simplify their difference that yields zero. This finalizes the verification. (One can observe that the $-$ sign option from the formula (8.20) would correspond to the $-$ sign in the formula for γ (4.2).

Among the polynomials $\hat{p}_n$ the property of type $\Pi 1$ can be attributed only to those with $n = 2k + 1$ and winding numbers $(2k + 1, 2k)$, in other words, to those with the signature $(0, 2k - 1)$.

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8.3. Akhiezer Polynomials on Symmetric Intervals $[-1, -\alpha] \cup [\alpha, 1]$

The problem of finding polynomials of degree n with the leading coefficient 1 and minimizing the uniform norm on the union of two symmetric intervals $[-1, -\alpha] \cup [\alpha, 1]$, for given $0 < \alpha < 1$, turned out to be of significant interest in radio engineering applications. Following the ideas of Chebyshev and Zolotarev, Akhiezer derived in 1928 the explicit formulae for such polynomials $A_n(x; \alpha)$ with the deviation $L_n(\alpha)$ [5, 6].

These formulas are especially simple in the case of even degrees $n = 2m$, when Akhiezer polynomials A_{2m} are obtained by a quadratic substitution from the Chebyshev polynomial T_m:

$$A_{2m}(x; \alpha) = \frac{(1 - \alpha^2)^m}{2^{2m-1}} T_m \left(\frac{2x^2 - 1 - \alpha^2}{1 - \alpha^2} \right), \tag{8.24}$$

with

$$L_{2m}(\alpha) = \frac{(1 - \alpha^2)^m}{2^{2m-1}}.$$

We are going to construct $\hat{p}_4(s)$ up to a nonessential constant factor as a composition of $A_4(x; \alpha)$ for certain α and an affine transformation. We are going to study the possibility of having an affine transformation

$$g : [-1, -\alpha] \cup [\alpha, 1] \to [-b^{-1}, \gamma^{-1}] \cup [0, a^{-1}], \quad g(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\gamma < -b$ ie $a > b$. For $n = 4$ such a caustic is (4.4):

$$\gamma = \frac{ab}{b - a}.$$

From $g(-1) = -b^{-1}$, $g(1) = a^{-1}$ we get

$$\hat{a} = \frac{a + b}{2ab}, \quad \hat{b} = \frac{b - a}{2ab}.$$

Then, from $g(\alpha) = 0$ we get

$$\alpha = \frac{a - b}{a + b}.$$

Finally, we calculate:

$$g(-\alpha) = \frac{a + b}{2ab} \frac{b - a}{a + b} + \frac{b - a}{2ab} = \frac{b - a}{ab}.$$

We recognize γ^{-1} on the right-hand side of the last relation.
This proves the following:

Proposition 2. *In this case the polynomial $\hat{p}_4(s)$ is equal, up to a constant multiplier, to*

$$\hat{p}_4(s) \sim T_2(2abs^2 + 2(a - b)s + 1), \tag{8.25}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial and $x = \dfrac{1}{a + b}(2abs + a - b)$.

Let us study the possibility of having an affine transformation

$$f : [-1, -\alpha] \cup [\alpha, 1] \to [-b^{-1}, 0] \cup [\gamma^{-1}, a^{-1}], \quad f(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\gamma > a$, i.e., $a < b$. For $n = 4$ such a caustic is

$$\gamma = \frac{ab}{b - a}.$$

From $f(-1) = -b^{-1}$, $f(1) = a^{-1}$ we get

$$\hat{a} = \frac{a+b}{2ab}, \quad \hat{b} = \frac{b-a}{2ab}.$$

Then, from $f(-\alpha) = 0$ we get

$$\alpha = \frac{b-a}{a+b}.$$

Finally, we calculate:

$$f(\alpha) = \frac{a+b}{2ab}\frac{b-a}{a+b} + \frac{b-a}{2ab} = \frac{b-a}{ab}.$$

We recognize γ^{-1} on the right-hand side of the last relation.
This proves the following proposition, which is the same as (8.25).

Proposition 3. *In this case the polynomial $\hat{p}_4(s)$ is equal, up to a constant multiplier, to*

$$\hat{p}_4(s) \sim T_2(2abs^2 + 2(a-b)s + 1), \tag{8.26}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial and $x = \dfrac{1}{a+b}\big(2abs + a - b\big)$.

Let us study the possibility to have an affine transformation

$$h : [-1, -\alpha] \cup [\alpha, 1] \to [\gamma^{-1}, -b^{-1}] \cup [0, a^{-1}], \quad h(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\gamma \in (-b, 0)$. For $n = 4$ such a caustic is

$$\gamma = -\frac{ab}{a+b}.$$

From $h(1) = a^{-1}$, $h(\alpha) = 0$ we get

$$\hat{a} = \frac{1}{1-\alpha}\frac{1}{a}, \quad \hat{b} = -\frac{\alpha}{1-\alpha}\frac{1}{a}.$$

Then, from $h(-\alpha) = -\frac{1}{b}$ we get

$$\frac{\alpha}{1-\alpha} = \frac{a}{2b}.$$

i. e.,

$$\alpha = \frac{a}{a+2b}.$$

Finally, we calculate:

$$h(-1) = -\Big(1 + \frac{\alpha}{1-\alpha}\Big)\frac{1}{a} - \frac{\alpha}{1-\alpha}\frac{1}{a},$$

$$h(-1) = -\Big(1 + \frac{a}{2b}\Big)\frac{1}{a} - \frac{a}{2b}\frac{1}{a} = -\frac{1}{a} - \frac{1}{b} = -\frac{a+b}{ab}.$$

We recognize γ^{-1} on the right-hand side of the last relation.
This proves the following:

Proposition 4. *In this case the polynomial $\hat{p}_4(s)$ is equal, up to a constant multiplier, to*

$$\hat{p}_4(s) \sim T_2\left(\frac{2a^2bs^2 + 2a^2s - (a+b)}{a+b}\right), \tag{8.27}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial.

Let us study the possibility to have an affine transformation

$$l : [-1, -\alpha] \cup [\alpha, 1] \to [-b^{-1}, 0] \cup [a^{-1}, \gamma^{-1}], \quad l(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\gamma \in (0, a)$. For $n = 4$ such a caustic is

$$\gamma = \frac{ab}{a + b}.$$

From $l(-1) = -b^{-1}$, $l(-\alpha) = 0$ we get

$$\hat{a} = \frac{1}{1 - \alpha} \frac{1}{b}, \quad \hat{b} = \frac{\alpha}{1 - \alpha} \frac{1}{b}.$$

Then, from $l(\alpha) = \dfrac{1}{a}$ we get

$$\frac{\alpha}{1 - \alpha} = \frac{b}{2a}.$$

i. e.,

$$\alpha = \frac{b}{b + 2a}.$$

Finally, we calculate:

$$l(1) = \left(1 + \frac{\alpha}{1 - \alpha}\right) \frac{1}{b} + \frac{\alpha}{1 - \alpha} \frac{1}{b},$$

$$l(1) = \left(1 + \frac{b}{2a}\right) \frac{1}{b} + \frac{b}{2a} \frac{1}{b} = \frac{1}{a} + \frac{1}{b} = \frac{a + b}{ab}.$$

We recognize γ^{-1} on the right-hand side of the last relation.

This proves the following:

Proposition 5. *In this case the polynomial $\hat{p}_4(s)$ is equal, up to a constant multiplier, to*

$$\hat{p}_4(s) \sim T_2 \left(\frac{2ab^2s^2 - 2b^2s - (a + b)}{a + b} \right), \tag{8.28}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial.

8.4. General Akhiezer Polynomials on Unions of Two Intervals

Following Akhiezer [1–3], let us consider the union of two intervals $[-1, \alpha] \cup [\beta, 1]$, where

$$\alpha = 1 - 2\operatorname{sn}^2(\frac{m}{n}K), \quad \beta = 2\operatorname{sn}^2\left(\frac{n - m}{n}K\right) - 1. \tag{8.29}$$

Define

$$TA_n(x, m, \kappa) = L\left(v^n(u) + \frac{1}{v^n(u)}\right), \tag{8.30}$$

where

$$v(u) = \frac{\theta_1\left(u - \frac{m}{n}K\right)}{\theta_1\left(u + \frac{m}{n}K\right)}, \quad x = \frac{\operatorname{sn}^2(u)\operatorname{cn}^2(\frac{m}{n}K) + \operatorname{cn}^2(u)\operatorname{sn}^2(\frac{m}{n}K)}{\operatorname{sn}^2(u) - \operatorname{sn}^2(\frac{m}{n}K)},$$

and

$$L = \frac{1}{2^{n-1}} \left(\frac{\theta_0(0)\theta_3(0)}{\theta_0(\frac{m}{n}K)\theta_3(\frac{m}{n}K)} \right), \quad \kappa^2 = \frac{2(\beta - \alpha)}{(1 - \alpha)(1 + \beta)}.$$

Akhiezer proved the following result:

Theorem 9 (Akhiezer).

(a) *The function $TA_n(x, m, \kappa)$ is a polynomial of degree n in x with the leading coefficient 1 and the second coefficient equal to $-n\tau_1$, where*

$$\tau_1 = -1 + 2\frac{\mathrm{sn}(\frac{m}{n}K)\,\mathrm{cn}(\frac{m}{n}K)}{\mathrm{dn}(\frac{m}{n}K)}\left(\frac{1}{sn(\frac{2m}{n}K)} - \frac{\theta'(\frac{m}{n}K)}{\theta(\frac{m}{n}K)}\right).$$

(b) *The maximum of the modulus of TA_n on the union of the two intervals $[-1, \alpha] \cup [\beta, 1]$ is L.*

(c) *The function TA_n takes values $\pm L$ with alternating signs at $\mu = n - m + 1$ consecutive points of the interval $[-1, \alpha]$ and at $\nu = m + 1$ consecutive points of the interval $[\beta, 1]$. In addition,*

$$TA_n(\alpha, m, \kappa) = TA_n(\beta, m, \kappa) = (-1)^m L,$$

and for any $x \in (\alpha, \beta)$ the following inequality holds:

$$(-1)^m TA_n(x, m, \kappa) > L.$$

(d) *Let F be a polynomial of degree n in x with the leading coefficient 1, such that:*

i) $\max|F(x)| = L$ *for* $x \in [-1, \alpha] \cup [\beta, 1]$;

ii) $F(x)$ *takes values $\pm L$ with alternating signs at $n - m + 1$ consecutives points of the interval $[-1, \alpha]$ and at $m+1$ consecutive points of the interval $[\beta, 1]$.*

Then $F(x) = TA_n(x, m, \kappa)$.

Let us determine the affine transformations when the caustic is an ellipse.

Case $\gamma \in (-b, 0)$. For $h : [-1, \alpha] \cup [\beta, 1] \to [\gamma^{-1}, -b^{-1}] \cup [0, a^{-1}]$, $h(x) = \hat{a}x + \hat{b}$, we get

$$\hat{a} = \frac{1}{\beta - \alpha}\frac{1}{b}, \quad \hat{b} = \frac{-\beta}{\beta - \alpha}\frac{1}{b}, \quad \frac{1 - \beta}{\beta - \alpha} = \frac{b}{a}.$$

Thus,

$$\gamma = \frac{\beta - 1}{1 + \beta}a = \frac{\alpha - \beta}{\beta + 1}b. \tag{8.31}$$

Example 7. For $n = 3$ and $m = 2$. From (8.29), one gets

$$\alpha = 1 - 2\,\mathrm{sn}^2\frac{2}{3}K, \quad \beta = 2\,\mathrm{sn}^2\frac{K}{3} - 1.$$

It follows that

$$\frac{b}{a} = t = \frac{1 - \beta}{\beta - \alpha} = \frac{1 - \mathrm{sn}^2\frac{K}{3}}{\mathrm{sn}^2\frac{2}{3}K + \mathrm{sn}^2\frac{K}{3} - 1}. \tag{8.32}$$

Thus,

$$\gamma = b\frac{\alpha - \beta}{\beta + 1} = b\frac{1 - \mathrm{sn}^2\frac{K}{3} - \mathrm{sn}^2\frac{2}{3}K}{\mathrm{sn}^2\frac{K}{3}}. \tag{8.33}$$

From the addition formula:

$$\mathrm{sn}\frac{2}{3}K = \mathrm{sn}\left(K - \frac{K}{3}\right) = \frac{\mathrm{sn}K\,\mathrm{cn}\frac{-K}{3}\,\mathrm{dn}\frac{-K}{3} + \mathrm{sn}\frac{-K}{3}\,\mathrm{cn}K\,\mathrm{dn}K}{1 - \kappa^2\,\mathrm{sn}^2\frac{-K}{3}\,\mathrm{sn}^2K}.$$

Hence,

$$\operatorname{sn}^2 \frac{2}{3}K = \frac{1 - \operatorname{sn}^2 \frac{K}{3}}{1 - \kappa^2 \operatorname{sn}^2 \frac{K}{3}} = \frac{\operatorname{sn}^2 \frac{2}{3}K - 1}{\kappa^2 \operatorname{sn}^2 \frac{2}{3}K - 1}.$$

Let $\operatorname{sn}\dfrac{K}{3} = Z$:

$$\kappa^2 = \frac{2Z - 1}{Z^3(2 - Z)}.$$

Also,

$$\alpha = 2Z^2 - 4Z + 1, \quad \beta = 2Z^2 - 1.$$

Equation (8.32) implies that

$$t = \frac{1 - Z^2}{2Z - 1}. \tag{8.34}$$

Thus, we have two expressions for γ. One is from the Cayley condition (4.2) and the other is from (8.31). We want to show that these two expressions are identical, that is,

$$b\frac{\alpha - \beta}{\beta + 1} = -\frac{ab}{(a + b)^2}(-a + b + 2\sqrt{a^2 + ab + b^2}). \tag{8.35}$$

In order to do so, we first express both the left-hand side and the right-hand side of the above equation in terms of $t = \dfrac{b}{a}$ and then transform both sides in terms of Z. We show that the left- and the right-hand sides yield the same expression:

$$\frac{Z^2 - Z + 1}{2Z - 1},$$

therefore, (8.35) holds.

Case $\gamma \in (0, a)$. For $l : [-1, \alpha] \cup [\beta, 1] \to [-b^{-1}, 0] \cup [a^{-1}, \lambda^{-1}]$, $l(x) = \hat{a}x + \hat{b}$, we get

$$\hat{a} = \frac{1}{\alpha + 1}\frac{1}{b}, \quad \hat{b} = \frac{-\alpha}{\alpha + 1}\frac{1}{b}, \quad \frac{\alpha + 1}{\beta - \alpha} = \frac{a}{b}.$$

Thus,

$$\gamma = \frac{\alpha + 1}{1 - \alpha}b. \tag{8.36}$$

Example 8. For $n = 3$, and $m = 1$. From (8.29), one gets

$$\alpha = 1 - 2\operatorname{sn}^2\frac{K}{3}, \quad \beta = 2\operatorname{sn}^2\frac{2K}{3} - 1.$$

$$\frac{b}{a} = t = \frac{\beta - \alpha}{\alpha + 1} = \frac{\operatorname{sn}^2\frac{2}{3}K + \operatorname{sn}^2\frac{K}{3} - 1}{1 - \operatorname{sn}^2\frac{K}{3}}. \tag{8.37}$$

Thus,

$$\gamma = \frac{1 - \operatorname{sn}^2\frac{K}{3}}{\operatorname{sn}^2\frac{K}{3}}b.$$

From the addition formula:

$$\operatorname{sn}\frac{2K}{3} = \operatorname{sn}\left(K - \frac{K}{3}\right) = \frac{\operatorname{sn}K \operatorname{cn}\frac{-K}{3} \operatorname{dn}\frac{-K}{3} + \operatorname{sn}\frac{-K}{3} \operatorname{cn}K \operatorname{dn}K}{1 - \kappa^2 \operatorname{sn}^2\frac{-K}{3} \operatorname{sn}^2 K}.$$

Hence,

$$\operatorname{sn}^2 \frac{2}{3}K = \frac{1 - \operatorname{sn}^2 \frac{K}{3}}{1 - \kappa^2 \operatorname{sn}^2 \frac{K}{3}}.$$

Let $\operatorname{sn} \frac{K}{3} = Z$:

$$\kappa^2 = \frac{2Z - 1}{Z^3(2 - Z)}.$$

Also, $\beta = -2Z^2 + 4Z - 1$, and $\alpha = 1 - 2Z^2$. Equation (8.37) implies that

$$t = -\frac{2Z - 1}{Z^2 - 1}, \tag{8.38}$$

Thus, we have two expressions for γ. One is from the Cayley condition (4.2) and the other is from (8.36). Similarly, as in Example 7, we can show that these two expressions are identical.

Proposition 6. *For $n = 3$ and $\gamma \in (-b, 0)$, the polynomial $\hat{p}_3$ is up to a nonessential factor equal to*

$$\hat{p}_3 \sim TA_3 \left(2a \left(1 - \operatorname{sn}^2 \frac{K}{3} \right) s + 2 \operatorname{sn}^2 \frac{K}{3} - 1; 2, \kappa \right).$$

For $n = 3$ and $\gamma \in (0, a)$, the polynomial $\hat{p}_3$ is up to a nonessential factor equal to

$$\hat{p}_3 \sim TA_3 \left(2b \left(1 - \operatorname{sn}^2 \frac{K}{3} \right) s + 1 - 2 \operatorname{sn}^2 \frac{K}{3}; 1, \kappa \right).$$

Now, using the Akhiezer theorem, part (c) (see Theorem 9), one can compare and see that the number of internal extremal points coincides with $n_1 - 1$ and $n_2 - 1$ as proposed in Theorem 8. These numbers match Fig. 22 and the table from Section 4.1.

9. PERIODIC LIGHT-LIKE TRAJECTORIES AND CHEBYSHEV POLYNOMIALS

The light-like billiard trajectories, by definition, have at each point the velocity v satisfying $\langle v, v \rangle = 0$. Their caustic is the conic at infinity $\mathcal{C}_\infty$. Since successive segments of such trajectories are orthogonal to each other, the light-like trajectories can close only after an even number of reflections. In [13]*Theorem 3.3, it is proved that a light-like billiard trajectory within $\mathcal{E}$ is periodic with even period n if and only if

$$\operatorname{arccot} \sqrt{\frac{a}{b}} \in \left\{ \frac{k\pi}{n} \mid 1 \leqslant k < \frac{n}{2}, \left(k, \frac{n}{2} \right) = 1 \right\}. \tag{9.1}$$

For k not being relatively prime with $n/2$, the corresponding light-like trajectories are also periodic, and their period is a divisor of n.

Applying the limit $\gamma \to +\infty$ in Corollary 2, we get the following proposition.

Proposition 7. *A light-like trajectory within ellipse $\mathcal{E}$ is periodic with period $n = 2m$ if and only if there exist real polynomials $\hat{p}_m(s)$ and $\hat{q}_{m-1}(s)$ of degrees m and $m - 1$, respectively, if and only if*

- $\hat{p}_m^2(s) - \left(s - \frac{1}{a} \right) \left(s + \frac{1}{b} \right) \hat{q}_{m-1}^2(s) = 1$; *and*

- $\hat{q}_{m-1}(0) = 0.$

The first condition from Proposition 7 is the standard Pell equation describing extremal polynomials on one interval $[-1/b, 1/a]$, thus, the polynomials $\hat{p}_m$ can be obtained as Chebyshev polynomials composed with an affine transformation $[-1/b, 1/a] \to [-1, 1]$. The additional condition $\hat{q}_{m-1}(0) = 0$, which is equivalent to $\hat{p}_m'(0) = 0$ implies an additional constraint on parameters a and b. We have the following

Proposition 8. *The polynomials $\hat{p}_m$ and the parameters a, b have the following properties:*

- $\hat{p}_m(s) = T_m\left(\dfrac{2ab}{a+b}s + \dfrac{a-b}{a+b}\right)$, *where T_m is defined by (8.5);*

- *the condition $\hat{q}_{m-1}(0) = 0$ is equivalent to (9.1).*

Proof. The increasing affine transformation $h : [-1/b, 1/a] \to [-1, 1]$ is given by the formula $h(s) = (2abs + a - b)/(a + b)$. The internal extremal points of the Chebyshev polynomial T_m of degree m on the interval $[-1, 1]$ are given by

$$x_k = \cos\left(\frac{k}{m}\pi\right), \quad k = 1, \ldots, m - 1,$$

according to the formula (8.5). The second item follows from $h(0) = x_k$. This is equivalent to

$$\frac{a-b}{a+b} \in \left\{\cos\left(\frac{k}{m}\pi\right) \mid k = 1, \ldots, m-1\right\},$$

which is equivalent to (9.1). □

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