ISSN 1560-3547, Regular and Chaotic Dynamics, 2019, Vol. 24, No. 4, pp. 353–369. © Pleiades Publishing, Ltd., 2019.

The Kepler Problem: Polynomial Algebra of Nonpolynomial First Integrals

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Received April 09, 2019; revised June 03, 2019; accepted June 09, 2019

Abstract—The sum of elliptic integrals simultaneously determines orbits in the Kepler problem and the addition of divisors on elliptic curves. Periodic motion of a body in physical space is defined by symmetries, whereas periodic motion of divisors is defined by a fixed point on the curve. The algebra of the first integrals associated with symmetries is a well-known mathematical object, whereas the algebra of the first integrals associated with the coordinates of fixed points is unknown. In this paper, we discuss polynomial algebras of nonpolynomial first integrals of superintegrable systems associated with elliptic curves.

MSC2010 numbers: 70H12, 33E05, 37E99

DOI: 10.1134/S1560354719040014

Keywords: algebra of first integrals, divisor arithmetic

1. INTRODUCTION

The main point of interest in integrable systems relies on the fact that they can be integrated by quadratures. For many known integrable systems these quadratures involve various sums of Abelian integrals, which are inextricably entwined with the arithmetic of divisors. In physics, we describe first integrals of dynamical systems in terms of physical variables, and usually these first integrals are related to symmetries, including dynamical ones. For the Kepler problem the corresponding first integrals are well-known polynomials in momenta [3, 8, 19, 20, 25, 26].

In algebraic geometry we describe the evolution of divisors in terms of coordinates of divisors. The corresponding constants of motion are nothing more than the coordinates of fixed points, which are algebraic functions on original physical variables. In fact, algebraic first integrals for the Kepler problem have been obtained by Euler as a byproduct of his study of the algebraic orbits appearing in two fixed centers problem [8].

Algebras of the polynomial first integrals of superintegrable systems can be associated with orthogonal polynomials, see, e. g., [5, 6, 12–14, 23, 24] and references therein. For instance, it could be the Racah – Wilson algebra, Bannai – Ito algebra, Askey – Wilson algebra, etc. We suppose that the polynomial algebra of nonpolynomial first integrals arising in divisor arithmetic on elliptic and hyperelliptic curves may be associated with elliptic and hyperelliptic functions. It could be Weierstrass functions, Jacobi functions, Abelian functions, etc.

In 1762 Euler wrote a paper entitled "*Problem: a body is attracted to two given fixed centers inversely proportional to the square of the distance; find in which case the curve described by the body will be algebraic*" [8]. In this paper he separated algebraic orbits from transcendental ones using elliptic coordinates on a plane and an addition law for the corresponding elliptic integrals. Algebraic orbits were interesting because if one of the centers were absent, the body would move in algebraic orbits, as a solution of the Kepler problem.

Indeed, let us consider the motion of the body attracted to two fixed centers by forces inversely proportional to the squares of the distance

$$R = \frac{\alpha}{r^2} \qquad \text{and} \qquad Q = \frac{\beta}{q^2}.$$

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In elliptic coordinates s and u the equations of motion are reduced to differential equations

$$\frac{ds}{\sqrt{Hs^4+(\alpha+\beta)s^3+cs^2-h^2(\alpha+\beta)s-Hh^4-(b^2+c)h^2}} = \frac{du}{\sqrt{Hu^4+(\alpha-\beta)u^3+cs^2-h^2(\alpha-\beta)u-Hh^4-(b^2+c)h^2}}$$

and

$$dt = \frac{s^2ds}{4\sqrt{Hs^4+(\alpha+\beta)s^3+cs^2-h^2(\alpha+\beta)s-Hh^4-(b^2+c)h^2}} - \frac{u^2du}{4\sqrt{Hu^4+(\alpha-\beta)u^3+cs^2-h^2(\alpha-\beta)u-Hh^4-(b^2+c)h^2}},$$

which we copied from page 106 of Lagrange's textbook [26]. Here H and h are integrals of motion, which are second-order polynomials in momenta, and b and c are geometric parameters describing the positions of fixed centers.

At $\beta=0$ and $x_1=s$, $x_2=u$ these equations become well-known Abel's quadratures for the two-body Kepler problem

$$\int\frac{dx_1}{\sqrt{f(x_1)}}+\int\frac{dx_2}{\sqrt{f(x_2)}}=\mathrm{const}, \tag{1.1}$$

and

$$\int\frac{x_1^2dx_1}{\sqrt{f(x_1)}}+\int\frac{x_2^2dx_2}{\sqrt{f(x_2)}}=4t, \tag{1.2}$$

on the elliptic curve X defined by an equation of the form

$$X:\quad \Phi(x,y)=y^2-f(x)=0,\qquad f(x)=a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \tag{1.3}$$

on a projective plane. The first equation of (1.1) determines the trajectories of motion, whereas the second equation of (1.2) defines time [26].

Equations (1.1), (1.2) describe the motion of a body in the Kepler problem and, simultaneously, the evolution of points $P_1(t)=(x_1,y_1)$, $P_2(t)=(x_2,y_2)$ around the fixed point $P_3=(x_3,y_3)$ on X governed by the arithmetic equation

$$P_1(t)+P_2(t)=P_3. \tag{1.4}$$

According to Abel's theorem [1, 2, 11, 17, 18], the trajectories of points $P_{1,2}(t)$ on X are uniquely determined by Abel's sum (1.1) in the same way as trajectories of a body in the Kepler problem. Subsequently, the periodic motion of points along the plane curve X generates periodic motion in the phase space of the Kepler system and vice versa.

According to [9], the coordinates of the fixed point x_3 and y_3 are algebraic functions on the coordinates of movable points $x_{1,2}$ and $y_{1,2}$ which are constants of divisor motion along the elliptic curve X (1.4). These algebraic functions on elliptic coordinates $u_{1,2}$ and momenta $p_{u_{1,2}}$ are also first integrals in the Kepler problem. In [8] Euler used these algebraic first integrals and their combination

$$C=2a_4x_3^2+a_3x_3+a_2-2\sqrt{a_4}y_3 \tag{1.5}$$

in order to separate algebraic orbits from transcendental ones in the problem of two fixed centers.

In the Kepler case the first integral C (1.5) is a square of the component of angular momentum

$$C=-\big(p_1q_2-(q_1-\kappa)p_2\big)^2.$$

Of course, this first integral is related to the well-studied rotational symmetry [3, 19, 20, 25]. The Poisson algebras of polynomial first integrals for the Kepler problem and other dynamical systems separable in elliptic, parabolic and polar coordinates are well studied objects, see, e. g., [5, 6, 12–14, 23, 24].

In [41–44] we presented a few families of superintegrable systems with additional first integrals which are rational and algebraic functions on coordinates and momenta. In this paper our aim is to calculate the algebra of nonpolynomial first integrals x_3 and y_3 and to discuss various representations of this algebra. This algebra occurs in a standard arithmetic of divisors on elliptic curves and, therefore, it could belong to a family of algebras associated with the arithmetic of divisors on more complicated hyperelliptic curves.

2. THE KEPLER PROBLEM

In the original physical problem, configuration space is 6-dimensional, and phase space is 12-dimensional. A discussion of the traditional topics, such as symmetries, conservation of angular momentum, conservation of Laplace – Runge – Lenz vector, regularization and so on, may be found in [3, 19, 20, 25] and many other papers and textbooks.

Our aim is to come back to Euler's calculations in order to get a family of superintegrable systems with nonpolynomial first integrals, which cannot be obtained using symmetries. Following Euler [8], we start with the planar two-center problem. Reduction of the original phase space to the orbital plane, which Euler described by using a picture, may be found in the Lagrange textbook [26].

2.1. Motion in an Orbit

Let us introduce elliptic coordinates on an orbital plane. If r and r' are the distances from a point on the plane to the fixed centers, then the elliptic coordinates $u_{1,2}$ are

$$r + r' = 2u_1, \qquad r - r' = 2u_2.$$

If the centers are taken to be fixed at $-\kappa$ and κ on the OX-axis of the Cartesian coordinate system, then we have the standard Euler definition of elliptic coordinates on the plane

$$q_1 = \frac{u_1 u_2}{\kappa} \quad \text{and} \quad q_2 = \frac{\sqrt{(u_1^2 - \kappa^2)(\kappa^2 - u_2^2)}}{\kappa}. \tag{2.1}$$

Coordinates $u_{1,2}$ are curvilinear orthogonal coordinates, which take values only in the intervals

$$u_2 < \kappa < u_1,$$

i. e., they are locally defined coordinates. The corresponding momenta are given by

$$\begin{aligned} p_1 &= \frac{u_1 u_2 (p_{u_1} u_1 - p_{u_2} u_2) - \kappa^2 (p_{u_1} u_2 - p_{u_2} u_1)}{\kappa (u_1^2 - u_2^2)}, \\ p_2 &= \frac{(p_{u_1} u_1 - p_{u_2} u_2)\sqrt{u_1^2 - \kappa^2}\sqrt{\kappa^2 - u_2^2}}{\kappa (u_1^2 - u_2^2)}. \end{aligned} \tag{2.2}$$

For the planar Kepler problem with one center of attraction at point $(\kappa, 0)$, which is a partial case of Euler's two-center problem, the Hamiltonian and the first integral are equal to

$$2H = I_1 = p_1^2 + p_2^2 + \frac{\alpha}{r}, \qquad I_2 = \frac{\alpha (r^2 - r'^2)}{4r} - (\kappa^2 + q_2^2) p_1^2 - 2 q_1 q_2 p_1 p_2 - q_1^2 p_2^2. \tag{2.3}$$

In elliptic coordinates these integrals of motion have the following form:

$$\begin{aligned} I_1 &= \frac{(u_1^2 - \kappa^2) p_{u_1}^2}{u_1^2 - u_2^2} + \frac{(u_2^2 - \kappa^2) p_{u_2}^2}{u_2^2 - u_1^2} + \frac{\alpha}{u_1 + u_2}, \\ I_2 &= \frac{u_2^2 (u_1^2 - \kappa^2) p_{u_1^2}}{u_2^2 - u_1^2} + \frac{u_1^2 (u_2^2 - \kappa^2) p_{u_2}^2}{u_1^2 - u_2^2} + \frac{\alpha u_1 u_2}{u_1 + u_2}. \end{aligned} \tag{2.4}$$

Substituting the solutions of these equations with respect to p_{u_1} and p_{u_2} into the equations of motion

$$\frac{du_1}{dt} = \{u_1, H\} = \frac{(u_1^2 - \kappa^2) p_{u_1}}{u_1^2 - u_2^2}, \qquad \frac{du_2}{dt} = \{u_2, H\} = \frac{(u_2^2 - \kappa^2) p_{u_2}}{u_2^2 - u_1^2},$$

we obtain differential equations of the form

$$\frac{du_1}{\sqrt{(u_1^2-\kappa^2)(I_1u_1^2-\alpha u_1+I_2)}} = \frac{dt}{u_1^2-u_2^2},$$
$$\frac{du_2}{\sqrt{(u_2^2-\kappa^2)(I_1u_2^2-\alpha u_2+I_2)}} = -\frac{dt}{u_1^2-u_2^2},$$

After integration of the sum of these equations one gets a sum of Abelian integrals

$$\int\frac{du_1}{\sqrt{(u_1^2-\kappa^2)(I_1u_1^2-\alpha u_1+I_2)}}+\int\frac{du_2}{\sqrt{(u_2^2-\kappa^2)(I_1u_2^2-\alpha u_2+I_2)}}=\mathrm{const} \tag{2.5}$$

involving holomorphic differentials on the elliptic curve X (1.3) defined by the equation

$$X:\quad \Phi(x,y)=y^2-f(x)=0,\qquad f(x)=I_1x^4-\alpha x^3+(I_2-I_1\kappa^2)x^2+\kappa^2\alpha x-I_2\kappa^2. \tag{2.6}$$

Here $I_{1,2}$ are values of the integrals of motion, for terminology and discussion, see Lagrange's textbook [26] and comments by Darboux and Serret [4, 31].

2.2. Motion in an Elliptic Curve

Using the sum of Abelian integrals (2.5), we can transfer from classical mechanics to algebraic geometry and, in particular, to divisors arithmetic on elliptic curves. Indeed, the coordinates of movable points $P_{1,2}(t)$ in the equation of motion along the elliptic curve X (1.4) are

$$x_1=u_1,\quad y_1=(u_1^2-\kappa^2)p_{u_1}\qquad\text{and}\qquad x_2=u_2,\quad y_2=(u_2^2-\kappa^2)p_{u_2}.$$

Because

$$u_2<\kappa<u_1\quad\Rightarrow\quad x_1\neq x_2\quad\Rightarrow\quad (x_3,y_3)\neq(\infty,\infty),$$

the abscissa x_3 and the ordinate y_3 of the fixed point P_3 are well-defined finite functions on $T^*\mathbb{R}^2$.

In order to calculate affine coordinates of the fixed point P_3, we have to consider the intersection of X and the parabola Y with a fixed leading coefficient

$$Y:\qquad y=\mathcal{P}(x),\qquad \mathcal{P}(x)=\sqrt{a_4}x^2+b_1x+b_0,$$

see [1, 17, 18] for details. Solving equations

$$y_1=\sqrt{a_4}x_1^2+b_1x_1+b_0\qquad\text{and}\qquad y_2=\sqrt{a_4}x_2^2+b_1x_2+b_0$$

with respect to b_1 and b_0, we calculate standard interpolation by Lagrange for the polynomial

$$\mathcal{P}(x)=\sqrt{a_4}x^2+b_1x+b_0=\sqrt{a_4}(x_1-x)(x_2-x)+\frac{(x-x_2)y_1}{x_1-x_2}+\frac{(x-x_1)y_2}{x_2-x_1}. \tag{2.7}$$

Substituting $y=\mathcal{P}(x)$ into $f(x)-y^2=0$, we obtain Abel's polynomial

$$\begin{aligned}\psi&=f(x)-\mathcal{P}^2(x)\\&=(a_3-2b_1\sqrt{a_4})x^3+(a_2-2b_0\sqrt{a_4}-b_1^2)x^2+(a_1-2b_0b_1)x+a_0-b_0^2\\&=(a_3-2b_1\sqrt{a_4})(x-x_1)(x-x_2)(x-x_3).\end{aligned}$$

Evaluating coefficients of this polynomial, we determine the abscissa of the fixed point P_3 in (1.4)

$$x_3=-x_1-x_2-\frac{2b_0\sqrt{a_4}+b_1^2-a_2}{2b_1\sqrt{a_4}-a_3} \tag{2.8}$$

and its ordinate

$$y_3=-\mathcal{P}(x_3)=-\sqrt{a_4}x_3^2-b_1x_3-b_0, \tag{2.9}$$

where b_1 and b_0 are functions of coordinates of the movable points x_1,x_2 and y_1,y_2 defined by Eq. (2.7).

Now we come back from divisor arithmetics to classical mechanics. For the Kepler problem we have

$$a_4 = I_1, \quad a_3 = -\alpha, \quad a_2 = (I_2 - \kappa^2 I_1), \quad a_1 = \kappa^2\alpha, \quad a_0 = -\kappa^2 I_2,$$

so the abscissa of the fixed point P_3 is equal to

$$x_3 = \frac{2\Big(\kappa^2(p_{u_1}u_2 - p_{u_2}u_1) - u_1u_2(p_{u_1}u_1 + p_{u_2}u_2)\Big)\Big(\sqrt{I_1}(u_1^2 - u_2^2) - \kappa^2(p_{u_1} - p_{u_2}) + p_{u_1}u_1^2 - p_{u_2}u_2^2\Big) - \alpha(u_1 - u_2)^2(\kappa^2 + u_1u_2)}{\big(u_1^2 - u_2^2\big)\Big(2\sqrt{I_1}\big(\kappa^2(p_{u_1} - p_{u_2}) - p_{u_1}u_1^2 + p_{u_2}u_2^2\big) + 2\kappa^2(p_{u_1}^2 - p_{u_2}^2) - 2p_{u_1}^2u_1^2 + 2p_{u_2}^2u_2^2 - \alpha(u_1 - u_2)\Big)}.$$

The ordinate y_3 (2.7) is equal to

$$y_3 = -\sqrt{I_1}(u_1 - x_3)(u_2 - x_3) - \frac{(x_3 - u_2)(\kappa^2 - u_1^2)p_{u_1}}{u_1 - u_2} - \frac{(x_3 - u_1)(\kappa^2 - u_2^2)p_{u_2}}{u_2 - u_1}.$$

Here I_1 is given by (2.4) and, therefore, x_3 and y_3 are algebraic functions on $u_{1,2}$ and $p_{u_{1,2}}$.

In [8, 9] Euler introduced the algebraic first integral C (1.5), which is nothing more than a square of angular momentum in the Kepler case:

$$C = 2a_4x_3^2 + a_3x_3 + a_2 - 2\sqrt{a_4}y_3 = \frac{(u_1^2 - \kappa^2)(u_2^2 - \kappa^2)(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^2}$$
$$= -\big(p_1q_2 - (q_1 - \kappa)p_2\big)^2.$$

It is well known that the existence of this first integral C is related to the rotational symmetry of the orbital plane around the center of attraction. The algebraic first integrals x_3 and y_3 have no obvious physical meaning, but they have a trivial geometric description as affine coordinates of the fixed point on the elliptic curve X.

2.3. Symmetry Breaking

Let us consider noncanonical transformations of momenta preserving symmetries of configuration space, but breaking symmetry between divisors [16, 36–39, 41–43].

It is easy to see that the transformation of momenta

$$p_{u_1} \to \frac{p_{u_1}}{m} \qquad \text{and} \qquad p_{u_2} \to \frac{p_{u_2}}{n}, \tag{2.10}$$

where m and n are rational numbers, preserves the only symmetry of the potential part of first integrals and breaks the symmetry of whole integrals of motion (2.3), which now have the form

$$\begin{aligned} 2H = I_1 &= \frac{u_1^2 - \kappa^2}{u_1^2 - u_2^2}\left(\frac{p_{u_1}}{m}\right)^2 + \frac{u_2^2 - \kappa^2}{u_2^2 - u_1^2}\left(\frac{p_{u_2}}{n}\right)^2 + \frac{\alpha}{u_1 + u_2}, \\ I_2 &= \frac{u_2^2(u_1^2 - \kappa^2)}{u_2^2 - u_1^2}\left(\frac{p_{u_1}}{m}\right)^2 + \frac{u_1^2(u_2^2 - \kappa^2)}{u_1^2 - u_2^2}\left(\frac{p_{u_2}}{n}\right)^2 + \frac{\alpha u_1u_2}{u_1 + u_2}. \end{aligned} \tag{2.11}$$

In Cartesian coordinates on the plane the Hamiltonians (2.11) read as

$$H = \frac{(m^2 + n^2)(p_1^2 + p_2^2)}{4m^2n^2} + \frac{(m^2 - n^2)\Big((\kappa^2 - q_1^2 + q_2^2)(p_1^2 - p_2^2) + 4q_1q_2p_1p_2\Big)}{4m^2n^2rr'} + \frac{\alpha}{2r}.$$

According to [41, 42], these Hamiltonians (2.11) are superintegrable Hamiltonians because this noncanonical transformation sends the original sum of elliptic integrals (2.5) to the sum

$$m\int\frac{du_1}{\sqrt{(u_1^2 - \kappa^2)(I_1u_1^2 - \alpha u_1 + I_2)}} + n\int\frac{du_2}{\sqrt{(u_2^2 - \kappa^2)(I_1u_2^2 - \alpha u_2 + I_2)}} = \text{const} \tag{2.12}$$

i. e., to the sum of elliptic integrals with integer coefficients

$$m_1n_2\int\frac{du_1}{\sqrt{(u_1^2 - \kappa^2)(I_1u_1^2 - \alpha u_1 + I_2)}} + n_1m_2\int\frac{du_2}{\sqrt{(u_2^2 - \kappa^2)(I_1u_2^2 - \alpha u_2 + I_2)}} = \text{const}.$$

Here we present the rational numbers $m = m_1/m_2$ and $n = n_1/n_2$ as the ratio of integer numbers. The corresponding first integrals of motion on the elliptic curve X were obtained in Problem 83 of Euler's textbook [9].

Below, without loss of generality we consider only the positive integer numbers m and n. In this case, the sum of elliptic integrals (2.12) generates the well-studied arithmetic equation for divisors on elliptic curves

$$[m]P_1(t) + [n]P_2(t) = P_3, \tag{2.13}$$

see [9, 27, 32]. Here $[k]P$ means scalar multiplication of a point on an elliptic curve on the integer number $k \in \mathbb{Z}$, and we denote the coordinates of $[k]P = [k](x, y)$ as $([k]x, [k]y)$, whereas notations for the coordinates of P_3 in (2.13) remain the same, x_3 and y_3.

In order to get the coordinates of the fixed point P_3 in (2.13), we have to:

1) Multiply divisors $P_{1,2}$ by integer numbers m and n using a recursion procedure proposed by Euler [9] or using standard expressions for scalar multiplication on elliptic curves, see [27, 32, 40] and references therein.

2) Add divisors $[m]P_1$ and $[n]P_2$. Because points $[m]P_1$ and $[n]P_2$ belong to the intersection divisor of X and Y, we can use the equation of parabola Y

$$[m]y_1 = \sqrt{a_4}\cdot([m]x_1)^2 + b_1\cdot[m]x_1 + b_0 \qquad \text{and} \qquad [n]y_2 = \sqrt{a_4}\cdot([n]x_2)^2 + b_1\cdot[n]x_2 + b_0$$

in order to calculate its coefficients b_1 and b_0. After that we substitute a_i, b_i and $[m]x_1$, $[m]y_1$ and $[n]x_2$, $[n]y_2$ into (2.8) and (2.9) and obtain the coordinates of the fixed point P_3 in (2.13):

$$x_3 = -[m]x_1 - [n]x_2 - \frac{2b_0\sqrt{a_4} + b_1^2 - a_2}{2b_1\sqrt{a_4} - a_3}, \qquad y_3 = -\sqrt{a_4}x_3^2 - b_1x_3 - b_0. \tag{2.14}$$

Following [9], we can also determine Euler's first integral of equation of motion (2.13) on X:

$$\begin{aligned} C_{mn} &= 2a_4x_3 + a_3x_3 + a_2 - 2\sqrt{a_4}y_3 \\ &= \left(\frac{[m]y_1 - [n]y_2}{[m]x_1 - [n]x_2}\right)^2 - a_4\big([m]x_1 + [n]x_2\big)^2 - a_3\big([m]x_1 + [n]x_2\big). \end{aligned} \tag{2.15}$$

3) Identify affine coordinates on the projective plane with elliptic coordinates on phase space

$$x_1 = u_1, \quad y_1 = (u_1^2 - \kappa^2)\frac{p_{u_1}}{m} \qquad \text{and} \qquad x_2 = u_2, \quad y_2 = (u_2^2 - \kappa^2)\frac{p_{u_2}}{n} \tag{2.16}$$

so that the constants of divisor motion on the elliptic curve X become first integrals of the Hamiltonian vector field in $T^*\mathbb{R}^2$.

At $m = n$ the first integral C_{mn} (2.15) is a square of angular momentum relating to rotational symmetry. At $m \neq n$ all first integrals x_3, y_3 (2.14) and C_{mn} (2.15) are algebraic functions in phase space. Some explicit expressions of these first integrals may be found in [42] .

Now we are ready to formulate the main result in this note.

Proposition 1. *Functions I_1, I_2 (2.11) and x_3, y_3 (2.14) in phase space $T^*\mathbb{R}^2$ can be considered as a representation of the following algebra of the first integrals*

$$\begin{aligned} &\{I_1, I_2\} = 0, && \{I_1, x_3\} = 0, && \{I_1, y_3\} = 0, \\ &\{I_2, x_3\} = \Phi_y(x_3, y_3), && \{I_2, y_3\} = -\Phi_x(x_3, y_3), && \{x_3, y_3\} = \kappa^2 - x_3^2 \end{aligned} \tag{2.17}$$

labelled by two integer numbers m and n. Here

$$\Phi_y(x, y) = \frac{\partial\Phi(x, y)}{\partial y} = 2y \quad \textit{and} \quad \Phi_x(x, y) = \frac{\partial\Phi(x, y)}{\partial x} = -(4I_1x^3 - 3\alpha x^2 + 2(I_2 - \kappa I_1)x + \alpha\kappa^2)$$

are derivatives of function $\Phi(x, y)$ from the definition of the elliptic curve X (2.6), and $\{.,.\}$ is the standard canonical Poisson bracket

$$\{u_1, u_2\} = 0, \quad \{p_{u_1}, p_{u_2}\} = 0, \quad \{u_i, p_{u_j}\} = \delta_{ij}.$$

The Poisson brackets (2.17) are derived from the brackets

$$\{I_i, I_j\} = \{\omega_i, \omega_j\} = 0, \qquad \{I_i, \omega_j\} = \delta_{ij},$$

between the action variables $I_{1,2}$ (2.11) and the angle variables

$$\begin{aligned}\omega_1 &= m\int \frac{u_1^2 du_1}{\sqrt{(u_1^2-\kappa^2)(I_1u_1^2-\alpha u_1+I_2)}} + n\int \frac{u_2^2 du_2}{\sqrt{(u_2^2-\kappa^2)(I_1u_2^2-\alpha u_2+I_2)}},\\ \omega_2 &= m\int \frac{du_1}{\sqrt{(u_1^2-\kappa^2)(I_1u_1^2-\alpha u_1+I_2)}} + n\int \frac{du_2}{\sqrt{(u_2^2-\kappa^2)(I_1u_2^2-\alpha u_2+I_2)}}.\end{aligned}$$

Here we use indefinite integrals determined only up to an additive constant following Euler [9], Abel [1], Jacobi [22] and Stäckel [33], see also [2, 17] for a discussion.

In (2.17) the form of the bracket $\{x_3, y_3\}$ coincides with the form of original brackets $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Two remaining nontrivial brackets can be rewritten in the following form:

$$\{\Phi(x,y), x_3\} = \frac{\partial\Phi}{\partial I_2}\frac{\partial\Phi}{\partial y}\bigg|_{P_3} \quad \text{and} \quad \{\Phi(x,y), y_3\} = -\frac{\partial\Phi}{\partial I_2}\frac{\partial\Phi}{\partial x}\bigg|_{P_3}, \tag{2.18}$$

which is reminiscent of Hamiltonian equations of motion. The first time brackets (2.17) appeared when we studied superintegrable systems associated with the elliptic curve in the short Weierstrass form [42, 44]. Below we discuss similar algebras of the first integrals for other superintegrable systems associated with the elliptic curve.

3. HARMONIC OSCILLATOR

Let us consider a 2D harmonic oscillator with the following Hamiltonian and additional integral of motion:

$$2H = I_1 = p_1^2 + p_2^2 - \alpha^2(q_1^2+q_2^2), \qquad I_2 = (p_1^2 - \alpha^2 q_1^2)\kappa^2 - (p_1q_2 + p_2q_1)^2,$$

which is a shifted square of angular momentum. In the elliptic coordinates (2.1) and (2.2) these constants of motion are equal to

$$\begin{aligned}I_1 &= \frac{(u_1^2-\kappa^2)p_{u_1}^2}{u_1^2-u_2^2} + \frac{(u_2^2-\kappa^2)p_{u_2}^2}{u_2^2-u_1^2} + \alpha^2(\kappa^2-u_1^2-u_2^2),\\ I_2 &= -\frac{u_2^2(u_1^2-\kappa^2)p_{u_1}^2}{u_1^2-u_2^2} - \frac{u_1^2(u_2^2-\kappa^2)p_{u_2}^2}{u_2^2-u_1^2} + \alpha^2u_1^2u_2^2.\end{aligned} \tag{3.1}$$

Rewriting the equations of motion

$$\frac{du_1}{dt} = \{u_1, H\} = \frac{(u_1^2-\kappa^2)p_{u_1}}{u_1^2-u_2^2}, \qquad \frac{du_2}{dt} = \{u_2, H\} = \frac{(u_2^2-\kappa^2)p_{u_2}}{u_2^2-u_1^2}$$

in the form

$$\frac{du_1}{p_{u_1}} = \frac{dt}{u_1^2-u_2^2}, \qquad \frac{du_2}{p_{u_2}} = -\frac{dt}{u_1^2-u_2^2},$$

we can eliminate time and obtain the equation

$$\frac{du_1}{p_{u_1}} + \frac{du_2}{p_{u_2}} = 0.$$

Substituting the solutions of Eqs. (3.1) with respect to $p_{u_{1,2}}$ into this expression and integrating, we obtain a standard equation defining the form of the trajectories [26]:

$$\int \frac{du_1}{\sqrt{(u_1^2-\kappa^2)(\alpha^2u_1^4+(I_1-\alpha^2\kappa^2)u_1^2+I_2)}} + \int \frac{du_2}{\sqrt{(u_2^2-\kappa^2)(\alpha^2u_2^4+(I_1-\alpha^2\kappa^2)u_2^2+I_2)}} = \text{const.}$$

This equation is reduced to (1.1) using Euler's substitution $u_i^2 = x_i$, which allows us to consider the evolution of divisors on the elliptic curve X

$$X: \quad \Phi(x,y) = y^2 - f(x) = 0, \quad f(x) = \alpha^2 x^4 + (I_1 - 2\alpha^2\kappa^2)x^3 + (\alpha^2\kappa^4 - I_1\kappa^2 + I_2)x^2 - \kappa^2 I_2 x \tag{3.2}$$

governed by Eq. (1.4)

$$P_1(t) + P_2(t) = P_3.$$

The coordinates of the movable points $P_{1,2}(t)$ are

$$x_1 = u_1^2, \quad y_1 = (u_1^2 - \kappa^2)u_1 p_{u_1} \qquad \text{and} \qquad x_2 = u_2^2, \quad y_2 = (u_2^2 - \kappa^2)u_2 p_{u_2}.$$

The abscissa of the fixed point x_3 (2.8) reads as

$$x_3 = -\frac{\Big((u_1^2-\kappa^2)u_2p_{u_1} - (u_2^2-\kappa^2)u_1p_{u_2} + \alpha u_1u_2(u_1^2-u_2^2)\Big)^2}{(u_1^2-u_2^2)\Big((\kappa^2-u_1^2)p_{u_1}^2 - (\kappa^2-u_2^2)p_{u_2}^2 + 2\alpha(u_1(\kappa^2-u_1^2)p_{u_1} - u_2(\kappa^2-u_2^2)p_{u_2}) + \alpha^2(\kappa^2-u_1^2-u_2^2)(u_1^2-u_2^2)\Big)}.$$

The ordinate y_3 (2.9) is a more lengthy rational function in elliptic coordinates, which we do not present for brevity, whereas Euler's integral (1.5) is the following simple polynomial:

$$C = -(p_1q_2 + p_2q_1)^2 - \kappa^2 I_1 + \alpha^2\kappa^4.$$

The existence of the polynomial first integrals I_1, I_2 and C is related to symmetries of equations of motion in the original physical space. The existence of nonpolynomial first integrals x_3 and y_3 is related to motion along the elliptic curve X around the fixed point P_3.

The symmetry breaking transformation (2.10) generates polynomial integrals of motion

$$\begin{aligned} I_1 &= \frac{u_1^2 - \kappa^2}{u_1^2 - u_2^2}\left(\frac{p_{u_1}}{m}\right)^2 + \frac{u_2^2 - \kappa^2}{u_2^2 - u_1^2}\left(\frac{p_{u_2}}{n}\right)^2 + \alpha^2(\kappa^2 - u_1^2 - u_2^2) \\ I_2 &= \frac{u_2^2(u_1^2 - \kappa^2)}{u_2^2 - u_1^2}\left(\frac{p_{u_1}}{m}\right)^2 + \frac{u_1^2(u_2^2 - \kappa^2)}{u_1^2 - u_2^2}\left(\frac{p_{u_2}}{n}\right)^2 + \alpha^2 u_1^2 u_2^2. \end{aligned} \tag{3.3}$$

In Cartesian coordinates on the plane superintegrable Hamiltonians in (3.3) read as

$$H = \frac{(m^2+n^2)(p_1^2+p_2^2)}{4m^2n^2} + \frac{(m^2-n^2)\Big((\kappa^2 - q_1^2 + q_2^2)(p_1^2 - p_2^2) + 4q_1q_2p_1p_2\Big)}{4m^2n^2rr'} - \frac{\alpha^2(q_1^2+q_2^2)}{2},$$

where r, r' and κ appear in the definition of elliptic coordinates. The corresponding first integrals x_3, y_3 (2.14) and C_{mn} (2.15) are rational functions at $m \neq n$. Some particular expressions for these first integrals may be found in [42].

Proposition 2. *Functions I_1, I_2 (3.3) and x_3, y_3 (2.14) on $T^*\mathbb{R}^2$ can be considered as a representation of the following algebra of the first integrals:*

$$\begin{aligned} &\{I_1, I_2\} = 0, && \{I_1, x_3\} = 0, && \{I_1, y_3\} = 0, \\ &\{I_2, x_3\} = 2\Phi_y(x_3, y_3), && \{I_2, y_3\} = -2\Phi_x(x_3, y_3), && \{x_3, y_3\} = 2x_3(\kappa^2 - x_3^2), \end{aligned} \tag{3.4}$$

labelled by two integer numbers m and n. Here

$$\Phi_y(x,y) = 2y, \qquad -\Phi_x(x,y) = 4\alpha^2x^3 + 3(I_1 - 2\alpha^2\kappa^2)x^2 + 2(\alpha^2\kappa^4 - \kappa^2 I_1 + I_2)x - \kappa^2 I_2$$

are derivatives of function $\Phi(x,y)$ from the definition of the elliptic curve X (3.2), and $\{.,.\}$ is the standard canonical Poisson bracket.

As in Section 2, the algebra of the first integrals (3.4) is derived from the Poisson brackets between the corresponding action-angle variables. We also have a computer-assisted proof of this proposition at $m = 1, 2, 3$ and $n = 1, 2, 3$.

This algebra of the first integrals (3.4) slightly differs from (2.17) because in the Kepler problem we take $x_{1,2} = u_{1,2}$, whereas for the oscillator we have to put $x_{1,2} = u_{1,2}^2$ and, therefore, we have different Poisson brackets between the coordinates of movable points.

3.1. Smorodinsky – Winternitz System

In order to obtain the so-called Smorodinsky – Winternitz system [10], we have to start with the elliptic curve X defined by the equation $\Phi(x,y) = y^2 - f(x) = 0$ with

$$f(x) = \alpha^2 x^4 + (I_1 - 2\alpha^2\kappa^2)x^3 + (\alpha^2\kappa^4 - I_1\kappa^2 + I_2 - 2\beta - \gamma)x^2 - \kappa^2(I_2 - 4\beta + \delta)x - 2\beta\kappa^4 \quad (3.5)$$

instead of (3.2). Equation (2.13)

$$[m]P_1(t) + [n]P_2(t) = P_3$$

determines the evolution of two moving points around a third fixed point in the intersection divisor. Coefficients $I_{1,2}$ of polynomial $f(x)$ together with coordinates (x_3, y_3) of the fixed point P_3 are constants of the divisor motion.

The constants of the divisor motion give rise to the first integrals on phase space, which can be calculated using the standard algorithm:

- identify the affine coordinates of movable points $P_{1,2}(t)$ on the projective plane with elliptic coordinates and momenta in phase space

$$x_1 = u_1^2, \quad y_1 = (u_1^2 - \kappa^2)u_1\frac{p_{u_1}}{m} \qquad \text{and} \qquad x_2 = u_2^2, \quad y_2 = (u_2^2 - \kappa^2)u_2\frac{p_{u_2}}{n};$$

- solve a pair of equations $\Phi(x_1, y_1) = 0$ and $\Phi(x_2, y_2)$ with respect to I_1, I_2;
- calculate first integrals associated with affine coordinates (x_3, y_3) (2.14) of the fixed point P_3.

After that we can verify that functions I_1, I_2 and x_3, y_3 on $T^*\mathbb{R}^2$ satisfy the Poisson brackets (3.4).

For the curve X (3.5) one gets the following Hamiltonian:

$$H = \frac{I_1}{2} = T_{mn} - \frac{\alpha^2}{2}(q_1^2 + q_2^2) + \frac{\beta}{q_1^2} + \frac{\gamma}{q_2^2}.$$

Here the potential part is independent of the integer numbers m and n, whereas the kinetic energy T_{mn} is

$$\begin{aligned} T_{mn} &= \frac{u_1^2 - \kappa^2}{u_1^2 - u_2^2}\left(\frac{p_{u_1}}{m}\right)^2 + \frac{u_2^2 - \kappa^2}{u_2^2 - u_1^2}\left(\frac{p_{u_2}}{n}\right)^2 \\ &= \frac{(m^2 + n^2)(p_1^2 + p_2^2)}{4m^2n^2} + \frac{(m^2 - n^2)\Big((\kappa^2 - q_1^2 + q_2^2)(p_1^2 - p_2^2) + 4q_1q_2p_1p_2\Big)}{4m^2n^2rr'}, \end{aligned}$$

where r, r' and κ appear in Euler's definition of elliptic coordinates on the plane. At $m = n = 1$ this Hamiltonian coincides with the Hamiltonian of the Smorodinsky – Winternitz system [10].

4. DRACH SYSTEM

In 1935 Jules Drach classified Hamiltonian systems in $T^*\mathbb{R}^2$ with third-order integrals of motion [7]. Below we consider the so-called (h) Drach system associated with an elliptic curve, see [35–37] for details of classification. Possible generalizations of the Drach systems are discussed in [28].

The (h) Drach system is defined by the Hamiltonian

$$H = I_1 = p_1p_2 - 2\alpha(q_1 + q_2) - \beta\left(\frac{q_1}{2\sqrt{q_2}} + \frac{3\sqrt{q_2}}{2}\right) - \frac{\gamma}{2\sqrt{q_2}}$$

and the first integral

$$I_2 = (q_1 + q_2)p_1p_2 - q_1p_1^2 - q_2p_2^2 - \alpha(q_1 - q_2)^2 - \frac{\beta(q_1 - q_2)^2}{2\sqrt{q_2}} - \frac{\gamma(q_1 - q_2)}{2\sqrt{q_2}}.$$

After canonical point transformation of variables

$$q_1 = \frac{(u_1-u_2)^2}{4}, \quad p_1 = \frac{p_{u_1}-p_{u_2}}{u_1-u_2}, \quad q_2 = \frac{(u_1+u_2)^2}{4}, \quad p_2 = \frac{p_{u_1}+p_{u_2}}{u_1+u_2}$$

the integrals of motion look like

$$I_1 = \frac{p_{u_1}^2}{u_1^2-u_2^2} + \frac{p_{u_2}^2}{u_2^2-u_1^2} - \alpha(u_1^2+u_2^2) - \frac{\beta(u_1^2+u_1u_2+u_2^2)}{u_1+u_2} - \frac{\gamma}{u_1+u_2}$$

$$I_2 = \frac{u_2^2p_{u_1}^2}{u_1^2-u_2^2}\frac{u_1^2p_{u_2}^2}{u_2^2-u_1^2} - \alpha u_1^2u_2^2 - \frac{\beta u_1^2u_2^2}{u_1+u_2} + \frac{\gamma u_1u_2}{u_1+u_2}.$$

Solving these equations with respect to p_{u_1} and p_{u_2}, we obtain the separated relations

$$\Phi_i(u_i,p_{u_i}) = p_{u_i}^2 - \Big(\alpha u_i^4 + \beta u_i^3 + I_1u_i^2 + \gamma u_i - I_2\Big), \qquad i=1,2. \tag{4.1}$$

Following [33], we determine the Stäckel matrix S with entries

$$S_{ij} = \frac{\partial\Phi_j}{\partial I_i} \tag{4.2}$$

and Stäckel angle variables

$$\omega_1 = \frac{1}{2}\int\frac{S_{11}du_1}{p_{u_1}} + \frac{1}{2}\int\frac{S_{12}du_2}{p_{u_2}}, \quad \omega_2 = \frac{1}{2}\int\frac{S_{21}du_1}{p_{u_1}} + \frac{2}{2}\int\frac{S_{22}du_2}{p_{u_2}},$$

which can be rewritten in a standard form for the Stäckel systems with n degrees of freedom

$$\omega_j = -\sum_{i=1}^n \int^{P_i} \frac{\partial\Phi(x,y)/\partial I_j}{\partial\Phi(x,y)/\partial y}\,dx\,, \tag{4.3}$$

using the separated relations (4.1), the definition of the Stäckel matrix (4.2) and the definition of points $P_i=(x_i,y_i)$ on the hyperelliptic curve X, see [34].

In action-angle variables $I_{1,2}$ and $\omega_{1,2}$ the equations of motion and the symplectic form look like

$$\dot I_i = 0\,, \qquad \dot\omega_i = \frac{\partial H}{\partial I_i}, \qquad \Omega = dI_1\wedge d\omega_1 + dI_2\wedge d\omega_2.$$

Because $H=I_1$, the differential equations are trivially reduced to quadratures, for instance,

$$I_{1,2} = \mathrm{const}, \qquad \omega_2 = -\frac{1}{2}\sum_{i=1}^2\int^{P_i}\frac{dx_1}{\sqrt{\alpha x^4+\beta x^3+I_1x^2+\gamma x-I_2}} = \mathrm{const}.$$

The relation $\omega_2=\mathrm{const}$ involves the sum of Abelian integrals with holomorphic differentials on X and, therefore, it defines the swing of two points around a third fixed point on the elliptic curve (1.4)

$$P_1(t)+P_2(t)=P_3.$$

In the Drach case the coordinates of moving points are a simple function on physical variables

$$x_1=u_1, \quad y_1=p_{u_1}, \qquad x_2=u_2, \quad y_2=p_{u_2},$$

and, therefore, the abscissa of the fixed point P_3 is a quite observable rational function

$$x_3 = -\frac{2(u_1^2-u_2^2)(u_2p_{u_1}-u_1p_{u_2})\sqrt{\alpha}+u_1u_2(u_1-u_2)^2\beta-(u_1-u_2)^2\gamma-2(p_{u_1}-p_{u_2})(u_2p_{u_1}-u_1p_{u_2})}{\Big(2(u_1^2-u_2^2)(u_1+u_2)\alpha-2(p_{u_1}-p_{u_2})\sqrt{\alpha}+(u_1-u_2)\beta\Big)(u_1^2-u_2^2)},$$

similar to Euler's integral (1.5), which is the following polynomial in momenta:

$$C = \frac{(p_{u_1}-p_{u_2})^2}{(u_1-u_2)^2} - (u_1+u_2)^2\alpha - (u_1+u_2)\beta.$$

Let us apply the symmetry breaking transformation (2.10) to this superintegrable Stäckel system. The action variables $I_{1,2}$ associated with the equation

$$[m]P_1(t)+[n]P_2(t)=P_3$$

are equal to

$$\begin{aligned} I_1&=\frac{p_{u_1}^2/m^2}{u_1^2-u_2^2}+\frac{p_{u_2}^2/n^2}{u_2^2-u_1^2}-\alpha(u_1^2+u_2^2)-\frac{\beta(u_1^2+u_1u_2+u_2^2)}{u_1+u_2}-\frac{\gamma}{u_1+u_2},\\ I_2&=\frac{u_2^2p_{u_1}^2/m^2}{u_1^2-u_2^2}+\frac{u_1^2p_{u_2}^2/n^2}{u_2^2-u_1^2}-\alpha u_1^2u_2^2-\frac{\beta u_1^2u_2^2}{u_1+u_2}+\frac{\gamma u_1u_2}{u_1+u_2}. \end{aligned} \tag{4.4}$$

In original Cartesian coordinates the Hamiltonian in (4.4) has the form

$$\begin{aligned} I_1=&\frac{(m^2+n^2)p_1p_2}{2m^2n^2}+\frac{(m^2-n^2)(\sqrt{q_1}+\sqrt{q_2})^2(q_1p_1^2+q_2p_2^2)}{4n^2m^2(\sqrt{q_1q_2}+q_1)(\sqrt{q_1q_2}+q_2)}\\ &-2\alpha(q_1+q_2)-\beta\left(\frac{q_1}{2\sqrt{q_2}}+\frac{3\sqrt{q_2}}{2}\right)-\frac{\gamma}{2\sqrt{q_2}}. \end{aligned}$$

Following [29], we can say that at $m\neq n$ these Hamiltonians describe the motions of the body with a position-dependent mass.

The corresponding Stäckel angle variables

$$\omega_1=\frac{m}{2}\int\frac{S_{11}du_1}{p_{u_1}}+\frac{n}{2}\int\frac{S_{12}du_2}{p_{u_2}},\quad \omega_2=\frac{m}{2}\int\frac{S_{21}du_1}{p_{u_1}}+\frac{n}{2}\int\frac{S_{12}du_2}{p_{u_2}}$$

involve a holomorphic differential on the elliptic curve, which allows us to calculate the coordinates of the fixed point using the arithmetic equation (2.14).

At $m=2$ and $n=1$ the abscissa of the fixed point P_3 remains a quite observable rational function if $\beta=\gamma=0$

$$x_3=-u_2-(u_1^2-u_2^2)p_{u_1}\left(\frac{1}{2\sqrt{\alpha}u_1(u_1^2-u_2^2)+u_2p_{u_1}-2u_1p_{u_2}}+\frac{1}{2\sqrt{\alpha}u_1(u_1^2-u_2^2)-u_2p_{u_1}+2u_1p_{u_2}}\right).$$

This expression was obtained using doubling of the point $P_1=(u_1,p_{u_1}/2)$ and addition (2.14) of points $[2]P_1$ and $P_2=(u_2,p_{u_2})$ on the elliptic curve X.

At $m=3$ and $n=1$ the abscissa of the fixed point P_3 is a bulky function even if $\beta=\gamma=0$

$$\begin{aligned} x_3=-&\frac{(u_2p_{u_1}+3u_1p_{u_2})(u_2p_{u_1}-3u_1p_{u_2})^2}{3\sqrt{\alpha}\big((4u_1^2-u_2^2)p_{u_1}^2-6u_1u_2p_{u_1}p_{u_2}-9u_1^2p_{u_2}^2\big)(u_1^2-u_2^2)}\\ &+6\sqrt{\alpha}(u_1^2-u_2^2)u_1^2p_{u_1}^2\left(\frac{(u_1-u_2)^2}{A_-}-\frac{(u_1+u_2)^2}{A_+}\right), \end{aligned}$$

where

$$A_+=\Big(9\alpha u_1^2(u_1^2-u_2^2)^2+(u_2p_{u_1}-3u_1p_{u_2})\big((2u_1+u_2)p_{u_1}+3u_1p_{u_2}\big)\Big)\big((2u_1+u_2)p_{u_1}+3u_1p_{u_2}\big)$$

and

$$A_-=\Big(9\alpha u_1^2(u_1^2-u_2^2)^2-(u_2p_{u_1}-3u_1p_{u_2})\big((2u_1-u_2)p_{u_1}-3u_1p_{u_2}\big)\Big)\big((2u_1-u_2)p_{u_1}-3u_1p_{u_2}\big).$$

This expression was obtained using tripling of the point $P_1=(u_1,p_{u_1}/3)$ and addition (2.14) of the points $[3]P_1$ and $P_2=(u_2,p_{u_2})$ on the elliptic curve X.

Proposition 3. *Functions I_1,I_2 (4.4) and x_3,y_3 (2.14) on $T^*\mathbb{R}^2$ can be considered as a representation of the following algebra of the first integrals:*

$$\begin{aligned} &\{I_1,I_2\}=0, &&\{I_1,x_3\}=0, &&\{I_1,y_3\}=0,\\ &\{I_2,x_3\}=\Phi_y(x_3,y_3), &&\{I_2,y_3\}=-\Phi_x(x_3,y_3), &&\{x_3,y_3\}=1 \end{aligned} \tag{4.5}$$

labelled by two integer numbers m and n. Here

$$\Phi_y(x,y) = 2y, \qquad -\Phi_x(x,y) = 4\alpha x^3 + 3\beta x^2 + 2I_1 x + \gamma$$

are derivatives of function $\Phi(x,y)$ from the definition of the elliptic curve X (4.1), and $\{.,.\}$ is the canonical Poisson bracket.

This algebra is derived from the Poisson bracket between the corresponding action-angle variables. We also have a computer-assisted proof of this proposition at $m = 1,2,3$ and $n = 1,2,3$.

Thus, the so-called (h) Drach system belongs to a family of two-dimensional superintegrable systems associated with elliptic curves of the form $X : \Phi(x,y) = y^2 - f^{(k)}(x)$, where

$$f^{(1)}(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + I_1 x + I_2, \qquad f^{(2)}(x) = \alpha x^4 + \beta x^3 + I_2 x^2 + \gamma x + I_2,$$
$$f^{(3)}(x) = \alpha x^4 + I_2 x^3 + \beta x^2 + \gamma x + I_2, \qquad f^{(4)}(x) = I_1 x^4 + \alpha x^3 + \beta x^2 + \gamma x + I_2,$$

and the equation of motion

$$[m]P_1(t) + [n]P_2(t) = P_3, \qquad m,n \in \mathbb{Z}.$$

For all these superintegrable systems the algebra of the first integrals has the standard form (4.5) which directly follows from the Poisson brackets between action-angle variables.

5. 3D SUPERINTEGRABLE STÄCKEL SYSTEM ON AN ELLIPTIC CURVE

Let us consider a Stäckel system associated with the symmetric product $X \times X \times X$ of the elliptic curve X defined by an equation of the form

$$X: \quad \Phi(x,y) = y^2 - f(x) = 0, \qquad f(x) = \alpha x^4 + \beta x^3 + I_1 x^2 + I_2 x + I_3. \tag{5.1}$$

If we identify the coordinates of the points on each copy of X in $X \times X \times X$ with canonical coordinates in $T^*\mathbb{R}^3$

$$x_1 = u_1\,, \quad y_1 = p_{u_1}, \quad x_2 = u_2, \quad y_2 = p_{u_2}, \quad x_3 = u_3\,, \quad y_3 = p_{u_3},$$

we obtain the action variables I_1, I_2 and I_3:

$$\begin{aligned}
I_1 &= \frac{p_{u_1}^2}{(u_1-u_3)(u_1-u_2)} + \frac{p_{u_2}^2}{(u_2-u_3)(u_2-u_1)} + \frac{p_{u_3}^2}{(u_3-u_1)(u_3-u_2)} \\
&\quad - (u_1^2+u_2^2+u_3^2+u_1u_2+u_1u_3+u_2u_3)\alpha - (u_1+u_2+u_3)\beta, \\
I_2 &= -\frac{(u_2+u_3)p_{u_1}^2}{(u_1-u_3)(u_1-u_2)} - \frac{(u_1+u_3)p_{u_2}^2}{(u_2-u_3)(u_2-u_1)} - \frac{(u_1+u_2)p_{u_3}^2}{(u_3-u_1)(u_3-u_2)} \\
&\quad + (u_1+u_2)(u_1+u_3)(u_2+u_3)\alpha + (u_1u_2+u_1u_3+u_2u_3)\beta, \\
I_3 &= \frac{u_2u_3p_{u_1}^2}{(u_1-u_3)(u_1-u_2)} + \frac{u_1u_3p_{u_2}^2}{(u_2-u_3)(u_2-u_1)} + \frac{u_1u_2p_{u_3}^2}{(u_3-u_1)(u_3-u_2)} \\
&\quad - u_1u_2u_3(u_1+u_2+u_3)\alpha - u_1u_2u_3\beta
\end{aligned} \tag{5.2}$$

solving the separated relations $\Phi(u_i, p_{u_i}, I_1, I_2, I_3) = 0$ with respect to I_1, I_2 and I_3. Substituting the solutions of the same separated relations with respect to p_{u_1}, p_{u_2} and p_{u_3} into the Stäckel definition (4.3), we get standard angle variables

$$\begin{aligned}
\omega_1 &= -\int \frac{u_1^2 du_1}{\sqrt{f(u_1)}} - \int \frac{u_2^2 du_2}{\sqrt{f(u_2)}} - \int \frac{u_3^2 du_3}{\sqrt{f(u_3)}}, \\
\omega_2 &= -\int \frac{u_1 du_1}{\sqrt{f(u_1)}} - \int \frac{u_2 du_2}{\sqrt{f(u_2)}} - \int \frac{u_3 du_3}{\sqrt{f(u_3)}}, \\
\omega_3 &= -\int \frac{du_1}{\sqrt{f(u_1)}} - \int \frac{du_2}{\sqrt{f(u_2)}} - \int \frac{du_3}{\sqrt{f(u_3)}}.
\end{aligned}$$

The equation of motion $\omega_3 = \mathrm{const}$ involves a holomorphic differential on the elliptic curve and, therefore, it is equivalent to an arithmetic equation for divisors on X

$$P_1(t) + P_2(t) + P_3(t) = P_4. \tag{5.3}$$

This equation describes the swing of parabola Y

$$Y: \quad y = \mathcal{P}(x), \qquad \mathcal{P}(x) = b_2(t)x^2 + b_1(t)x + b_0(t)$$

around some fixed point P_4 on X. Because four points P_1, P_2, P_3 and $-P_4$ form an intersection divisor of X and Y, we can calculate three coefficients b_2, b_1 and b_0 by solving three equations

$$y_1 = b_2x_1^2 + b_1x_1 + b_0, \quad y_2 = b_2x_2^2 + b_1x_2 + b_0, \quad y_3 = b_2x_3^2 + b_1x_3 + b_0.$$

Substituting $y = \mathcal{P}(x)$ into the definition $y^2 - f(x) = 0$ of X, we obtain Abel's polynomial

$$\psi(x) = f(x) - \mathcal{P}^2(x) = (a_4 - b_2^2)(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

Evaluating the coefficients of this polynomial, we find the coordinates of the fixed point

$$x_4 = -x_1 - x_2 - x_3 - \frac{a_3 - 2b_1b_2}{a_4 - b_2^2}, \quad y_4 = -\mathcal{P}(x_4),$$

which are constants of divisor motion (5.3) on the elliptic curve X.

The corresponding rational functions on phase space $T^*\mathbb{R}^3$

$$x_4 = -u_1 - u_2 - u_3 - \frac{\beta - 2b_1b_2}{\alpha - b_2^2}, \qquad y_4 = -(b_2x_4^2 + b_1x_4 + b_0), \tag{5.4}$$

where

$$b_2 = \frac{(u_2 - u_3)p_{u_1} + (u_3 - u_1)p_{u_2} + (u_1 - u_2)p_{u_3}}{(u_1 - u_2)(u_1 - u_2)(u_2 - u_3)},$$
$$b_1 = -\frac{(u_2^2 - u_3^2)p_{u_1} - (u_3^2 - u_1^2)p_{u_2} - (u_1^2 - u_2^2)p_{u_3}}{(u_1 - u_2)(u_1 - u_2)(u_2 - u_3)},$$
$$b_0 = \frac{u_2u_3(u_2 - u_3)p_{u_1} + u_1u_3(u_3 - u_1)p_{u_2} + u_1u_2(u_1 - u_2)p_{u_3}}{(u_1 - u_2)(u_1 - u_2)(u_2 - u_3)},$$

are first integrals of the dynamical system determined by the Hamiltonian $H(I_1, I_2, I_3)$ and canonical Poisson brackets.

After the symmetry breaking transformation (2.10)

$$p_{u_1} \to \frac{p_{u_1}}{m}, \qquad p_{u_2} \to \frac{p_{u_2}}{n}, \qquad p_{u_3} \to \frac{p_{u_3}}{k}$$

the equation of motion (5.3) on X becomes

$$[m]P_1(t) + [n]P_2(t) + [k]P_3(t) = P_4, \qquad m, n, k \in \mathbb{Z}.$$

The affine coordinates of the constant part of the intersection divisor are given by

$$x_4 = -[m]x_1 - [n]x_2 - [k]x_3 - \frac{a_3 - 2b_1b_2}{a_4 - b_2^2}, \quad y_4 = -\mathcal{P}(x_4),$$

where the parabola $Y : y = \mathcal{P}(x)$ is now defined by using Lagrange interpolation by movable points $[m]P_1$, $[n]P_2$ and $[k]P_3$ on the elliptic curve X, where

$$P_1 = (u_1, p_{u_1}/m), \qquad P_2 = (u_2, p_{u_2}/n), \qquad P_3 = (u_3, p_{u_3}/k).$$

Proposition 4. *Functions I_1, I_2, I_3 (5.2) and x_4, y_4 (5.4) in phase space $T^*\mathbb{R}^3$ can be considered as a representation of the following algebra of the first integrals:*

$$\begin{array}{llll} \{I_1, I_2\} = 0, & \{I_1, I_3\} = 0, & \{I_1, x_3\} = 0, & \{I_1, y_3\} = 0, \\ \{I_2, I_3\} = 0, & \{I_2, x_3\} = 0, & & \{I_2, y_3\} = 0, \\ \{I_3, x_4\} = \Phi_y(x_4, y_4), & \{I_3, y_4\} = -\Phi_x(x_4, y_4), & & \{x_4, y_4\} = 1. \end{array} \tag{5.5}$$

Here

$$\Phi_y(x,y)=\frac{\partial\Phi(x,y)}{\partial y}=2y \quad \text{and} \quad \Phi_x(x,y)=\frac{\partial\Phi(x,y)}{\partial x}=-(4\alpha x^3+3\beta x^2+2I_1x+I_2)$$

are derivatives of function $\Phi(x,y)$ from the definition of the elliptic curve X (5.1) and $\{.,.\}$ is the canonical Poisson bracket.

This algebra is derived from the Poisson bracket between the corresponding action-angle variables. We also have a computer-assisted proof of this proposition at $m=n=k$.

The algebra of the first integrals (5.5) slightly differs from the algebra (2.17) in the Kepler case. Abel's subalgebra of (5.5) consists of two elements I_1 and I_2, whereas Abel's subalgebra of (2.17) has only one central element I_1.

Summing up, we can construct six families of superintegrable systems using elliptic curves of the form $X:\Phi(x,y)=y^2-f^{(k)}(x)$, where

$$\begin{aligned}
&f^{(1)}(x)=\alpha x^4+\beta x^3+I_1x^2+I_2x+I_3, && f^{(2)}(x)=\alpha x^4+I_1x^3+\beta x^2+I_2x+I_3,\\
&f^{(3)}(x)=\alpha x^4+I_1x^3+I_2x^2+\beta x+I_3, && f^{(4)}(x)=I_1x^4+\alpha x^3+\beta x^2+I_2x+I_3,\\
&f^{(5)}(x)=I_1x^4+\alpha x^3+I_2x^2+\beta x+I_3, && f^{(6)}(x)=I_1x^4+I_2x^3+\alpha x^2+\beta I_2x+I_3,
\end{aligned}$$

and the intersection divisor equation of motion

$$[m]P_1(t)+[n]P_2(t)+[k]P_3(t)=P_4, \qquad m,n,k\in\mathbb{Z}.$$

For all these superintegrable systems the algebra of the first integrals has the standard form (5.5) which directly follows from the Poisson brackets between action-angle variables. Following [29], we can say that the Hamiltonians I_1, I_2 and I_3 describe the motion of the body in $T^*\mathbb{R}^3$ with a position-dependent mass.

6. CONCLUSION

The equations of motion

$$\dot z_i=\{H,z_i\}, \qquad z_i\in T^*\mathbb{R}^n$$

for the Stäckel systems on a symmetrized product $X\times\cdots\times X$ of the hyperelliptic curve X are equivalent to the equation of motion

$$\mathrm{div}X\cdot Y(t)=0$$

describing the evolution of the intersection divisor of X and the auxiliary curve $Y(t)$. For superintegrable Stäckel systems, the intersection divisor can be divided into moving and fixed parts

$$\mathrm{div}X\cdot Y(t)=D(t)+D'=0,$$

according to Abel's theorem. It is clear that the constants of divisor motion are the coordinates of fixed part D' of the intersection divisor, and integrals of motion in phase space are some functions on these constants of divisor motion. Thus, the algebra of the first integrals in phase space can be obtained from the algebra of the constants of divisor motion, which is easily obtained from the Poisson brackets between the Stäckel action-angle variables.

In this note we calculate the algebra of the constants of divisor motion associated with the Kepler problem, harmonic oscillator, Drach system, Stäckel systems with two and three degrees of freedom and some of their deformations associated with symmetry breaking transformations of the Stäckel matrices. All these systems are related to various elliptic curves, but we can rewrite the corresponding algebras of nonpolynomial integrals in a common form.

Scalar multiplication of points on elliptic curves

$$\varphi:\quad X\to X, \qquad \varphi(P)=[m]P$$

generates a noncanonical transformation of phase space

$$\psi:\quad T^*\mathbb{R}^n\to T^*\mathbb{R}^n, \qquad \psi(u_i)=u_i, \quad \psi(p_{u_i})=m_ip_{u_i},$$

which changes the form of the kinetic part of the Hamiltonian, but preserves its superintegrability property. Because multiplication of points on X is a special case of isogenies between elliptic curves, we suppose that isogeny arithmetics also generates noncanonical transformations of phase space

$$\psi: \qquad T^*\mathbb{R}^n \to T^*\mathbb{R}^n$$

preserving superintegrability. If this conjecture is true, than the isogeny volcanoes [30] could generate superintegrable system volcanoes. In a forthcoming publication, we will discuss this conjecture and application of Landen – Gauss transformations (2-isogenies) and Vélu's transformations [32, 45] to the construction of superintegrable systems associated with elliptic curves.

For superintegrable Stäckel systems on hyperelliptic curves of genus two we have affine coordinates of divisors, Mumford's coordinates of divisors, modified Jacobian coordinates, Chudnovski-Jacobian coordinates, mixed coordinates, etc. First integrals associated with these coordinates could be algebraic, rational or polynomial functions in phase space satisfying various polynomial or nonpolynomial relations. It is interesting to study these relations associated with various constants of the divisor motion.

The class of superintegrable or degenerate systems is closely related to the class of bi-Hamiltonian systems with equations of motion

$$\frac{d}{dt}z_i = \{I_1, z_i\} = \{I_2, z_i\}',$$

see [15] and references therein. Thus, we have two algebras of divisor motion constants with respect to compatible Poisson brackets $\{.,.\}$ and $\{.,.\}'$. We suppose that both algebras of the first integrals are similar to the "Hamiltonian equation of motion" with respect to the "Hamiltonian" $\Phi(x, y)$ (2.18).

FUNDING

This work was supported by the Russian Science Foundation (project 18-11-00032).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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