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# Combinatorial Ricci Flow for Degenerate Circle Packing Metrics

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**Abstract**—Chow and Luo [3] showed in 2003 that the combinatorial analogue of the Hamilton Ricci flow on surfaces converges under certain conditions to Thruston's circle packing metric of constant curvature. The combinatorial setting includes weights defined for edges of a triangulation. A crucial assumption in [3] was that the weights are nonnegative. Recently we have shown that the same statement on convergence can be proved under a weaker condition: some weights can be negative and should satisfy certain inequalities [4].
On the other hand, for weights not satisfying conditions of Chow – Luo's theorem we observed in numerical simulation a degeneration of the metric with certain regular behaviour patterns [5].
In this note we introduce degenerate circle packing metrics, and under weakened conditions on weights we prove that under certain assumptions for any initial metric an analogue of the combinatorial Ricci flow has a unique limit metric with a constant curvature outside of singularities.

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## 1. INTRODUCTION

The Ricci flow on a two-dimensional closed surface was introduced by R. Hamilton in [1]. The solution for the Ricci flow exists for all time, and after a suitable normalization the solution converges to a constant curvature metric as time goes to infinity [1, 2].

The combinatorial Ricci flow for triangulated surfaces was introduced by Chow and Luo in [3]. They gave a complete description of the asymptotic behavior of the solution to the combinatorial Ricci flow under certain assumptions. Both the Euclidean and the hyperbolic background geometry were considered.

Let us give a brief description of their results. Consider a closed surface $X$ with a triangulation $T$. Let $V = \{A_1, \ldots, A_N\}$ be the set of vertices of $T$. The set of all edges and faces is denoted by $E$ and $F$, respectively. A *weight* is a function $w : E \to (-1, 1]$. Given a triple $(X, T, w)$, Thurston defined *a circle packing metric* in the following way (see [6]). To each vertex $A_j \in V$ assign a number $r_j > 0$. For the Euclidean background metric assume that each face is a flat Euclidean triangle. For an edge $e_{ij}$ joining vertices $A_i$ and $A_j$ define its length

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j w_{ij}},$$

where $w_{ij} = w_{ji} = w(A_iA_j)$. A *circle packing metric* is a collection $r = \{r_j > 0 \ : \ j = 1, \ldots, N\}$. Clearly, only those $r$ should be considered which define the collection $\{l_{ij}\}$ so that the triangle

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inequalities are satisfied on each face of triangulation. It is not difficult to show that for nonnegative weights any choice of $r = \{r_j > 0 \ : \ j = 1, \ldots, N\}$ satisfies this condition, see [6].

*The curvature* $K_i$ at $A_i$ is defined to be

$$K_i = 2\pi - \sum_{\triangle A_i A_j A_k \in F} \angle A_k A_i A_j. \tag{1.1}$$

*The combinatorial Ricci flow* (as appeared in [3] for the first time) in the Euclidean background is the system of ODE

$$\frac{dr_i}{dt} = -K_i r_i, \ i = 1, \ldots, N. \tag{1.2}$$

For the Euclidean background metric it is useful to consider *the normalized Ricci flow*

$$\frac{dr_i}{dt} = -(K_i - K^{av}) r_i, \ i = 1, \ldots, N, \tag{1.3}$$

where $K^{av} = \frac{2\pi\chi(X)}{N}$. A function $r_i(t)$ is a solution to the Ricci flow (1.2) iff $e^{K^{av}} r_i(t)$ is a solution to the normalized Ricci flow (1.3). The circle packing metrics of constant curvature $K_j = K^{av}$ are the equilibrium points of (1.3). The product $\prod_{i=1}^{N} r_i$ is a first integral of the normalized Ricci flow.

Assume $I \subset V$ is a proper subset of vertices. Let $F_I$ be the subcomplex formed by simplices with vertices from $I$. Let $Lk(I)$ be the set of pairs $(v, e)$, $v \in I$, $e \in E$, such that both end points of $e$ are not in $I$ and $v, e$ form a triangle.

**Theorem 1 (see [3]).** *Suppose* $(X, T, w)$ *is a triangulation* $T$ *of a closed connected surface* $X$ *with a fixed weight function* $w \geqslant 0$. *Then for any initial metric* $r(0)$ *the solution* $r(t)$ *to the normalized combinatorial Ricci flow* (1.3)*:*

(*a*) *exists for all* $t \geqslant 0$*;*

(*b*) *converges to a metric of constant curvature* $K^{av} = \frac{2\pi\chi(X)}{N}$ *iff for any proper subset* $I \subset V$

$$2\pi|I|\chi(X)/|V| > -\sum_{(e,v)\in Lk(I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_I); \tag{1.4}$$

(*c*) *converges to a metric of constant curvature exponentially fast under conditions* (1.4) *of statement* (*b*).

For surfaces of negative Euler characteristic it is natural to consider a piece-wise hyperbolic metric. Namely, assume that each face is a triangle with the hyperbolic metric of constant curvature $-1$. The length of the edge $e_{ij}$ joining vertices $A_i$ and $A_j$ is defined by the equation

$$\cosh l_{ij} = \cosh r_i \cosh r_j + \sinh r_i \sinh r_j w_{ij}.$$

It is not difficult to show that for nonnegative weights any choice of $r = \{r_j > 0 \ : \ j = 1, \ldots, N\}$ defines the collection $\{l_{ij}\}$ so that the triangle inequalities are satisfied on each face of triangulation, see [6].

The Ricci flow is the system of ODE

$$\frac{dr_i}{dt} = -K_i \sinh r_i, \ i = 1, \ldots, N. \tag{1.5}$$

*The curvature* (or, to be more precise, the defect of the curvature) at $A_i$ is defined by the formula (1.1).

**Theorem 2 (see [3]).** *Suppose $(X, T, w)$ is a triangulation $T$ of a closed connected surface $X$ of negative Euler characteristic with a fixed weight function $w \geqslant 0$. Then for any initial metric $r(0)$ the solution $r(t)$ to the normalized Ricci flow* (1.5)*:*

(*a*) *exists for all $t \geqslant 0$;*

(*b*) *converges iff the following two conditions hold:*

(*b1*) *if for three edges $e_1, e_2, e_3$ forming a null-homotopic loop in $X$ one has $\sum_{j=1}^{3} \arccos w(e_j) \geqslant \pi$, then these edges form the boundary of a triangle of $T$,*

(*b2*) *if for four edges $e_1, e_2, e_3, e_4$ forming a null-homotopic loop in $X$ one has $\sum_{j=1}^{4} \arccos w(e_j) \geqslant 2\pi$, then these edges form the boundary of the union of two adjacent triangles of $T$;*

(*c*) *converges to a hyperbolic metric with all $K_i = 0$, provided conditions* (*b1*) *and* (*b2*) *hold.*

The proofs of both theorems are based on representation of the Ricci flow as a negative gradient flow of some convex function. In its own turn the convexity of the function is based on the following fact from elementary geometry (Lemma 13.7.3 from Thurston's book [6], see also [3] Lemma 2.2). Denote by $\theta_i, \theta_j$ and $\theta_k$ the inner angles of the triangle $\Delta A_i A_j A_k$. Then under the assumption $w \in [0, 1]$ we have $\partial\theta_p/\partial r_p < 0$ and $\partial\theta_p/\partial r_q > 0$ for $p, q \in \{i, j, k\}$, $p \neq q$. In our paper [4] we proved that the same inequalities are true if for any face of the triangulation either all weights are nonnegative or only one weight is negative, for instance, $\alpha < 0$, and the other two, $\beta$ and $\gamma$, are positive, so that $\alpha + \beta\gamma \geqslant 0$. This gives a partial answer to the question from [3] for what choices of a weight function the statements of Theorems 1 and 2 can be proved. The same conditions were independently found by Zhou, see [7], and were used by Xu in [8].

On the other hand, in our paper [5] we found several examples for which the statement of Theorem 1 does not hold. Namely, for the tetrahedron $A_0A_1A_2A_3$ with the weights $w_{01} = w_{23} = -0.8$, $w_{02} = w_{12} = w_{03} = w_{13} = -0.7$ there exist 5 different equilibrium points of (1.2), that is, 5 different constant curvature metrics. One of them is a sink, and the other four are saddle points. In every saddle point there are two nonattracting separatrices. One of them tends to the sink and along the other the metric degenerates in such a way that one radius $r_j$ tends to zero, while the other three radii tend to infinity, and at the same time the curvatures have finite limits. These observations suggest that there should exist a limiting circle packing metric which is in some sense degenerate.

In the present paper we define a degenerate circle packing metric and prove analogues of convergence theorems for both the Euclidean and the hyperbolic background geometry under weakened assumptions on the weight function $w$. Apparently, these metrics are the limits which were discussed in problem 3 of Section 7 of [3].

## 2. BASIC DEFINITIONS

Suppose $T$ is a triangulation of a closed surface $X$. We assume that a lift of a closed face or an edge to the universal cover $\tilde{X}$ is an embedding. Denote the sets of vertices, edges and faces of $T$ by $V$, $E$, $F$, respectively. Divide the set of vertices into a disjoint union $V = V_n \sqcup V_d$, such that there is no edge connecting two vertices from $V_d$. Without loss of generality we can assume $V_n = \{A_1, \ldots, A_M\}$ and $V_d = \{A_{M+1}, \ldots, A_N\}$. Vertices from $V_n$ are called *nondegenerate* and vertices from $V_d$ are called *degenerate*. Call a cell of $T$ (that is, an edge or a face) *nondegenerate* iff all its vertices are nondegenerate, and *degenerate* otherwise. Denote the set of (non)degenerate edges and faces by $E_d$ ($E_n$) and $F_d$ ($F_n$), respectively. Clearly, $E = E_n \sqcup E_d$ and $F = F_n \sqcup F_d$. Sometimes it is useful to denote a subset of vertices and the corresponding subset of indices by the same symbol.

A weight is a function $w : E_n \to (-1, 1]$. Fix a triple $(X, T, w)$. A (degenerate) circle packing metric is defined by a collection of numbers $r = (r_1, r_2, \ldots, r_N)$, where $r_j > 0$ for $1 \leqslant j \leqslant M$ and $r_j = 0$ for $M + 1 \leqslant j \leqslant N$. This definition differs from the classical circle packing metric where all

$r_j$ are positive, see [3, 6]. For the Euclidean background define the length of an edge connecting two vertices $A_i$ and $A_j$ by the formula

$$l_{ij}^2 = r_i^2 + r_j^2 + 2r_i r_j w_{ij}. \tag{2.1}$$

For a degenerate edge one of the numbers $r_i$ or $r_j$ is zero, therefore the last term is assumed to be zero, although the weight $w_{ij}$ is not defined. Moreover, if $r_i = 0$, then $l_{ij} = r_j$. The curvature $K_i$ at the vertex $A_i$ is defined as usual by formula (1.1).

The curvature at a degenerate vertex does not depend on $r$ and can be expressed in terms of the weight $w$. Indeed, let $A_i \in V_d$ and $\triangle A_i A_j A_k \in F$. Then $A_j, A_k \in V_n$. By the cosine law $\cos\angle A_j A_i A_k = -w_{jk}$, hence $\angle A_j A_i A_k = \pi - \arccos(w_{jk})$. Therefore, for the curvature $K_i$ we have the expression

$$K_i = 2\pi - \sum_{\triangle A_i A_j A_k \in F} \big(\pi - \arccos(w_{jk})\big). \tag{2.2}$$

The combinatorial Ricci flow is the system of ODE

$$\frac{dr_i}{dt} = -K_i r_i,\ i = 1, \ldots, M. \tag{2.3}$$

For $i = M+1, M+2, \ldots, N$ one has $r_i = \frac{dr_i}{dt} = 0$, therefore in (2.3) one can assume $1 \leqslant i \leqslant N$.

For a degenerate metric define the *averaged curvature* $K^{av}$:

$$K^{av} = \frac{1}{M}\left(2\pi\chi(X) - \sum_{j=M+1}^{N} K_j\right). \tag{2.4}$$

The *normalized combinatorial Ricci flow* is the system of ODE

$$\frac{dr_i}{dt} = -(K_i - K^{av})r_i,\ i = 1, \ldots, M. \tag{2.5}$$

The normalized and nonnormalized Ricci flows are in a certain sense equivalent.

**Lemma 1.** *Functions $r_i(t), i = 1, \ldots, M,$ are the solution of* (2.3) *iff functions $e^{K^{av}t} r_i(t)$ are the solution of* (2.5).

*Proof.* Let $\hat{r}_i(t) = e^{K^{av}t} r_i(t)$. Then

$$\frac{d\hat{r}_i(t)}{dt} = K^{av} e^{K^{av}t} r_i(t) + e^{K^{av}}\big(-K_i r_i(t)\big) = (K^{av} - K_i)e^{K^{av}} r_i(t) = -(K_i - K^{av})\hat{r}_i(t).$$

□

The normalized combinatorial Ricci flow has the following useful property.

**Lemma 2.** *The product $\prod_{j=1}^{M} r_j(t)$ is a first integral for* (2.5).

*Proof.*

$$\frac{d\left(\prod_{j=1}^{M} r_j(t)\right)}{dt} = \left(\sum_{j=1}^{M} -(K_j - K^{av})\right)\prod_{j=1}^{M} r_j(t) = 0.$$

□

For the hyperbolic background geometry the length of the edge $e_{ij}$ joining vertices $A_i$ and $A_j$ is defined by the equation

$$\cosh l_{ij} = \cosh r_i \cosh r_j + \sinh r_i \sinh r_j w_{ij}. \tag{2.6}$$

As in the Euclidean case for a degenerate edge one of the radii $r_i$ or $r_j$ is zero, so the last term is assumed to be zero, though the weight $w_{ij}$ is undefined. Clearly, for $r_i = 0$ one has $l_{ij} = r_j$. The curvature $K_i$ at the vertex $A_i$ is defined as usual by formula (1.1). The curvature at a degenerate vertex $A_i$, $M+1 \leqslant i \leqslant N$, is given by (2.2).

The (hyperbolic) combinatorial Ricci flow is the system of ODE

$$\frac{dr_i}{dt} = -K_i \sinh r_i,\ i = 1, \ldots, M. \tag{2.7}$$

For $i = M+1, M+2, \ldots, N$ one has $r_i = \frac{dr_i}{dt} = 0$, hence in (2.7) one can assume $1 \leqslant i \leqslant N$.

**Remark 1.** Equations (2.1) and (2.6) explain why it is natural to assume that no pair of degenerate vertices is connected by an edge. Indeed, otherwise the length of such an edge should be zero, and this edge should be shrunk to a point. A face with three degenerate vertices should be shrunk to a point, while a triangle with exactly two degenerate vertices should be shrunk to a segment. This means that the initial triangulation $T$ should be modified.

It is very important to determine the combinatorial type of the triangulation which appears at the limit as time tends to infinity when Theorem 1 (or Theorem 2) is not applicable. We do not address this question here.

Now we investigate the space of metrics for a triple $(X, T, w)$. Denote by $\mathcal{R}_w$ the set of all $r \in \mathbb{R}^M \times (0, \ldots, 0) \subset \mathbb{R}^N$ such that for every face of the triangulation the triangle inequalities hold.

**Lemma 3.** *Suppose any face of the triangulation satisfies one of the following conditions:*

(*a*) *the face is nondegenerate and all the weights of its edges are nonnegative;*

(*b*) *the face is nondegenerate, exactly one weight* $\alpha$ *of its edges is negative, the other two weights,* $\beta$ *and* $\gamma$*, are positive, and* $\alpha + \beta\gamma \geqslant 0$*;*

(*c*) *the face is degenerate and the weight of the nondegenerate edge of the face is not equal to* 1.

*Then for both the Euclidean and the hyperbolic background geometry one has* $\mathcal{R}_w = \mathbb{R}^M_+$.

*Proof.* We need to check the triangle inequalities for any $r$. For a face satisfying (a) it was done in Lemma 13.7.2 [6]. For a face satisfying (b) it was proved in [4]. For a face $\triangle A_i A_j A_k$ with the degenerate vertex $A_i$ it follows from the equalities $l_{ij} = r_j$, $l_{ik} = r_k$ and (2.1) or (2.6). □

*We say that condition* (*W*) *is satisfied for a weight function* $w$ *iff the condition of Lemma 3 is satisfied.*

The following statement plays a key role in the proof of the main properties of the combinatorial Ricci flow.

**Lemma 4.** *Let* $\triangle A_i A_j A_k$ *be a face of the triangulation* $T$*, and* $\theta_i$, $\theta_j$, $\theta_k$ *be the angles of this face at the vertices* $A_i$, $A_j$, $A_k$*, respectively. Suppose condition* $(W)$ *is satisfied.*

(*a*) *Suppose* $\triangle A_i A_j A_k \in F_n$. *Then*

(*a*1) $\frac{\partial \theta_p}{\partial r_p} < 0$, $p \in \{i, j, k\}$;

(*a*2) $\frac{\partial \theta_p}{\partial r_q} > 0$, $p, q \in \{i, j, k\}$, $p \neq q$;

(*a*3) $\frac{\partial(\theta_i+\theta_j+\theta_k)}{\partial r_p} = 0$ *for the Euclidean case and* $\frac{\partial(\theta_i+\theta_j+\theta_k)}{\partial r_p} < 0$ *for the hyperbolic case,* $p \in \{i, j, k\}$.

(*b*) *Suppose* $A_i \in V_d$. *Then*

(*b*1) $\frac{\partial \theta_j}{\partial r_j} < 0$ *and* $\frac{\partial \theta_k}{\partial r_k} < 0$;

(*b*2) $\frac{\partial \theta_j}{\partial r_k} > 0$ *and* $\frac{\partial \theta_k}{\partial r_j} > 0$;

(*b*3) $\frac{\partial(\theta_j+\theta_k)}{\partial r_p} = \frac{\partial(\theta_i+\theta_j+\theta_k)}{\partial r_p} = 0$ *for the Euclidean case, and* $\frac{\partial(\theta_j+\theta_k)}{\partial r_p} = \frac{\partial(\theta_i+\theta_j+\theta_k)}{\partial r_p} < 0$ *for the hyperbolic case,* $p \in \{j, k\}$.

*Moreover, in both cases the partial derivatives* $\frac{\partial \theta_n}{\partial r_m}$ *are elementary functions of* $r_i, r_j, r_k$*, where* $n, m \in \{i, j, k\}$.

*Proof.* (*a*) For a nondegenerate face with nonnegative weights see Lemma 13.7.3 from [6]. Other cases of nondegenerate faces were considered in [4].

(*b*) In the Euclidean case by the cosine law

$$r_k^2 = r_j^2 + l_{jk}^2 - 2r_j l_{jk}\cos\theta_j = r_j^2 + r_j^2 + r_k^2 + 2w_{jk}r_jr_k - 2r_j\sqrt{r_j^2 + r_k^2 + 2w_{jk}r_kr_j}\cos\theta_j.$$

Since

$$\frac{\partial\cos\theta_j}{\partial r_j} = -\sin\theta_j\frac{\partial\theta_j}{\partial r_j},$$

the sign of the derivative $\frac{\partial\theta_j}{\partial r_j}$ is opposite to the sign of the derivative $\frac{\partial\cos\theta_j}{\partial r_j}$. By the formula for a derivative of a fraction the sign of $\frac{\partial\cos\theta_j}{\partial r_j}$ coincides with the sign of

$$\begin{aligned}(r_j + w_{jk}r_k)'_{r_j}(r_j^2 + r_k^2 + 2w_{jk}r_jr_k) - \frac{1}{2}(r_j + w_{jk}r_k)(r_j^2 + r_k^2 + 2w_{jk}r_kr_j)'_{r_j}\\ = (r_j^2 + r_k^2 + 2w_{jk}r_jr_k) - (r_j + w_{jk}r_k)^2\\ = r_j^2 + r_k^2 + 2w_{jk}r_jr_k - r_j^2 - r_k^2w_{jk}^2 - 2r_jr_kw_{jk} = r_k^2(1 - w_{jk}^2).\end{aligned}$$

Therefore,

$$\frac{\partial\theta_j}{\partial r_j} < 0.$$

Similarly, the sign of $\frac{\partial\cos\theta_j}{\partial r_k}$ coincides with the sign of

$$\begin{aligned}(r_j + w_{jk}r_k)'_{r_k}(r_j^2 + r_k^2 + 2w_{jk}r_jr_k) - \frac{1}{2}(r_j + w_{jk}r_k)(r_j^2 + r_k^2 + 2w_{jk}r_kr_j)'_{r_k}\\ = w_{jk}(r_j^2 + r_k^2 + 2w_{jk}r_jr_k) - (r_j + w_{jk}r_k)(r_k + w_{jk}r_j)\\ = w_{jk}r_j^2 + w_{jk}r_k^2 + 2w_{jk}^2r_jr_k - r_jr_k - w_{jk}r_j^2 - w_{jk}r_k^2 - w_{jk}^2r_kr_j = r_jr_k(w_{jk}^2 - 1).\end{aligned}$$

Therefore,

$$\frac{\partial\theta_j}{\partial r_k} > 0.$$

The hyperbolic case is considered in the same way. A detailed calculation of the derivatives can be found in the proof of Lemma 5. □

**Lemma 5.** *Suppose the vertices $A_j$ and $A_k$ of a face $\triangle A_iA_jA_k$ are nondegenerate. Then*

$$\begin{aligned}\frac{\partial\theta_k}{\partial r_j}r_j &= \frac{\partial\theta_j}{\partial r_k}r_k \qquad \textit{for the Euclidean case,}\\ \frac{\partial\theta_k}{\partial r_j}\sinh r_j &= \frac{\partial\theta_j}{\partial r_k}\sinh r_k \qquad \textit{for the hyperbolic case.}\end{aligned} \tag{2.8}$$

*Proof.* For a nondegenerate face, equality (2.8) holds for three pairs of vertices. This case was considered in Lemma 2.3 of [3].

Now assume the vertex $A_i$ is degenerate. For the Euclidean background geometry using the notation from Lemma 4 we have

$$\cos\theta_j = \frac{r_j + w_{jk}r_k}{\sqrt{r_j^2 + r_k^2 + 2w_{jk}r_jr_k}}.$$

Similarly to the proof of Lemma 4, we obtain

$$-\sin\theta_j\frac{\partial\theta_j}{\partial r_k} = \frac{\partial\cos\theta_j}{\partial r_k} = \frac{r_jr_k(w_{jk}^2 - 1)}{(r_j^2 + r_k^2 + 2w_{jk}r_jr_k)^{3/2}}.$$

Consequently,

$$r_k \frac{\partial \theta_j}{\partial r_k} = -\frac{r_k}{\sin\theta_j} \cdot \frac{r_j r_k (w_{jk}^2 - 1)}{(r_j^2 + r_k^2 + 2w_{jk} r_j r_k)^{3/2}}$$

and

$$r_j \frac{\partial \theta_k}{\partial r_j} = -\frac{r_j}{\sin\theta_k} \cdot \frac{r_j r_k (w_{jk}^2 - 1)}{(r_j^2 + r_k^2 + 2w_{jk} r_j r_k)^{3/2}}.$$

To complete the proof in the Euclidean case, we note that by the sine law one has

$$\frac{r_j}{\sin\theta_k} = \frac{r_k}{\sin\theta_j}.$$

For the hyperbolic background we proceed as follows. For a triangle $\triangle A_i A_j A_k$ denote by $x_i, x_j, x_k$ the lengths of the edges opposite to the vertices $A_i, A_j, A_k$, respectively. By the sine law, the quantity $A_{ijk} = \sinh x_i \sinh x_j \sin\theta_k$ is symmetric in indices $i, j, k$. By Lemma A1 of [3] we have two equalities

$$\frac{\partial \theta_k}{\partial x_k} = \frac{\sinh x_k}{A_{ijk}} \text{ and } \frac{\partial \theta_k}{\partial x_i} = -\frac{\partial \theta_k}{\partial x_k} \cos\theta_j.$$

In our setting $x_i = l_{jk}, x_j = r_k, x_k = r_j$. Note also $\frac{\partial l_{jk}}{\partial r_j} = \frac{1}{\sinh l_{jk}} \frac{\partial \cosh l_{jk}}{\partial r_j}$. Then

$$\begin{aligned}
\frac{\partial \theta_k}{\partial r_j} \sinh r_j &= \sinh r_j \frac{\sinh x_k}{A_{ijk}} \left(1 - \cos\theta_j \frac{\partial x_i}{\partial r_j}\right) \\
&= \frac{1}{A_{ijk} \sinh^2 l_{jk}} \big( \sinh^2 r_j \sinh^2 l_{jk} \\
&\qquad - \sinh r_j (\cosh r_j \cosh l_{jk} - \cosh r_k)(\sinh r_j \cosh r_k + w_{jk} \cosh r_j \sinh r_k) \big) \\
&= \frac{(1 - w_{jk}^2) \sinh^2 r_j \sinh^2 r_k}{A_{ijk} \sinh^2 l_{jk}},
\end{aligned}$$

where the rightmost quantity is clearly symmetric in indices $j$ and $k$. □

**Proposition 1.** *Let $(X, T, w)$ be a closed surface $X$ with a triangulation $T$ and a weight function $w$. Suppose $r_i(t), i = 1, \dots, M$, satisfy the equations $\frac{dr_i}{dt} = -L_i(r_1, \dots, r_M) s(r_i)$, where $s(r_i) = r_i$ in the Euclidean case and $s(r_i) = \sinh r_i$ in the hyperbolic case. Then the time derivative of the inner angle $\theta_i^{jk}$ at the vertex $A_i \in V_n$ of a face $\triangle A_i A_j A_k$ can be written as*

(1) $\frac{d\theta_i^{jk}}{dt} = -B_{ij}(L_j - L_i) - B_{ik}(L_k - L_i) - B_i \lambda L_i$ *for* $\triangle A_i A_j A_k \in F_n$*;*

(2) $\frac{d\theta_i^{jk}}{dt} = -B_{ij}(L_j - L_i) - B_i \lambda L_i$ *for* $\triangle A_i A_j A_k \in F_d$ *with* $A_k \in V_d$.

*Assume, in addition, that $w$ satisfies condition $(W)$. Then $B_{ij} = B_{ji}$ and $B_i$ are positive elementary functions for all $i, j = 1, \dots, M, i \neq j$. Here $\lambda = 0$ for the Euclidean background and $\lambda = -1$ for the hyperbolic background.*

*Proof.* Case (1) was proved in [3], Proposition 2.4. For convenience we recall the proof:

$$\begin{aligned}
\frac{d\theta_i^{jk}}{dt} &= \frac{\partial \theta_i^{jk}}{\partial r_i} r_i' + \frac{d\theta_i^{jk}}{\partial r_j} r_j' + \frac{d\theta_i^{jk}}{\partial r_k} r_k' = -\frac{\partial \theta_i^{jk}}{\partial r_i} L_i s(r_i) - \frac{d\theta_i^{jk}}{\partial r_j} L_j s(r_j) - \frac{d\theta_i^{jk}}{\partial r_k} L_k s(r_k) \\
&= -\frac{d\theta_i^{jk}}{\partial r_j} (L_j - L_i) s(r_j) - \frac{d\theta_i^{jk}}{\partial r_k} (L_k - L_i) s(r_k) - \left( \frac{\partial \theta_i^{jk}}{\partial r_i} s(r_i) + \frac{\partial \theta_i^{jk}}{\partial r_j} s(r_j) + \frac{\partial \theta_i^{jk}}{\partial r_k} s(r_k) \right) L_i \\
&= -\frac{\partial \theta_i^{jk}}{\partial r_j} (L_j - L_i) s(r_j) - \frac{\partial \theta_i^{jk}}{\partial r_k} (L_k - L_i) s(r_j) - \left( \frac{\partial \theta_i^{jk}}{\partial r_i} s(r_i) + \frac{\partial \theta_j^{ik}}{\partial r_i} s(r_i) + \frac{\partial \theta_k^{ij}}{\partial r_i} s(r_i) \right) L_i \\
&= -B_{ij}(L_j - L_i) - B_{jk}(L_k - L_i) - B_i L_i \lambda.
\end{aligned}$$

For case (2) assume $A_k \in V_d$. Then

$$\frac{d\theta_i^{jk}}{dt} = \frac{\partial\theta_i^{jk}}{\partial r_i} r_i' + \frac{\partial\theta_i^{jk}}{\partial r_j} r_j' = -\frac{\partial\theta_i^{jk}}{\partial r_i} L_i s(r_i) - \frac{\partial\theta_i^{jk}}{\partial r_j} s(r_j) L_j$$

$$= -\frac{\partial\theta_i^{jk}}{\partial r_j}(L_j - L_i)s(r_j) - \left(\frac{\partial\theta_i^{jk}}{\partial r_i} L_i s(r_i) + \frac{\partial\theta_i^{jk}}{\partial r_j} L_i s(r_j)\right)$$

$$= -\frac{\partial\theta_i^{jk}}{\partial r_j}(L_j - L_i)s(r_j) - \left(\frac{\partial\theta_i^{jk}}{\partial r_i} L_i s(r_i) + \frac{\theta_j^{ik}}{\partial r_i} L_i s(r_i)\right)$$

$$= -\frac{\partial\theta_i^{jk}}{\partial r_j}(L_j - L_i)s(r_j) - \left(\frac{\partial(\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij})}{\partial r_i}\right) L_i s(r_i)$$

$$= -B_{ij}(L_j - L_i) - B_i L_i \lambda.$$

□

From Proposition 1 we deduce the evolution of curvatures at nondegenerate vertices.

**Proposition 2.** *Assume $w$ satisfies condition $(W)$. Then in the assumptions of Proposition 1 for $1 \leqslant i \leqslant M$ one has*

$$\frac{dK_i}{dt} = \sum_{i \backsim j, j \leqslant M} C_{ij}(L_j - L_i) + \lambda C_i L_i,$$

*where $C_{ij} = C_{ji}$ and $C_i$ are positive elementary functions in $r_1, \ldots, r_M$, and the summation is over all nondegenerate vertices $A_j$ adjacent to $A_i$.*

*Proof.* The statement follows directly from Proposition 1 since

$$\frac{dK_i}{dt} = -\sum_{\triangle A_i A_j A_k \in F} \frac{d\theta_i^{jk}}{dt}.$$

□

## 3. EXISTENCE OF THE SOLUTION TO THE RICCI FLOW FOR $t \in [0, \infty)$

Let $\overline{M}(t) = \max\big(K_1(t), \ldots, K_M(t)\big)$ and $\underline{M}(t) = \min\big(K_1(t), \ldots, K_M(t)\big)$. The solution to the combinatorial Ricci flow with any given initial metric exists for all $t \in [0, +\infty)$. This can be proved by the same argument as in [3] Section 3.4 using the following maximum principle.

**Proposition 3.** *Let $r(t) = \big(r_1(t), \ldots, r_M(t)\big)$ be a solution to the Ricci flow (2.5) or (2.7) on an interval. Then*

(1) *for the Euclidean geometry the function $\overline{M}(t)$ is nonincreasing and the function $\underline{M}(t)$ is nondecreasing;*

(2) *for the hyperboilc geometry the function $\max\big(0, \overline{M}(t)\big)$ is nonincreasing and the function $\min\big(0, \underline{M}(t)\big)$ is nondecreasing.*

**Proposition 4.** *Let $r(t) = \big(r_1(t), \ldots, r_M(t)\big)$ is a solution to the normalized Ricci flow (2.5). Suppose the curve $r(t)$ is contained in a compact subset of $\mathbb{R}_+^M$. Then $r(t)$ converges to a point in $\mathbb{R}_+^M$ such that the corresponding curvatures at nondegenerate vertices are equal to $K^{av} = \frac{1}{M}\Big(2\pi\chi(X) - \sum\limits_{j \geqslant M+1} K_j\Big)$. The convergence is exponentially fast.*

*Proof.* Consider the function

$$g(t)=\sum_{j=1}^{M}\big(K_j(t)-K^{av}\big)^2=\sum_{j=1}^{M}K_j^2(t)-M(K^{av})^2.$$

For its derivative we have

$$g'=2\sum_{j=1}^{M}K_j(t)K_j'(t).$$

Let us calculate $K_j'(t)$. By Proposition 2 for $L_j=(K_j-K^{av})$ we obtain

$$K_j'(t)=\sum_{j\backsim i,i\leqslant M}C_{ij}\big((K_j-K^{av})-(K_i-K^{av})\big)=\sum_{j\backsim i,i\leqslant M}C_{ij}(K_j-K_i).$$

Therefore,

$$g'(t)=2\sum_{j=1}^{M}\sum_{i\backsim j,i\leqslant M}C_{ji}K_j(K_i-K_j). \tag{3.1}$$

Interchanging indices $i$ and $j$, we obtain another expression

$$g'(t)=2\sum_{i=1}^{M}\sum_{j\backsim i,j\leqslant M}C_{ij}K_i(K_j-K_i). \tag{3.2}$$

The sum of (3.1) and (3.2) gives

$$g'(t)=-2\sum_{i\backsim j\leqslant M}C_{ij}(K_i-K_j)^2.$$

The curve $r(t)$ lies in a compact subset of $\mathbb{R}_+^M$, and the functions $C_{ij}$ are positive. Hence, there exists a constant $c_1>0$ such than for any adjacent vertices $A_i\backsim A_j$ one has the inequality $C_{ij}\big(r(t)\big)\geqslant c_1>0$.

The Cauchy inequality

$$\frac{\sum\limits_{j=1}^{M}y_j}{M}\leqslant\sqrt{\frac{\sum\limits_{j=1}^{M}y_j^2}{M}}$$

for $y_j=K_i-K_j$ gives

$$\left(\frac{\sum\limits_{j=1}^{M}(K_j-K_i)}{M}\right)^2\leqslant\frac{\sum\limits_{j=1}^{M}(K_i-K_j)^2}{M}.$$

Therefore,

$$(K_i-K^{av})^2\leqslant\frac{\sum\limits_{j=1}^{M}(K_i-K_j)^2}{M}.$$

Taking a sum on $i$ from 1 to $M$, we obtain

$$\sum_{i=1}^{M}(K_i-K^{av})^2\leqslant\frac{1}{M}\sum_{i,j=1}^{M}(K_i-K_j)^2. \tag{3.3}$$

Any two nondegenerate vertices $A_i$ and $A_j$ can be connected by a sequence of edges joining consequently nondegenerate vertices $A_i, A_{i_1}, \ldots, A_{i_p}, A_j$. Hence for the constant $c_2 = p+1 > 0$ we have

$$(K_i - K_j)^2 \leqslant c_2 \left((K_i - K_{i_1})^2 + (K_{i_1} - K_{i_2})^2 + \ldots + (K_{i_p} - K_j)^2\right).$$

Since the number of edges in such a sequence is bounded by $|V_n| - 1$, there exists a constant $c_3 > 0$ such that

$$\sum_{i,j=1}^{M} (K_i - K_j)^2 \leqslant c_3 \sum_{i \backsim j \leqslant M} (K_i - K_j)^2. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\sum_{i=1}^{M} (K_i - K^{av})^2 \leqslant \frac{c_3}{M} \sum_{i \backsim j \leqslant M} (K_i - K_j)^2,$$

hence,

$$\sum_{i \backsim j \leqslant M} C_{ij}(K_i - K_j)^2 \geqslant c_1 \sum_{i \backsim j \leqslant M} (K_i - K_j)^2 \geqslant \frac{c_1 M}{c_3} \sum_{i=1}^{M} (K_i - K^{av})^2.$$

Therefore, there exists a constant $c_4 > 0$ such that

$$g' = -2 \sum_{i \backsim j \leqslant M} C_{ij}(K_i - K_j)^2 \leqslant -c_4 \sum_{i=1}^{M} (K_i - K^{av})^2 = -c_4 g.$$

Now we have the inequality

$$g(t) \leqslant c_5 e^{-c_4 t}$$

for some positive constants $c_4$ and $c_5$. Finally, we obtain

$$(K_i - K^{av})^2 \leqslant \sum_{i=1}^{M} (K_i - K^{av})^2 \leqslant c_5 e^{-c_4 t}.$$

□

**Proposition 5.** *Let $r(t) = \big(r_1(t), \ldots, r_M(t)\big)$ be a solution to the Ricci flow* (2.7)*. Suppose curve $r(t)$ is contained in a compact subset $\mathbb{R}_+^M$. Then $r(t)$ converges exponentially fast to a point $(r_1, \ldots, r_M) \in \mathbb{R}_+^M$ such that the corresponding curvatures $K_1, \ldots, K_M$ vanish.*

*Proof.* The proof is the same as that of Proposition 3.7 from [3]. □

## 4. RICCI FLOW AS A NEGATIVE GRADIENT FLOW

Consider the following change of variables. For the Euclidean background geometry define $u_j = \ln r_j$, and for the hyperbolic background geometry define $u_j = \ln \tanh r_j/2$. Then both Ricci flows, (2.3) and (2.7), take the form

$$\frac{du_j}{dt} = -K_j, \; j = 1, \ldots, M. \tag{4.1}$$

Under assumption $(W)$ for the Euclidean background $u = (u_1, \ldots, u_M)$ belongs to $\mathcal{U} = R^M$, and for the hyperbolic background $u$ belongs to $\mathcal{U} = (-\infty, 0)^M \subset \mathbb{R}^M$. By Lemma 5

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}\;, \; i, j = 1, \ldots M.$$

Hence, the 1-form $\Omega = \sum\limits_{j=1}^{M} K_j du_j$ is closed. Since in both cases $\mathcal{U}$ is simply connected, there exists a function $F(u_1, \ldots, u_M) : U \to \mathbb{R}$ such that $dF = \Omega$.

**Proposition 6.** *Assume the weight function satisfies condition* $(W)$. *Then*

$(a)$ *for hyperbolic background the function* $F(u_1, \ldots, u_M)$ *is strictly convex;*

$(b)$ *for the Euclidean background the function* $F(u_1, \ldots, u_M)$ *is strictly convex on any plane* $\sum\limits_{j=1}^{M} u_j = \text{const}$.

*Proof.* The proof is the same as that of Proposition 3.9 from [3]. □

From this proposition it easily follows that in the hyperbolic background the metric is determined by its curvatures, and that in the Euclidian background the metric is determined by its curvatures up to a scalar multiple. This gives us rigidity of circle packings with degenerations.

## 5. DEGENERATION

**Proposition 7.** *Suppose* $X$ *is a closed surface with a triangulation* $T$ *and a weight* $w$, *satisfying condition* $(W)$. *Let* $I$ *be a proper subset of* $V_n$. *Denote by* $D_I$ *the set of all degenerate vertices adjacent to a vertex from* $I$. *Consider a sequence of metrics* $r^{(n)} = (r_i^{(n)} : i = 1, \ldots, M)$ *in the Euclidean or in the hyperbolic backgound geometry such that* $\lim\limits_{n\to\infty} r_i^{(n)} = 0$ *for* $i \in I$ *and* $\lim\limits_{n\to\infty} r_i^{(n)} > 0$ *for* $i \in \{1, \ldots, M\} \setminus I$. *Then*

$$\lim_{n\to\infty} \sum_{i\in I} K_i\big(r^{(n)}\big) + \sum_{j\in D_I} K_j = - \sum_{(e,v)\in Lk(I\cup D_I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_{I\cup D_I}). \tag{5.1}$$

*Moreover, for any metric* $r$ *in the Euclidean or in the hyperbolic backgound geometry and any proper subset* $I \in V_n$ *we have*

$$\sum_{i\in I} K_i(r) + \sum_{j\in D_I} K_j > - \sum_{(e,v)\in Lk(I\cup D_I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_{I\cup D_I}). \tag{5.2}$$

*Proof.* Equality (5.1) can be obtained by the same argument as the equality in Proposition 4.1 from [3] or at the end of the proof of Theorem 13.7.1 in [6] taking into account that the curvatures at the degenerate vertices are fixed. But to prove (5.2) for the weight function $w$ satisfying conditions $(W)$, we need to look at the details of the proof.

Denote by $\theta_i^{jk}$ the inner angle at the vertex $A_i$ of a triangle $\triangle A_i A_j A_k$ of the triangulation $T$. The faces of the triangulation having a vertex in $I \cup D_I$ are divided into three types $T_1, T_2, T_3$, where a face belongs to $T_s$ iff it has exactly $s$ vertices in $I \cup D_I$ counting multiplicities (in other words, the lift of the face to the universal cover $\pi : \tilde{X} \to X$ has exactly $s$ vertices in $\pi^{-1}(I)$). Write the curvature at a vertex $A_i$ as $2\pi - a_i$. Then $\sum\limits_{i\in I\cup D_I} a_i$ is equal to

$$\sum_{\substack{A_i \in I\cup D_I,\\ \triangle A_iA_jA_k \in T_1}} \theta_i^{jk} + \sum_{\substack{A_i, A_j \in I\cup D_I,\\ \triangle A_iA_jA_k \in T_2}} (\theta_i^{jk} + \theta_j^{ik}) + \sum_{\substack{A_i, A_j, A_k \in I\cup D_I,\\ \triangle A_iA_jA_k \in T_3}} (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}).$$

Applying the arguments from the proof of Proposition 4.1 of [3], we obtain the statement about the limit behavior of $\sum\limits_{i\in I} K_i\big(r^{(n)}\big)$, namely, the formula (5.1).

For (5.2) note that $\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}$ in the third sum is not greater than $\pi$ for any metric $r$. Also, $\theta_i^{jk} + \theta_j^{ik}$ in the second sum is less than $\pi$ for any metric $r$. In the first sum there are two types of terms. For $A_i \in D_I$ we have $\theta_i^{jk} = \pi - \arccos w_{jk}$. Finally, assuming that the weight $w$ satisfies condition $(W)$ for $A_i \in I$, we have $\theta_i^{jk} < \pi - \arccos w_{jk}$ by monotonicity from Lemma 4. □

## 6. THE EUCLIDEAN CASE

**Theorem 3.** *Suppose $X$ is a closed surface with a triangulation $T$ and a weight $w$, satisfying condition $(W)$.*

*The solution to the normalized Ricci flow* (1.3) *converges for any initial metric iff for any proper subset $I \in V_n$*

$$|I|K^{av} + \sum_{j\in D_I} K_j > -\sum_{(e,v)\in Lk(I\cup D_I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_{I\cup D_I}). \tag{6.1}$$

*Furthermore, if the solution converges, then it converges exponentially fast to the metric with $K_i = K^{av}$, $i = 1,\dots,M$.*

*Proof.* Consider the set metrics

$$M_a = \{(r_1,\dots,r_M) \mid r_i > 0 \text{ for all } i = 1,\dots,M \text{ and } \prod_{i=1}^{M} r_i = a\},$$

where $a > 0$. Also consider the curvature map $\Xi : M_a \to \mathbb{R}^M$, $\Xi(r) = \big(K_1(r),\dots,K_M(r)\big)$. By Proposition 6 the map $\Xi$ is injective. Its image is contained in the hyperplane

$$\Pi = \left\{(K_1,\dots,K_M) \in \mathbb{R}^M \mid \sum_{i=1}^{M} K_i = 2\pi\xi(X) - \sum_{j=M+1}^{N} K_j\right\}.$$

Consider the convex open polytope $P_K \subset \Pi$, defined by the inequalities

$$\sum_{i\in I} K_i > -\sum_{(e,v)\in Lk(I\cup D_I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_{I\cup D_I}) - \sum_{j\in D_I} K_j,$$

where $I$ runs through all proper subsets $I \subset V_n$. Then we have an injective continuous map $\Xi : M_a \to P_K$, where both $M_a$ and $P_K$ are homeomorphic to $\mathbb{R}^{M-1}$. By the invariance of the domain theorem the map $\Xi$ is a homeomorphism of $M_a$ onto $\operatorname{im}\Xi$. Applying Proposition 7, we see that $\operatorname{im}\Xi = P_K$. Therefore, there exists a unique $r^{(0)} = \big(r_1^{(0)},\dots,r_m^{(0)}\big) \in M_a$ such that $\Xi(r^{(0)}) = (K^{av},\dots,K^{av}) \in P_K$, that is, the metric of constant curvature exists and is unique up to scalar multiplication.

Now we pass to convergence of the normalized Ricci flow. As we have already seen, after the change of variables $u_i = \ln r_i, i = 1,\dots,M$, the normalized Ricci flow takes the form

$$\frac{du_i}{dt} = -\big(K_i(u) - K^{av}\big). \tag{6.2}$$

Also, there is a function $F$ such that $\frac{\partial F}{\partial u_i} = K_i(u) - K^{av}$. Hence, the latter equation is the negative gradient flow of $F$. Now we fix $a = 1$. The restriction of $F$ on the hyperplane $U_0 = \{u \in \mathbb{R}^M \mid \sum_{i=1}^{M} u_i = 0\}$ is strictly convex. By the argument at the beginning of the proof $F$ has a unique critical point $u^{(0)} \in U_0$. Therefore, this point is a minimum of $F$. A gradient line of $F$ with initial point in $U_0$ is contained in $U_0$. Let us show that a (negative) gradient line converges to $u^{(0)}$.

Consider a negative gradient line $u(t)$, that is, $\frac{du}{dt} = -\operatorname{grad} F$. Choose a sequence $t_n \to +\infty$ and denote $u^{(n)} = u(t_n)$. Since $F$ is bounded from below and $\frac{dF(u(t))}{dt} = -\|\operatorname{grad} F|_{u=u(t)}\|^2$, it follows that after passing to a subsequence we have $\operatorname{grad} F(u^{(n)}) \to 0$. For a sequence $r^{(n)}$ corresponding to $u^{(n)}$ this means that $\lim_{n\to+\infty} K_i(r^{(n)}) = K^{av}$. Since there exists a neighborhood of the point $(K^{av},\dots,K^{av}) \in P_K$ such that its closure is compact and is contained in $P_K$, passing to a subsequence we see that $r^{(n)}$ belongs to a compact in $M_a$ and, therefore, contains a convergent subsequence. Hence, $u^{(n)}$ contains a convergent subsequence. Therefore, all (negative) gradient lines converge to a metric with $K_i = K^{av}$, $i = 1,\dots,M$. □

## 7. THE HYPERBOLIC CASE

**Theorem 4.** *Suppose $X$ is a closed surface of negative Euler characteristic with a triangulation $T$ and a weight $w$, satisfying condition $(W)$.*

*The solution to the hyperbolic Ricci flow (1.5) converges for any initial metric iff for any subset $I \in V_n$*

$$\sum_{j\in D_I} K_j > -\sum_{(e,v)\in Lk(I\cup D_I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_{I\cup D_I}). \tag{7.1}$$

*Furthermore, if the solution converges, then it converges exponentially fast to the metric with $K_i = 0$, $i = 1, \ldots, M$.*

*Proof.* Consider the set of all metrics

$$M = \{(r_1, \ldots, r_M) \mid r_i > 0 \text{ for all } i = 1, \ldots, M\}.$$

Also, consider the curvature map $\Xi : M \to \mathbb{R}^M$, $\Xi(r) = \big(K_1(r), \ldots, K_M(r)\big)$. By Proposition 6 the map $\Xi$ is injective. By the Gauss–Bonnet theorem the image of $\Xi$ is contained in the halfspace $L \subset \mathbb{R}^M$, given by the inequality

$$\sum_{i=1}^{M} K_i > -\sum_{j=M+1}^{N} K_j + 2\pi\chi(X).$$

Also, consider the subset $L_K \subset L$, defined by the inequalities

$$\sum_{i\in I} K_i > -\sum_{(e,v)\in Lk(I\cup D_I)} \big(\pi - \arccos w(e)\big) + 2\pi\chi(F_{I\cup D_I}) - \sum_{j\in D_I} K_j,$$

where $I$ runs through all proper subsets $I \subset V_n$. Then $\Xi : M \to L_K$ is an injective continuous map. Both of them are homeomorphic to $\mathbb{R}^M$.

To prove that there exists a metric $r^{(0)}$ with $K_i\big(r^{(0)}\big) = 0$ for $i = 1, \ldots, M$, we proceed as follows. Choose an initial metric $r(0)$ such that the curvatures $K_i(r)$ are close to $2\pi$ for $i = 1, \ldots, M$. This is possible by the following statement.

**Lemma 6 (see [3], Lemma 3.5).** *Let $A_i \in V_n$ and condition $(W)$ be satisfied. Then for any $\varepsilon > 0$ there exists a number $L$ such that for any $r_i > L$ and any $r_j$ and $r_k$ the inner angle $\theta_i^{jk}$ of the hyperbolic triangle $\triangle A_iA_jA_k$ is smaller than $\varepsilon$.*

Fix a number $\delta > 0$ such that $K_i\big(r(0)\big) < 2\pi - \delta$. Consider the solution $r(t)$ to the hyperbolic Ricci flow (1.5) with the initial value $r(0)$. By the maximum principle (see Proposition 3) $\min\Big(K_1\big(r(t)\big), \ldots, K_M\big(r(t)\big), 0\Big)$ is not decreasing as $t \to +\infty$. Therefore, $K_i(t) \geqslant 0$ for all $t \geqslant 0$ and $i = 1, \ldots, M$. On the other hand, $\max\Big(K_1\big(r(t)\big), \ldots, K_M\big(r(t)\big), 0\Big)$ is not increasing, thus $K_i\big(r(t)\big) < 2\pi - \delta$ for all $t \geqslant 0$ and $i = 1, \ldots, M$. Therefore, $K\big(r(t)\big)$ is contained in a compact subset of $L_K$. Hence, the curve $r(t)$ is contained in a compact subset of $M$. By Proposition 5 $r(t)$ converges to a metric $r^{(0)}$ with $K_i = 0$ for $i = 1, \ldots, M$. Therefore, $r^{(0)}$ exists.

Now we pass to convergence of the Ricci flow (1.5). As we have already seen, after the change of variables $u_i = \ln\tanh\frac{r_i}{2}, i = 1, \ldots, M$, the Ricci flow takes the form

$$\frac{du_i}{dt} = -K_i(u), \tag{7.2}$$

where $u \in U = (-\infty, 0)^M$. Also, there is a strictly convex function $F$ on $U$ such that $\frac{\partial F}{\partial u_i} = K_i(u)$. Hence, the latter equation is the negative gradient flow of $F$. By the argument at the beginning of the proof, $F$ has a unique critical point $u^{(0)} \in U$. Therefore, this point is a minimum of $F$. Let us show that a (negative) gradient line converges to $u^{(0)}$.

Consider a negative gradient line $u(t)$, that is, $\frac{du}{dt} = -\operatorname{grad} F$. Choose a sequence $t_n \to +\infty$ and denote $u^{(n)} = u(t_n)$. Since $F$ is bounded from below and $\frac{dF\left(u(t)\right)}{dt} = -\|\operatorname{grad} F|_{u=u(t)}\|^2$, it follows that after passing to a subsequence we have $\operatorname{grad} F\left(u^{(n)}\right) \to 0$. For a sequence $r^{(n)}$ corresponding to $u^{(n)}$ this means that $\lim\limits_{n\to+\infty} K_i\left(r^{(n)}\right) = 0$. Since there exists a neighborhood of the point $(0,\ldots,0) \in L_K$ such that its closure is compact and is contained in $L_K$, passing to a subsequence we see that $r^{(n)}$ belongs to a compact in $M$ and, therefore, contains a convergent subsequence. Hence, $u^{(n)}$ contains a convergent subsequence. Therefore, all (negative) gradient lines converge to a metric with $K_i = 0$, $i = 1,\ldots,M$. □


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First of all, note that by the sine theorem equality (2.8) can be written as $\frac{\partial\theta_k}{\partial r_j}\sin\theta_k = \frac{\partial\theta_j}{\partial r_k}\sin\theta_j$ for both the Euclidean and the hyperbolic geometries. Using the notation from the proof of Lemma 5, denote by $x_i, x_j, x_k$ the lengths of the edges opposite to the vertices $A_i, A_j, A_k$ of a triangle $\triangle A_iA_jA_k$, respectively. Assume that the angle $\theta_i$ is fixed. Then by an elementary geometry argument one can show that $\frac{\partial x_i}{\partial x_k} = \cos\theta_j$ in both the Euclidean and the hyperbolic cases. For example, one can consider the right-angled triangle $\triangle A_kA_jH$, where $H$ is the point on the geodesic line $A_iA_j$ such that the geodesic line $A_kH$ is perpendicular to $A_iA_j$.

In our setting $x_i = l_{jk}, x_j = r_k, x_k = r_j$, and $\frac{\partial l_{jk}}{\partial r_j} = \cos\theta_j$. Therefore, $\frac{\partial^2 l_{jk}}{\partial r_k\partial r_j} = -\sin\theta_j\cdot\frac{\partial\theta_j}{\partial r_k}$. Then finally from the Mixed Derivative Theorem we find that this is equal to $\frac{\partial^2 l_{jk}}{\partial r_j\partial r_k} = -\sin\theta_k\cdot\frac{\partial\theta_k}{\partial r_j}$. □

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## CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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