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Heteroclinic Transition Motions in Periodic Perturbations of Conservative Systems with an Application to Forced Rigid Body Dynamics

Kazuyuki Yagasaki*

Department of Applied Mathematics and Physics,
Graduate School of Informatics, Kyoto University,
Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
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Abstract—We consider periodic perturbations of conservative systems. The unperturbed systems are assumed to have two nonhyperbolic equilibria connected by a heteroclinic orbit on each level set of conservative quantities. These equilibria construct two normally hyperbolic invariant manifolds in the unperturbed phase space, and by invariant manifold theory there exist two normally hyperbolic, locally invariant manifolds in the perturbed phase space. We extend Melnikov's method to give a condition under which the stable and unstable manifolds of these locally invariant manifolds intersect transversely. Moreover, when the locally invariant manifolds consist of nonhyperbolic periodic orbits, we show that there can exist heteroclinic orbits connecting periodic orbits near the unperturbed equilibria on distinct level sets. This behavior can occur even when the two unperturbed equilibria on each level set coincide and have a homoclinic orbit. In addition, it yields transition motions between neighborhoods of very distant periodic orbits, which are similar to Arnold diffusion for three or more degree of freedom Hamiltonian systems possessing a sequence of heteroclinic orbits to invariant tori, if there exists a sequence of heteroclinic orbits connecting periodic orbits successively. We illustrate our theory for rotational motions of a periodically forced rigid body. Numerical computations to support the theoretical results are also given.

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1. INTRODUCTION

In this paper we consider periodic perturbations of conservative systems of the form

$$\dot{x} = f(x) + \varepsilon g(x, \nu t), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $n \geqslant 3$ is an integer, $\nu > 0$ is a constant, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are C^r $(r \geqslant 2)$ and $g(x, \theta)$ is 2π-periodic in θ. When $\varepsilon = 0$, Eq. (1.1) becomes

$$\dot{x} = f(x). \tag{1.2}$$

We assume that Eq. (1.2) has m conservative quantities $F_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, i. e.,

$$\mathrm{D}F_j(x) f(x) = 0, \quad j = 1, \ldots, m, \tag{1.3}$$

where $1 \leqslant m \leqslant n - 2$ and the derivatives $\mathrm{D}F_j(x)$, $j = 1, \ldots, m$, are row vectors. We easily see that if $x = x_0$ is an equilibrium in (1.2), then by (1.3)

$$\mathrm{D}F_j(x_0)\mathrm{D}f(x_0) = 0, \quad j = 1, \ldots, m. \tag{1.4}$$

*E-mail: yagasaki@amp.i.kyoto-u.ac.jp

Hence, $\mathrm{D}f(x_0)$ has a zero eigenvalue of geometric multiplicity m, and especially x_0 is nonhyperbolic, if $\mathrm{D}F_j(x_0)$, $j = 1, \ldots, m$, are linearly independent. Moreover, we assume that there exist two equilibria connected by a heteroclinic orbit on each level set $F(x) = c \in I$, where $F(x) = \big(F_1(x), \ldots, F_m(x)\big)^{\mathrm{T}}$ with T the transpose operator and $I \subset \mathbb{R}^m$ is a closed domain. See Section 2 for details on our assumptions.

For $\varepsilon > 0$ Eq. (1.1) is shown to have two $(m+1)$-dimensional normally hyperbolic, locally invariant manifolds in Section 2, where the concepts "normal hyperbolicity" and "local invariance" are briefly described. They are fully expounded in [5–8, 25]. We study the behavior of the stable and unstable manifolds of the locally invariant manifolds and discuss heteroclinic motions to them. In particular, when the locally invariant manifolds consist of nonhyperbolic periodic orbits so that they are exactly invariant, we show that periodic orbits near the unperturbed equilibria on different level sets can be connected by heteroclinic orbits (see Theorem 2 below). This behavior can occur even when there exists only one equilibrium with a homoclinic orbit on each level set in the unperturbed system (1.2). Moreover, it yields transition motions between neighborhoods of very distant periodic orbits, which are similar to Arnold diffusion [1] for three or more degree of freedom Hamiltonian systems possessing a sequence of heteroclinic orbits to invariant tori, if there exists a sequence of heteroclinic orbits connecting periodic orbits successively (see Remark 2 below). We also remark that it was reported in [26] that such diffusion motions for invariant tori can occur not only in nearly integrable Hamiltonian systems but also in strongly nonintegrable ones.

Our main example is a periodically forced rigid body

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 - \varepsilon\frac{\alpha}{I_1}\omega_2 \sin\nu t,\\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \varepsilon\frac{\alpha}{I_2}\omega_1 \sin\nu t + \varepsilon\frac{\beta_2}{I_2}\sin\nu t,\\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \varepsilon\frac{\beta_3}{I_3}\sin\nu t,
\end{aligned} \tag{1.5}$$

where $I_1, I_2, I_3, \alpha, \beta_2, \beta_3, \nu$ are nonnegative constants. Equation (1.5) represents a mathematical model of a quadrotor helicopter illustrated in Fig. 1 when one of the rotors becomes out of tune and its rotational speed is periodically modulated. See Section 5.1 for more details on the model. This example is also important from an application point of view since quadrotor helicopters are used or expected to be used in many areas. Chaotic motions in a similar system were previously studied in [21], but the occurrence of transition behavior was ignored there. Moreover, rigid bodies with attachments were also investigated in [17, 18, 27]. In particular, chaotic motions were discussed in [17], although the possibility of such transition motions as discussed here was not taken into account.

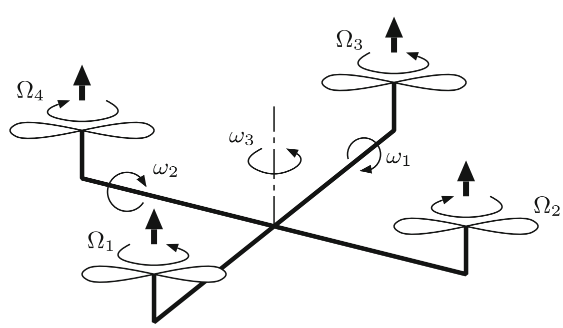

Fig. 1. Mathematical model for a quadrotor helicopter.

Homoclinic or heteroclinic motions in systems similar to (1.1) were previously studied by several researchers (e. g., [13, 16, 23]). So the reader may think that the problem treated here has already been settled in the field of dynamical systems. However, the previous results do not apply to (1.1) and (1.5) since especially the perturbed periodic orbits, if they exist, as well as the unperturbed equilibria, are nonhyperbolic and the perturbed system is not Hamiltonian (even not conservative), although they could if the perturbed periodic orbits were hyperbolic or the perturbed system was Hamiltonian. Furthermore, it has not been reported or pointed out anywhere, to the author's knowledge, that periodic orbits near the unperturbed equilibria on different level sets can have heteroclinic connections and transition motions between neighborhoods of very distant periodic orbits can occur in systems such as (1.1) and (1.5). Thus, the result obtained here is quite new and is not just a straightforward extension of the previous ones.

The outline of this paper is as follows: In Section 2 we precisely state our assumptions and describe the unperturbed and perturbed phase space structures which immediately follow from the assumptions. In Section 3 we give a condition under which the stable and unstable manifolds of the two locally invariant manifolds intersect transversely. In Section 4, under an additional assumption that the locally invariant manifolds consist of periodic orbits, we show that the periodic orbits are connected by heteroclinic orbits and estimate the difference between the level sets of equilibria to which they are close when these transverse heteroclinic orbits exist. Finally, we illustrate the theory for (1.5) and give numerical computations for periodic orbits and heteroclinic orbits to demonstrate the theoretical results in Section 5.

2. PHASE SPACE STRUCTURE

We make the following assumptions on the unperturbed system (1.2):

(A1) There exist m C^r-conservative quantities $F_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, where $1 \leqslant m \leqslant n-2$.

(A2) There exists a nonempty closed domain $I \subset \mathbb{R}^m$ such that for any $c \in I$ Eq. (1.2) has two equilibria of nonhyperbolic saddle type, $x_\pm(c)$, at which $\mathrm{D}F_j(x)$, $j = 1, \ldots, m$, are linearly independent, on the level set $F(x) = c$. Moreover, $x_\pm(c)$ are C^r with respect to $c \in I^0 = I \setminus \partial I$.

By (A2), the zero eigenvalue of $\mathrm{D}f(x_\pm(c))$ is of geometric multiplicity m since $\mathrm{D}F_j(x_\pm(c))$, $j = 1, \ldots, m$, are the associated left eigenvectors, as stated in Section 1.

(A3) n_s and n_u eigenvalues of $\mathrm{D}f\big(x_\pm(c)\big)$ have negative and positive real parts, respectively, where $n_\mathrm{s} + n_\mathrm{u} + m = n$.

(A4) The two equilibria $x_\pm(c)$ are connected by a heteroclinic orbit $x^\mathrm{h}(t; c)$ satisfying

$$\lim_{t\to\pm\infty} x^\mathrm{h}(t; c) = x_\pm(c)$$

on each level set $F(x) = c$ for $c \in I$. Moreover, $x^\mathrm{h}(t; c)$ is C^r with respect to $c \in I^0 = I \setminus \partial I$ as well as it is C^{r+1} with respect to $t \in \mathbb{R}$. See Fig. 2.

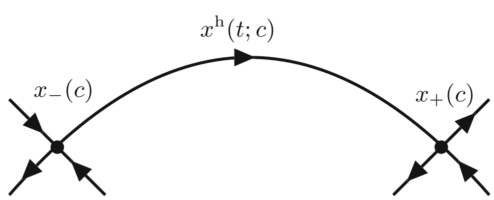

Fig. 2. Assumptions (A2) and (A4).

Consider the variational equation of (1.2) along the heteroclinic orbit $x^\mathrm{h}(t; c)$,

$$\dot{\xi} = \mathrm{D}f\big(x^\mathrm{h}(t; c)\big)\xi, \quad \xi \in \mathbb{R}^n. \tag{2.1}$$

We easily see that $\xi = \dot{x}^\mathrm{h}(t; c)$ is a solution to (2.1) which exponentially tends to 0 as $t \to \pm\infty$. By a fundamental result concerning asymptotic behavior of linear differential equations (e. g., Section 3.8 of [3]), Assumption (A3) means that there exist n_s linearly independent solutions to (2.1) which exponentially tend to 0 as $t \to \infty$ and n_u linearly independent solutions to (2.1) which exponentially tend to 0 as $t \to -\infty$ for each $c \in I$. We also assume the following.

(A5) The variational equation (2.1) has no other linearly independent solution than $\xi = \dot{x}^{\mathrm{h}}(t;c)$ such that it exponentially tends to 0 as $t \to \pm\infty$.

In (A2)–(A5), it is allowed that $x_+(c) = x_-(c)$ and $x^{\mathrm{h}}(t;c)$ becomes a homoclinic orbit to the equilibrium.

We introduce a new state variable $\theta \in \mathbb{S}^1 = \mathbb{R}/2\pi$ such that $\theta = \nu t \bmod 2\pi$, and rewrite (1.1) as

$$\dot{x} = f(x) + \varepsilon g(x, \theta), \quad \dot{\theta} = \nu, \tag{2.2}$$

for which the phase space is the $(n+1)$-dimensional space $\mathbb{R}^n \times \mathbb{S}^1$. When $\varepsilon = 0$, there exist two $(m+1)$-dimensional normally hyperbolic invariant manifolds with boundaries,

$$\mathscr{M}_0^{\pm} = \{(x_{\pm}(c), \theta) \mid \theta \in \mathbb{S}^1, c \in I\},$$

which consist of nonhyperbolic saddles. Here "normal hyperbolicity" means that the expansion and contraction rates of the flow generated by (2.2) with $\varepsilon = 0$ normal to $\mathscr{M}_0^{\pm}$ dominate those tangent to $\mathscr{M}_0^{\pm}$. See [5–8, 25] for details on this concept and its related matters. Their $(m + n_{\mathrm{s}} + 1)$- and $(m + n_{\mathrm{u}} + 1)$-dimensional stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}_0^+)$ and $W^{\mathrm{u}}(\mathscr{M}_0^-)$, intersect in the $(m+2)$-dimensional heteroclinic manifold

$$\mathscr{N} = \{(x^{\mathrm{h}}(t;c), \theta) \mid \theta \in \mathbb{S}^1, t \in \mathbb{R}, c \in I\},$$

i. e., $W^{\mathrm{s}}(\mathscr{M}_0^+) \cap W^{\mathrm{u}}(\mathscr{M}_0^-) \supset \mathscr{N}$.

We immediately have the following result for the perturbed system (1.1) by the invariant manifold theory [5–8, 25].

Proposition 1. *For $\varepsilon > 0$ sufficiently small, there exist $(m+1)$-dimensional normally hyperbolic, locally invariant manifolds $\mathscr{M}_\varepsilon^{\pm}$ in $O(\varepsilon)$-neighborhoods of $\mathscr{M}_0^{\pm}$. Moreover, their $(m + n_{\mathrm{s}} + 1)$ and $(m + n_{\mathrm{u}} + 1)$-dimensional stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}_\varepsilon^{\pm})$ and $W^{\mathrm{u}}(\mathscr{M}_\varepsilon^{\pm})$, are also $O(\varepsilon)$-close to $W^{\mathrm{s}}(\mathscr{M}_0^{\pm})$ and $W^{\mathrm{u}}(\mathscr{M}_0^{\pm})$, respectively, in neighborhoods of $\mathscr{M}_0^{\pm}$. See Fig.* 3.

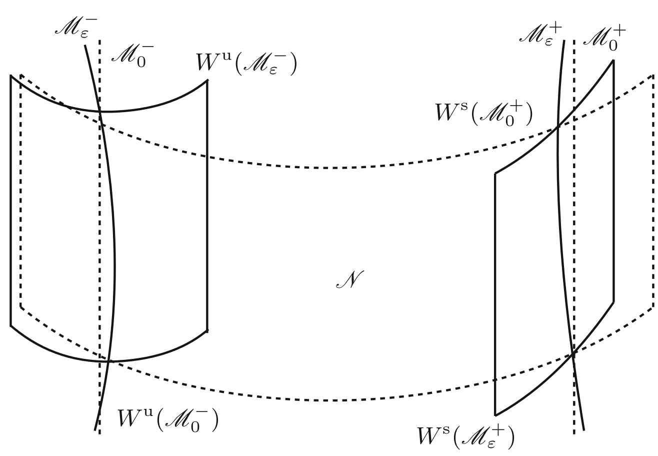

Fig. 3. Proposition 1.

Here "local invariance" of $\mathscr{M}_\varepsilon^{\pm}$ means that the trajectory $\big(x(t), \theta(t)\big)$ stays in $\mathscr{M}_\varepsilon^{\pm}$ only for $t \in (t_1, t_2)$ if it starts in $\mathscr{M}_\varepsilon^{\pm}$ at $t = 0$ and (t_1, t_2) is the maximal interval such that $\big(x(t), \theta(t)\big) \in U^{\pm}$ for $t \in (t_1, t_2)$, where $U^{\pm}$ are open domains containing $\mathscr{M}_\varepsilon^{\pm} \setminus \partial\mathscr{M}_\varepsilon^{\pm}$ with $\partial U^{\pm} \supset \partial\mathscr{M}_\varepsilon^{\pm}$. Thus, some trajectories starting in $\mathscr{M}_\varepsilon^{\pm}$ may escape $\mathscr{M}_\varepsilon^{\pm}$ through its boundary $\partial\mathscr{M}_\varepsilon^{\pm}$. The stable and unstable manifolds $W^{\mathrm{s}}(\mathscr{M}_\varepsilon^{\pm})$ and $W^{\mathrm{u}}(\mathscr{M}_\varepsilon^{\pm})$ inherit this property as a necessary consequence. See [5–8, 25] for details on this concept and its related matters.

3. HETEROCLINIC ORBITS TO LOCALLY INVARIANT MANIFOLDS

In this section we assume (A1)–(A5) and analyze the behavior of the stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}_\varepsilon^+)$ and $W^{\mathrm{u}}(\mathscr{M}_\varepsilon^-)$. We begin with describing solutions to the variational equation (2.1) and its adjoint equation.

Let $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$. Substituting $x = x^{\mathrm{h}}(t; c)$ into (1.2) and differentiating the resulting equation with respect to c_j, $j = 1, \ldots, m$, we easily see that

$$\xi = \frac{\partial x^{\mathrm{h}}}{\partial c_j}(t; c), \quad j = 1, \ldots, m, \tag{3.1}$$

are solutions to the variational equation (2.1). We also differentiate the relations

$$F\big(x^{\mathrm{h}}(t; c)\big) = c, \quad f\big(x_\pm(c)\big) = 0$$

with respect to c_j to obtain

$$\mathrm{D}F\big(x^{\mathrm{h}}(t; c)\big)\frac{\partial x^{\mathrm{h}}}{\partial c_j}(t; c) = e_j, \quad \mathrm{D}f\big(x_\pm(c)\big)\frac{\partial x_\pm}{\partial c_j}(c) = 0, \tag{3.2}$$

where $e_j \in \mathbb{R}^m$ is a vector of which the jth element is one and all the others are zero. Hence, the solutions (3.1) are linearly independent and tend to linearly independent eigenvectors for the zero eigenvalue of multiplicity m as $t \to \pm\infty$ since so do $\mathrm{D}F_j\big(x^{\mathrm{h}}(t; c)\big)$, $j = 1, \ldots, m$, by (A2).

Let $n_0 = n_{\mathrm{s}} + n_{\mathrm{u}}$. Using the above fact and Theorem 1 of [10], we immediately obtain the following lemma.

Lemma 1. *There exists a fundamental matrix* $\Phi(t; c) = \big(\phi_1(t; c), \ldots, \phi_n(t; c)\big)$ *to* (2.1) *such that*

$$\begin{aligned}
&\lim_{t\to\pm\infty} \phi_1(t; c) = 0, \quad \lim_{t\to\pm\infty} \phi_2(t; c) = \infty,\\
&\lim_{t\to+\infty} \phi_j(t; c) = 0, \quad \lim_{t\to-\infty} \phi_j(t; c) = \infty, \quad 3 \leqslant j \leqslant n_{\mathrm{s}} + 1,\\
&\lim_{t\to+\infty} \phi_j(t; c) = \infty, \quad \lim_{t\to-\infty} \phi_j(t; c) = 0, \quad n_{\mathrm{s}} + 1 < j \leqslant n_0,\\
&\lim_{t\to+\infty} |\phi_j(t; c)| < \infty, \quad \lim_{t\to-\infty} |\phi_j(t; c)| < \infty, \quad j > n_0,
\end{aligned} \tag{3.3}$$

where the convergence to 0 *and divergence for* $\phi_j(t; c)$, $j \leqslant n_0$, *are exponentially fast.*

We see that $\Psi(t; c) = \big(\Phi(t; c)^{-1}\big)^{\mathrm{T}}$ is a fundamental matrix to the adjoint equation for (2.1),

$$\dot{\xi} = -\mathrm{D}f\big(x^{\mathrm{h}}(t; c)\big)^{\mathrm{T}}\xi, \tag{3.4}$$

which we call the *adjoint variational equation* of (1.2) along $x^{\mathrm{h}}(t; c)$. Let $\Psi(t; c) = \big(\psi_1(t; c), \ldots, \psi_n(t; c)\big)$. We have

$$\begin{aligned}
&\lim_{t\to\pm\infty} \psi_1(t; c) = \infty, \quad \lim_{t\to\pm\infty} \psi_2(t; c) = 0,\\
&\lim_{t\to+\infty} \psi_j(t; c) = \infty, \quad \lim_{t\to-\infty} \psi_j(t; c) = 0, \quad 3 \leqslant j \leqslant n_{\mathrm{s}} + 1,\\
&\lim_{t\to+\infty} \psi_j(t; c) = 0, \quad \lim_{t\to-\infty} \psi_j(t; c) = \infty, \quad n_{\mathrm{s}} + 1 < j \leqslant n_0,\\
&\lim_{t\to+\infty} |\psi_j(t; c)| < \infty, \quad \lim_{t\to-\infty} |\psi_j(t; c)| < \infty, \quad j > n_0,
\end{aligned} \tag{3.5}$$

where the convergence to 0 and divergence for $\psi_j(t; c)$, $j \leqslant n_0$, are also exponentially fast.

Remark 1. From the argument at the beginning of this section, we take the solutions (3.1) as $\phi_{n_0+j}(t; c)$, $j = 1, \ldots, m$, in Lemma 1. Then $\psi_{n_0+j}(t; c)$, $j = 1, \ldots, m$, in (3.5) are given by

$$\xi = \mathrm{D}F_j\big(x^{\mathrm{h}}(t; c)\big)^{\mathrm{T}}, \quad j = 1, \ldots, m. \tag{3.6}$$

Actually, we differentiate (1.3) with respect to x to obtain

$$\mathrm{D}^2F_j(x)f(x) + \mathrm{D}F_j(x)\mathrm{D}f(x) = 0, \quad j = 1, \dots, m,$$

so that Eq. (3.6) gives linearly independent solutions to the adjoint variational equation (3.4) since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}F_j\big(x^{\mathrm{h}}(t;c)\big) = \mathrm{D}^2F_j\big(x^{\mathrm{h}}(t;c)\big)\dot{x}^{\mathrm{h}}(t;c) = \mathrm{D}^2F_j\big(x^{\mathrm{h}}(t;c)\big)f\big(x^{\mathrm{h}}(t;c)\big).$$

Moreover, it follows from (3.2) that they tend to linearly independent eigenvectors of $-\mathrm{D}f\big(x_\pm(c)\big)^{\mathrm{T}}$ for the zero eigenvalue of multiplicity m as $t \to \pm\infty$.

As in Lemma 4.5.2 of [11], we obtain the following lemma.

Lemma 2. *Let $(x, \theta) = \big(x^{\mathrm{s}}_\varepsilon(t; t_0, c), \nu t\big)$ and $\big(x^{\mathrm{u}}_\varepsilon(t; t_0, c), \nu t\big)$, respectively, denote orbits on the stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}^+_\varepsilon)$ and $W^{\mathrm{u}}(\mathscr{M}^-_\varepsilon)$, passing through points in an $O(\varepsilon)$-neighborhood of $\big(x^{\mathrm{h}}(0;c), \nu t_0\big)$. Then for any $\tau > 0$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$*

$$\begin{aligned} x^{\mathrm{s}}_\varepsilon(t; t_0, c) =& x^{\mathrm{h}}(t - t_0; c) + \varepsilon\xi^{\mathrm{s}}(t; t_0, c) + O(\varepsilon^2), \quad t \in [t_0, \tau), \\ x^{\mathrm{u}}_\varepsilon(t; t_0, c) =& x^{\mathrm{h}}(t - t_0; c) + \varepsilon\xi^{\mathrm{u}}(t; t_0, c) + O(\varepsilon^2), \quad t \in (-\tau, t_0], \end{aligned} \tag{3.7}$$

where these expressions hold with uniform validity in the indicated time intervals and $\xi^{\mathrm{s,u}}(t; t_0, c)$ are solutions of the first variational equation

$$\dot{\xi} = \mathrm{D}f\big(x^{\mathrm{h}}(t - t_0; c)\big)\xi + g\big(x^{\mathrm{h}}(t - t_0; c), \nu t\big). \tag{3.8}$$

The unperturbed stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}^+_0)$ and $W^{\mathrm{u}}(\mathscr{M}^-_0)$, intersect in the heteroclinic manifold $\mathscr{N}$. Their tangent spaces at $\big(x^{\mathrm{h}}(0;c), \theta\big)$,

$$T_{\left(x^{\mathrm{h}}(0;c),\theta\right)}W^{\mathrm{s}}(\mathscr{M}^+_0) \text{ and } T_{\left(x^{\mathrm{h}}(0;c),\theta\right)}W^{\mathrm{u}}(\mathscr{M}^-_0),$$

are spanned by

$$\{\tilde{\phi}_1(0;c), \tilde{\phi}_3(0;c), \dots, \tilde{\phi}_{n_{\mathrm{s}}+1}(0;c), \tilde{\phi}_{n_0+1}(0;c), \dots, \tilde{\phi}_n(0;c), \tilde{e}_{n+1}\}$$

and

$$\{\tilde{\phi}_1(0;c), \tilde{\phi}_{n_{\mathrm{s}}+2}(0;c), \dots, \tilde{\phi}_{n_0}(0;c), \tilde{\phi}_{n_0+1}(0;c), \dots, \tilde{\phi}_n(0;c), \tilde{e}_{n+1}\},$$

respectively, where $\tilde{e}_{n+1} \in \mathbb{R}^{n+1}$ is a vector of which $(n+1)$th element is one and all the others are zero, and

$$\tilde{\phi}_j(t;c)^{\mathrm{T}} = \big(\phi_j(t;c)^{\mathrm{T}}, 0\big) \in \mathbb{R}^n \times \mathbb{R}.$$

So the perturbed stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}^+_\varepsilon)$ and $W^{\mathrm{u}}(\mathscr{M}^-_\varepsilon)$, may separate in the direction of $\tilde{\phi}_2(0;c)$ near $\big(x^{\mathrm{h}}(0;c), \theta\big)$. Since $\psi_2(0;c)$ are normal to $\phi_j(0;c)$ for $j \neq 2$, the distance between $W^{\mathrm{s}}(\mathscr{M}^+_\varepsilon)$ and $W^{\mathrm{u}}(\mathscr{M}^-_\varepsilon)$ near $(x^{\mathrm{h}}(0;c), \nu t_0)$ is measured by

$$d_\varepsilon(t_0;c) = \frac{1}{|\psi_2(0;c)|}\langle\psi_2(0;c), x^{\mathrm{u}}_\varepsilon(t_0; t_0, c) - x^{\mathrm{s}}_\varepsilon(t_0; t_0, c)\rangle, \tag{3.9}$$

where the bracket $\langle\cdot,\cdot\rangle$ represents the inner product in $\mathbb{R}^n$ and $\nu t_0 = \theta$. See Fig. 4.

Let $\Delta^{\mathrm{s,u}}(t,c) = \langle\psi_2(t - t_0;c), \xi^{\mathrm{s,u}}(t; t_0, c)\rangle$. From (3.7) we have

$$d_\varepsilon(t_0;c) = \frac{\varepsilon}{|\psi_2(0;c)|}\big(\Delta^{\mathrm{u}}(t_0;c) - \Delta^{\mathrm{s}}(t_0;c)\big) + O(\varepsilon^2).$$

Differentiating $\Delta^{\mathrm{s,u}}(t;c)$ with respect to t yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Delta^{\mathrm{s,u}}(t;c) =& \langle\dot{\psi}_2(t - t_0;c), \xi^{\mathrm{s,u}}(t; t_0, c)\rangle + \langle\psi_2(t - t_0;c), \dot{\xi}^{\mathrm{s,u}}(t; t_0, c)\rangle \\ =& \langle\psi_2(t - t_0;c), g\big(x^{\mathrm{h}}(t - t_0;c), \nu t\big)\rangle, \end{aligned}$$

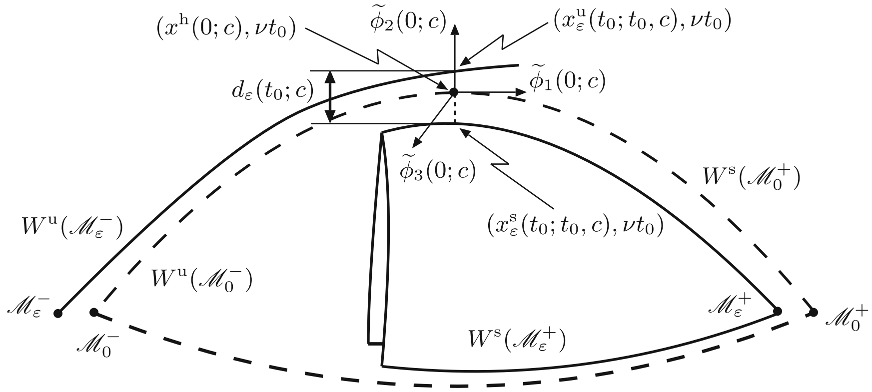

Fig. 4. Distance $d_\varepsilon(t_0;c)$.

where Eqs. (3.4) and (3.8) have been used. Employing "Melnikov's trick" as in the standard Melnikov technique (e. g., [11]) and noting that $\psi_2(t;c)$ exponentially tends to zero as $t\to\pm\infty$ and $|\xi^{\mathrm{s}}(t;t_0,c)|<\infty$ (resp. $|\xi^{\mathrm{u}}(t;t_0,c)|<\infty$) for $t\in[0,\infty)$ (resp. $t\in(-\infty,0]$), we obtain

$$\Delta^{\mathrm{s}}(t_0;c)=-\int_{t_0}^{\infty}\langle\psi_2(t-t_0;c),g\big(x^{\mathrm{h}}(t-t_0;c),\nu t\big)\rangle\mathrm{d}t,$$
$$\Delta^{\mathrm{u}}(t_0;c)=\int_{-\infty}^{t_0}\langle\psi_2(t-t_0;c),g\big(x^{\mathrm{h}}(t-t_0;c),\nu t\big)\rangle\mathrm{d}t.$$

Here we have used the fact that any arbitrarily large value can be chosen as $\tau>0$ in Lemma 2. Thus, we have

$$d_\varepsilon(t_0;c)=\frac{\varepsilon}{|\psi_2(0;c)|}M(\nu t_0;c)+O(\varepsilon^2),$$

where

$$M(\theta;c)=\int_{-\infty}^{\infty}\langle\psi_2(t;c),g\big(x^{\mathrm{h}}(t;c),\nu t+\theta\big)\rangle\mathrm{d}t,\tag{3.10}$$

which we call the *Melnikov function.* As in the standard Melnikov method [11, 24], if $M(\theta;c)$ has a simple zero, then so does $d_\varepsilon(t_0;c)$ for $\varepsilon>0$ sufficiently small by the implicit function theorem. Thus, we obtain the following theorem.

Theorem 1. *Suppose that $M(\theta;c)$ has a simple zero at $\theta=\theta_0$ for some $c\in I^0$. Then for $\varepsilon>0$ sufficiently small $W^{\mathrm{s}}(\mathscr{M}_\varepsilon^+)$ and $W^{\mathrm{u}}(\mathscr{M}_\varepsilon^-)$ intersect transversely.*

4. HETEROCLINIC ORBITS TO PERIODIC ORBITS

Let $\gamma_0^\pm(c)=\{(x_\pm(c),\theta)\in\mathbb{R}^n\times\mathbb{S}^1\mid\theta\in\mathbb{S}^1\}$ denote the unperturbed periodic orbits in (2.2) with $\varepsilon=0$. Throughout this section, we assume the following in addition to (A1)–(A5).

(A6) For $\varepsilon>0$ sufficiently small the locally invariant manifolds $\mathscr{M}_\varepsilon^\pm$ consist of m-parameter families of nonhyperbolic T-periodic orbits $\{\gamma_\varepsilon^\pm(c)\}_{c\in\tilde{I}}$ such that $\gamma_\varepsilon^\pm(c)\to\gamma_0^\pm(c)$ as $\varepsilon\to0$, where $\tilde{I}\subset I^0$ is a nonempty closed set and $T=2\pi/\nu$.

Assumption (A6) also means that $\mathscr{M}_\varepsilon^\pm$ are exactly invariant. We parameterize the perturbed periodic orbits as

$$c=F\big(x_\varepsilon^\pm(0;c)\big),\tag{4.1}$$

where $\gamma_\varepsilon^\pm(c)=\{(x_\varepsilon^\pm(\theta;c),\theta)\in\mathbb{R}^n\times\mathbb{S}^1\mid\theta\in\mathbb{S}^1\}$.

Lemma 3. *If Assumption* (A6) *holds, then*

$$\int_0^T \mathrm{D}F\big(x_\pm(c)\big)g\big(x_\pm(c),\nu t\big)\mathrm{d}t = 0.$$

Proof. For $c \in \tilde{I}$ we can write the periodic orbit as

$$x_\varepsilon^\pm(\theta;c) = x_\pm(c) + \varepsilon\xi^\pm(\theta;c) + O(\varepsilon^2)$$

with $\xi^\pm(2\pi;c) = \xi^\pm(0;c)$. Substituting the above expressions into (1.1) with $\theta = \nu t$, we have

$$\frac{\mathrm{d}x_\varepsilon^\pm}{\mathrm{d}t}(\nu t;c) = \varepsilon\Big(\mathrm{D}f\big(x_\pm(c)\big)\xi^\pm(\nu t;c) + g\big(x_\pm(c),\nu t\big)\Big) + O(\varepsilon^2),$$

so that

$$\begin{aligned}
&\mathrm{D}F_j\big(x_\pm(c)\big)\big(x_\varepsilon^\pm(2\pi;c) - x_\varepsilon^\pm(0;c)\big)\\
&= \varepsilon\int_0^T \mathrm{D}F_j\big(x_\pm(c)\big)\,\big(\mathrm{D}f\big(x_\pm(c)\big)\xi^\pm(\nu t;c) + g\big(x_\pm(c),\nu t\big)\big)\,\mathrm{d}t + O(\varepsilon^2)\\
&= \varepsilon\int_0^T \mathrm{D}F_j\big(x_\pm(c)\big)g\big(x_\pm(c),\nu t\big)\mathrm{d}t + O(\varepsilon^2) = 0, \quad j = 1,\dots,m,
\end{aligned}$$

since $f\big(x_\pm(c)\big) = 0$, $F\big(x_\varepsilon^\pm(2\pi;c)\big) = F\big(x_\varepsilon^\pm(0;c)\big) = c$, and $\mathrm{D}F_j\big(x_\pm(c)\big)$ is a left eigenvector of $\mathrm{D}f\big(x_\pm(c)\big)$ for the zero eigenvalue (see Eq. (1.4)). Thus, we obtain the result. □

Suppose that the Melnikov function $M(\theta;c)$ has a simple zero at $\theta = \theta_0$ so that $W^{\mathrm{s}}(\mathscr{M}_\varepsilon^+)$ and $W^{\mathrm{u}}(\mathscr{M}_\varepsilon^-)$ intersect transversely near $(x,\theta) = \big(x^{\mathrm{h}}(0;c),\theta_0\big)$ for $c \in \tilde{I}^0 = \tilde{I}\setminus\partial\tilde{I}$. Then there exists a heteroclinic orbit $\big(x_\varepsilon(t;t_0,c),\nu t\big)$ represented by

$$x_\varepsilon(t;t_0,c) = x^{\mathrm{h}}(t-t_0;c) + \varepsilon\tilde{\xi}(t;t_0,c) + O(\varepsilon^2) \tag{4.2}$$

on $[-\tau,\tau]$ for any $\tau > 0$, where $\nu t_0 = \theta_0 \bmod 2\pi$ and $\tilde{\xi}(t;t_0,c)$ is a solution to (3.8) such that $\tilde{\xi}(0;t_0,c) \in \operatorname{span}\{\phi_2(0;c),\dots,\phi_{n_0}(0;c)\}$, where $\operatorname{span}\{\zeta_1,\dots,\zeta_j\}$ represents the subspace spanned by vectors $\zeta_1,\dots,\zeta_j \in \mathbb{R}^n$ for $0 < j \leqslant n$. We solve (3.8) to obtain

$$\tilde{\xi}(t;t_0,c) = \Phi(t;c)\left(\int_0^t \Psi(s;c)^{\mathrm{T}} g\big(x^{\mathrm{h}}(s-t_0;c),\nu s\big)\mathrm{d}s + \tilde{\xi}(0;t_0,c)\right). \tag{4.3}$$

On the other hand, it follows from (A6) that the heteroclinic orbit tends to periodic orbits $\gamma_\varepsilon\big(c_\varepsilon^\pm(t_0,c)\big)$ on $\mathscr{M}_\varepsilon^\pm$ as $t \to \pm\infty$, where

$$c_\varepsilon^\pm(t_0,c) = \lim_{N\to\infty} F\big(x_\varepsilon(\pm NT;t_0,c)\big).$$

Let $W^{\mathrm{s}}\big(\gamma_\varepsilon^\pm(c)\big)$ and $W^{\mathrm{u}}\big(\gamma_\varepsilon^\pm(c)\big)$, respectively, denote the stable and unstable manifolds of $\gamma_\varepsilon^\pm(c)$. By the invariant manifold theorem for nonhyperbolic periodic orbits (e. g., [15]), we see that the stable and unstable manifolds, $W^{\mathrm{s}}\Big(\gamma_\varepsilon^+\big(c_\varepsilon^+(t_0,c)\big)\Big)$ and $W^{\mathrm{u}}\Big(\gamma_\varepsilon^-\big(c_\varepsilon^-(t_0,c)\big)\Big)$, are, respectively, $O(\varepsilon)$-close to the stable and unstable manifolds of the unperturbed periodic orbits, $W^{\mathrm{s}}\big(\gamma_0^+(c)\big)$ and $W^{\mathrm{u}}\big(\gamma_0^-(c)\big)$. Thus, the estimate (4.2) uniformly holds for $t \in (-\infty,\infty)$.

Since $\psi_{j+n_0}(t;c) = \mathrm{D}F_j(x^{\mathrm{h}}(t;c))^{\mathrm{T}}$, $j = 1,\dots,m$, as stated in Remark 1, we have

$$\mathrm{D}F\big(x^{\mathrm{h}}(t;c)\big)\Phi(t;c) = (0\ e_1\ \dots\ e_m).$$

Recall that $e_j \in \mathbb{R}^m$ is a vector whose jth element is one and all the others are zero. Using the above relation, (4.2) and (4.3), we compute

$$c_\varepsilon^+(t_0,c) - c = \lim_{N\to\infty} \varepsilon\mathrm{D}F\big(x^{\mathrm{h}}(NT-t_0;c)\big)\tilde{\xi}(NT;t_0,c) + O(\varepsilon^2)$$

$$=\varepsilon\sum_{j=1}^{m}D_j^+(t_0,c)e_j+O(\varepsilon^2),$$

where

$$D_j^+(t_0,c)=\lim_{N\to\infty}\int_0^{NT}\langle\psi_{j+n_0}(t;c),g\big(x^{\rm h}(t-t_0;c),\nu t\big)\rangle{\rm d}t$$
$$=\int_0^{\infty}{\rm D}F_j\big(x^{\rm h}(t;c)\big)\Big(g\big(x^{\rm h}(t-t_0;c),\nu t\big)-g\big(x_+(c),\nu t\big)\Big){\rm d}t.$$

Here we have used Remark 1 and Lemma 3. Note that

$$|g\big(x^{\rm h}(t-t_0;c),\nu t\big)-g\big(x_+(c),\nu t\big)|\to 0$$

exponentially as $t\to\infty$ since g is C^r. Similarly, we obtain

$$c_\varepsilon^-(t_0,c)-c=\varepsilon\sum_{j=1}^{m}D_j^-(t_0,c)e_j+O(\varepsilon^2),$$

where

$$D_j^-(t_0,c)=\int_0^{-\infty}{\rm D}F_j\big(x^{\rm h}(t;c)\big)\Big(g\big(x^{\rm h}(t-t_0;c),\nu t\big)-g\big(x_-(c),\nu t\big)\Big){\rm d}t.$$

Thus, we have

$$c_\varepsilon^+(t_0,c)-c_\varepsilon^-(t_0,c)=\varepsilon D(t_0;c)+O(\varepsilon^2),$$

where

$$D(t_0;c)=\sum_{j=1}^{m}\Big(D_j^+(t_0,c)-D_j^-(t_0,c)\Big)e_j. \tag{4.4}$$

Changing $c_\varepsilon^-(t_0,c)$ to c in the above argument, we obtain the following theorem.

Theorem 2. *Suppose that the Melnikov function $M(\theta;c)$ has a simple zero at $\theta=\theta_0$ for $c\in\tilde I\setminus\partial\tilde I$. Then for $\varepsilon>0$ sufficiently small $W^{\rm u}\big(\gamma_\varepsilon^-(c)\big)$ intersect $W^{\rm s}\big(\gamma_\varepsilon^+(c+\varepsilon\Delta c)\big)$, where $\Delta c=D(t_0;c)+O(\varepsilon)$ with $\nu t_0=\theta_0$. See Fig. 5.*

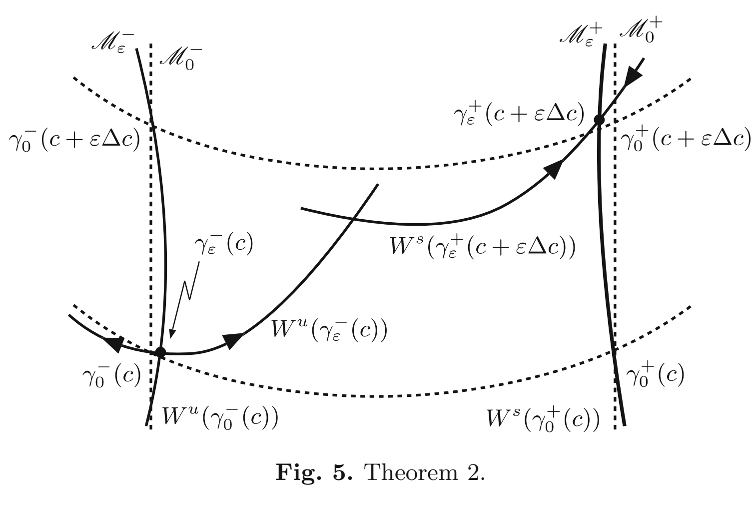

Fig. 5. Theorem 2.

Remark 2. Assume that Eq. (1.1) with $\varepsilon=0$ has another heteroclinic orbit $\tilde x^{\rm h}(t)$ such that

$$\lim_{t\to\pm\infty}\tilde x^{\rm h}(t)=x_\mp(c)$$

and the hypothesis of Theorem 2 holds at $\theta = \theta_k$ and $\tilde{\theta}_k$ with Δc_k and $\Delta \tilde{c}_k$, $k = 1, 2$, for $x^{\rm h}(t)$ and $\tilde{x}^{\rm h}(t)$, respectively. For any integer $N > 0$ and sequence $\{k_j \in \{1, 2\}\}_{j=-2N}^{2N}$, let $\{c_j\}_{j=-2N}^{2N}$ be a sequence with the length $4N$ satisfying

$$\begin{aligned} & c_j = c + O(\varepsilon) \in \tilde{I} \setminus \partial\tilde{I}, \quad j = -2N, \ldots, 2N, \\ & c_{2j+1} - c_{2j} = -\varepsilon\Delta c_{k_{2j}} + O(\varepsilon^2), \\ & c_{2j+2} - c_{2j+1} = \varepsilon\Delta\tilde{c}_{k_{2j+1}} + O(\varepsilon^2), \quad j = -N, \ldots, N-1. \end{aligned} \tag{4.5}$$

Then $W^{\rm u}\big(\gamma_\varepsilon^-(c_{2j})\big)$ and $W^{\rm u}\big(\gamma_\varepsilon^+(c_{2j+1})\big)$ intersect $W^{\rm s}\big(\gamma_\varepsilon^+(c_{2j+1})\big)$ and $W^{\rm s}\big(\gamma_\varepsilon^-(c_{2j+2})\big)$, respectively. Hence, there exists an orbit successively visiting neighborhoods of periodic orbits $\gamma_\varepsilon^-(c_{2j})$ and $\gamma_\varepsilon^+(c_{2j+1})$, $j = -N, \ldots, N$. This behavior is very similar to Arnold diffusion and that type of motions [1, 26] for three or more degree of freedom Hamiltonian systems.

5. EXAMPLE: PERIODICALLY FORCED RIGID BODY

5.1. Rotational Motion of the Quadrotor Helicopter

Consider the quadrotor helicopter illustrated in Fig. 1. Let ω_j and I_j, $j = 1, 2, 3$, respectively, denote the angular velocities and moments of inertia about the quadrotor's principal axes. We assume that $I_1 < I_2 < I_3$. Let Ω_j be the angular velocities of the jth rotors for $j = 1-4$ and write

$$U_1 = \Omega_4^2 - \Omega_2^2, \quad U_2 = \Omega_3^2 - \Omega_1^2, \quad U_3 = \Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_1^3,$$

and $\Omega = \Omega_2 + \Omega_4 - \Omega_1 - \Omega_3$. Equations of motion for rotational motion of the quadrotor are given by

$$\begin{aligned} \dot{\omega}_1 =& \frac{I_2 - I_3}{I_1}\omega_2\omega_3 - \frac{J}{I_1}\Omega\omega_2 + \frac{\ell b}{I_1}U_1, \\ \dot{\omega}_2 =& \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \frac{J}{I_2}\Omega\omega_1 + \frac{\ell b}{I_2}U_2, \\ \dot{\omega}_3 =& \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{\ell d}{I_3}U_3, \end{aligned} \tag{5.1}$$

where ℓ is the length from the center of mass to the rotational axis of the rotor, and J, b and d are the rotor's moment of inertia about the rotational axis, thrust factor and drag factor, respectively. See [2, 14] for the derivation of (5.1). In this model, by equations of motion for its translational motion, the quadrotor can hover only if

$$\Omega_j = \Omega_0 := \frac{1}{2}\sqrt{\frac{m_0 g}{b}}, \quad j = 1\text{-}4,$$

with m_0 the quadrotor's mass and g the gravitational acceleration.

Suppose that when all the rotors rotate with the angular velocity Ω_0, one of them, say the rotor No. 1, suddenly becomes out of tune so that its angular velocity is periodically modulated, $\Omega_1 = \Omega_0(1 + \varepsilon\sin\nu t)$. In this situation, Eq. (5.1) becomes (1.5) up to $O(\varepsilon)$, where

$$\alpha = J\Omega_0, \quad \beta_2 = 2\ell b\Omega_0^2, \quad \beta_3 = 2\ell d\Omega_0^2$$

are nonnegative constants.

5.2. Application of the Theory

We now illustrate our theory for (1.5). When $\varepsilon = 0$, Eq. (1.5) has a conservative quantity

$$\tilde{F}(\omega) = \frac{1}{2}\big(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\big)$$

and has two nonhyperbolic equilibria at $\omega = (0, \pm c, 0)$ connected by four heteroclinic orbits

$$\omega_\pm^{\rm h}(t; c) = \left(\pm c\sqrt{\frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}}\operatorname{sech} kct, c\tanh kct, \pm c\sqrt{\frac{I_2(I_2 - I_1)}{I_3(I_3 - I_1)}}\operatorname{sech} kct\right),$$

$$\tilde{\omega}^{\mathrm{h}}_{\pm}(t;c)=\left(\pm c\sqrt{\frac{I_2(I_3-I_2)}{I_1(I_3-I_1)}}\mathrm{sech}kct,-c\tanh kct,\mp c\sqrt{\frac{I_2(I_2-I_1)}{I_3(I_3-I_1)}}\mathrm{sech}kct\right)$$

on the level set $F_1(\omega):=\sqrt{2\tilde{F}(\omega)/I_2}=c>0$, where

$$k=\sqrt{\frac{(I_2-I_1)(I_3-I_2)}{I_3I_1}}.$$

See Fig. 6 for the unperturbed orbits of (1.5) with $\varepsilon=0$ on the level set $F_1(\omega)=c$. The variational equations of (1.5) with $\varepsilon=0$ along $\omega^{\mathrm{h}}_{\pm}(t;c)$ and $\tilde{\omega}^{\mathrm{h}}_{\pm}(t;c)$ have no other linearly independent solutions than $\xi=\dot{\omega}^{\mathrm{h}}_{\pm}(t;c)$ and $\dot{\tilde{\omega}}^{\mathrm{h}}_{\pm}(t;c)$, respectively, such that they tend to 0 exponentially as $t\to\pm\infty$ since the numbers of eigenvalues with positive and negative real parts are both one. Thus, Assumptions (A1)–(A5) hold with $n=3$, $m,n_{\mathrm{s}},n_{\mathrm{u}}=1$ and $I=[c_\ell,c_{\mathrm{r}}]$ for any $0<c_\ell<c_{\mathrm{r}}<\infty$. The solutions tending to 0 as $t\to\pm\infty$ for the corresponding adjoint variational equations are given by

$$\psi_2(t;c)=\big(I_1(I_1-I_2)\omega_1(t;c),0,I_3(I_3-I_2)\omega_3(t;c)\big)^{\mathrm{T}},$$

where $\omega(t;c)=\omega^{\mathrm{h}}_{\pm}(t;c)$ or $\tilde{\omega}^{\mathrm{h}}_{\pm}(t;c)$. Henceforth we use the notation repeatedly. Moreover,

$$\mathrm{D}F_1(\omega)=\frac{1}{cI_2}(I_1\omega_1,I_2\omega_2,I_3\omega_3).$$

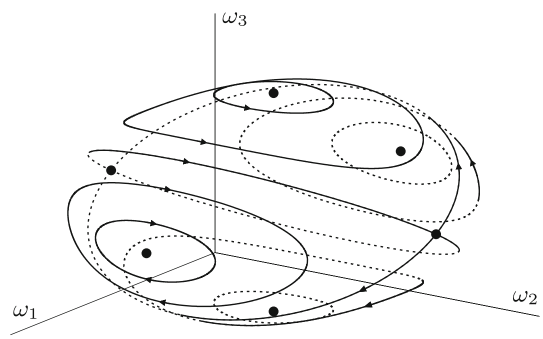

Fig. 6. Unperturbed orbits of (1.5) with $\varepsilon=0$ on the level set $F_1(\omega)=c$.

Fix the closed interval $I=[c_\ell,c_{\mathrm{r}}]$. When $\varepsilon>0$ is sufficiently small, there are two normally hyperbolic, locally invariant manifolds $\mathscr{M}^{\pm}_{\varepsilon}$ near

$$\mathscr{M}^{\pm}_0=\{(0,\pm c,0,\theta)\in\mathbb{R}^3\times\mathbb{S}^1\mid\theta\in\mathbb{S}^1,c\in I\}$$

in (1.5). We compute the Melnikov functions as

$$\begin{aligned}M(\theta;c)=&\int_{-\infty}^{\infty}[\alpha(I_2-I_1)\omega_1(t;c)\omega_2(t;c)+\beta_3(I_3-I_2)\omega_3(t;c)]\sin(\nu t+\theta_0)\mathrm{d}t\\=&\alpha(I_2-I_1)\cos\theta\int_{-\infty}^{\infty}\omega_1(t;c)\omega_2(t;c)\sin\nu t\,\mathrm{d}t\\&+\beta_3(I_3-I_2)\sin\theta\int_{-\infty}^{\infty}\omega_3(t;c)\cos\nu t\,\mathrm{d}t,\end{aligned}$$

so that

$$M_{\pm}(\theta;c)=\pm\alpha M_1(c)\cos\theta\pm\beta_3M_2(c)\sin\theta \tag{5.2}$$

for $\omega=\omega^{\mathrm{h}}_{\pm}$, and

$$\tilde{M}_{\pm}(\theta;c)=\pm\alpha M_1(c)\cos\theta\mp\beta_3M_2(c)\sin\theta \tag{5.3}$$

for $\omega = \tilde{\omega}^{\mathrm{h}}_{\pm}$, where

$$M_1(c) = \pi\nu\sqrt{\frac{I_1 I_2 I_3^2}{(I_3 - I_1)(I_3 - I_2)}}\,\mathrm{sech}\left(\frac{\pi\nu}{2kc}\right),$$
$$M_2(c) = \pi\sqrt{\frac{I_1 I_2 (I_3 - I_2)}{I_3 - I_1}}\,\mathrm{sech}\left(\frac{\pi\nu}{2kc}\right).$$

From Theorem 1 we obtain the following.

Theorem 3. *Assume that $\alpha \neq 0$ or $\beta_3 \neq 0$. Then the stable and unstable manifolds, $W^{\mathrm{s}}(\mathscr{M}_\varepsilon^{\mp})$ and $W^{\mathrm{u}}(\mathscr{M}_\varepsilon^{\pm})$, intersect transversely.*

We next apply Theorem 2. First, we easily see that $\mathscr{M}_\varepsilon^{\pm}$ satisfy (A6) if $\beta_2, \beta_3 = 0$. Actually, since $F_1(\omega)$ is still a conservative quantity, for $\varepsilon > 0$ sufficiently small $\mathscr{M}_\varepsilon^{\pm}$ is transversal to the level set $F_1(\omega) = c$. Hence, $\mathscr{M}_\varepsilon^{\pm} \cap \{(\omega, \theta) \mid F_1(\omega) = c, \theta \in \mathbb{S}^1\}$ are closed curves, which correspond to periodic orbits. More generally, we prove the following.

Proposition 2. *$\mathscr{M}_\varepsilon^{\pm}$ satisfy* (A6) *if and only if $\alpha = 0$ or $\beta_3 = 0$.*

Proof. We only give a proof for $\mathscr{M}_\varepsilon^{+}$ since the proof for $\mathscr{M}_\varepsilon^{-}$ is similar. For $\alpha = 0$ or $\beta_3 = 0$, Eq. (1.5) has the following symmetry.

Lemma 4. *Let $\omega = \big(\bar{\omega}_1(t), \bar{\omega}_2(t), \bar{\omega}_3(t)\big)$ be a solution to* (1.5) *with $\varepsilon > 0$.*

(i) *If $\alpha = 0$, then $\omega = \big(-\bar{\omega}_1(-t), \bar{\omega}_2(-t), \bar{\omega}_3(-t)\big)$ is a solution to* (1.5).

(ii) *If $\beta_3 = 0$, then $\omega = \big(\bar{\omega}_1(-t), \bar{\omega}_2(-t), -\bar{\omega}_3(-t)\big)$ is a solution to* (1.5).

Proof. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_j(-t) = -\dot{\omega}_j(-t), \quad j = 1, 2, 3,$$

we can immediately obtain the results. □

Let $\alpha = 0$ and assume that Eq. (1.5) has a solution $\bar{\omega}(t)$ such that $\bar{\omega}_1(0) = \bar{\omega}_1(T/2) = 0$. Then by Lemma 4 $\hat{\omega}(t) = (-\bar{\omega}_1(-t), \bar{\omega}_2(-t), \bar{\omega}_3(-t))$ is also a solution to (1.5). Obviously, $\hat{\omega}(0) = \bar{\omega}(0)$, so that $\hat{\omega}(t) \equiv \bar{\omega}(t)$. Hence, $\bar{\omega}(t)$ is T-periodic since $\bar{\omega}(-T/2) = \hat{\omega}(-T/2) = \bar{\omega}(T/2)$. Similarly, if Eq. (1.5) with $\beta_3 = 0$ has a solution $\bar{\omega}(t)$ such that $\bar{\omega}_3(0) = \bar{\omega}_3(T/2) = 0$, then it is also T-periodic. These facts are keys to our argument below in the proof.

We now prove the sufficiency part. We begin with the case of $\alpha = 0$. Substituting

$$\omega_1 = \varepsilon\zeta_1, \quad \omega_2 = c + \varepsilon\zeta_2, \quad \omega_3 = \varepsilon\zeta_3 \tag{5.4}$$

into (1.5), we have

$$\begin{aligned} &\dot{\zeta}_1 = -ca_1\zeta_3 + O(\varepsilon), \quad \dot{\zeta}_2 = \bar{\beta}_2 \sin\nu t + O(\varepsilon), \\ &\dot{\zeta}_3 = -ca_3\zeta_1 + \bar{\beta}_3 \sin\nu t + O(\varepsilon), \end{aligned} \tag{5.5}$$

where

$$a_1 = \frac{I_3 - I_2}{I_1}, \quad a_3 = \frac{I_2 - I_1}{I_3}, \quad \bar{\beta}_j = \frac{\beta_j}{I_j}, \quad j = 2, 3.$$

Solving (5.5), we obtain

$$\zeta_1(t) = \frac{ca_1\bar{\beta}_3}{\nu^2 + c^2 a_1 a_3}\sin\nu t + C_1\cosh c\sqrt{a_1 a_3}t + C_2 \sinh c\sqrt{a_1 a_3}t + O(\varepsilon),$$

where C_1, C_2 are constants. By the implicit function theorem, we see that for $\varepsilon > 0$ sufficiently small and each $c > 0$, there is a unique point (C_1, C_2) near $(0, 0)$ such that

$$\zeta_1(0) = C_1 + O(\varepsilon) = 0,$$
$$\zeta_1(T/2) = C_1 \cosh \frac{\pi c\sqrt{a_1 a_3}}{\nu} + C_2 \sinh \frac{\pi c\sqrt{a_1 a_3}}{\nu} + O(\varepsilon) = 0.$$

Hence, Eq. (1.5) has a periodic orbit in an $O(\varepsilon)$-neighborhood of $\mathscr{M}_0^+$ and it must be on $\mathscr{M}_\varepsilon^+$ for each $c > 0$ since it would escape from the neighborhood as $t \to \infty$ or $-\infty$ if not.

We turn to the case of $\beta_3 = 0$. Substituting (5.4) into (1.5), we have

$$\begin{aligned} \dot{\zeta}_1 &= -ca_1\zeta_3 - c\bar{\alpha}_1 \sin\nu t + O(\varepsilon),\\ \dot{\zeta}_2 &= \bar{\beta}_2 \sin\nu t + O(\varepsilon), \quad \dot{\zeta}_3 = -ca_3\zeta_1 + O(\varepsilon), \end{aligned} \tag{5.6}$$

where $\alpha_1 = \alpha/I_1$. Solving (5.6), we obtain

$$\zeta_3(t) = -\frac{c^2 a_3\bar{\alpha}_1}{\nu^2 + c^2 a_1 a_3} \sin\nu t + C_3 \cosh c\sqrt{a_1 a_3}t + C_4 \sinh c\sqrt{a_1 a_3}t + O(\varepsilon),$$

where C_3, C_4 are constants. We apply the above argument for ζ_3 instead of ζ_1 to see that Eq. (1.5) has a periodic orbit on $\mathscr{M}_\varepsilon^+$ for $\varepsilon > 0$ sufficiently small and each $c > 0$.

We next prove the necessity part by contradiction. Assume that $\alpha, \beta_3 \neq 0$ and that there exists a T-periodic orbit on $\mathscr{M}_\varepsilon^+$. Substituting (5.4) into (1.5), we have

$$\begin{aligned} \dot{\zeta}_1 &= -ca_1\zeta_3 - c\bar{\alpha}_1 \sin\nu t + O(\varepsilon),\\ \dot{\zeta}_2 &= \bar{\beta}_2 \sin\nu t + \varepsilon(a_2\zeta_3\zeta_1 + \bar{\alpha}_2\zeta_1 \sin\nu t) + O(\varepsilon^2),\\ \dot{\zeta}_3 &= -ca_3\zeta_1 + \bar{\beta}_3 \sin\nu t + O(\varepsilon), \end{aligned} \tag{5.7}$$

where

$$a_2 = \frac{I_3 - I_1}{I_2}, \quad \bar{\alpha}_2 = \frac{\alpha}{I_2}.$$

Solving (5.7), we obtain the expressions

$$\zeta_1(t) = \frac{c(\nu\bar{\alpha}_1 \cos\nu t + a_1\bar{\beta}_3 \sin\nu t)}{\nu^2 + c^2 a_1 a_3} + O(\varepsilon),$$
$$\zeta_3(t) = -\frac{\nu\bar{\beta}_3 \cos\nu t + c^2 a_3\bar{\alpha}_1 \sin\nu t)}{\nu^2 + c^2 a_1 a_3} + O(\varepsilon)$$

for the periodic orbit. Hence,

$$\zeta_2(T) - \zeta_2(0) = \varepsilon\frac{\pi c(a_2\bar{\alpha}_1 - a_1\bar{\alpha}_2)\bar{\beta}_3}{\nu(\nu^2 + c^2 a_1 a_3)} + O(\varepsilon^2),$$

which is nonzero since $I_2 > I_1$ so that $a_2\bar{\alpha}_1 - a_1\bar{\alpha}_2 > 0$. This is a contradiction. Thus, we complete the proof. □

We assume that $\alpha = 0$ or $\beta_3 = 0$ and apply Theorem 2. We have

$$D_1^\pm(t_0; c) = \frac{1}{cI_2} \int_{-t_0}^{\pm\infty} \big(\beta_2\Delta_\pm\omega_2(t; c) + \beta_3\omega_3(t; c)\big) \sin\nu(t + t_0)\,\mathrm{d}t,$$

where

$$\Delta_\pm\omega_2(t; c) = \begin{cases} \omega_2(t; c) \mp c & \text{for } \omega = \omega_\pm^{\mathrm{h}};\\ \omega_2(t; c) \pm c & \text{for } \omega = \tilde{\omega}_\pm^{\mathrm{h}}. \end{cases}$$

Thus, we compute (4.4) as

$$D_\pm(t_0; c) = \frac{2\beta_2}{cI_2}\left(\cos\nu t_0 \int_0^\infty \Delta_+\omega_2(t; c) \sin\nu t\,\mathrm{d}t - c\int_{-t_0}^0 \sin\nu(t + t_0)\mathrm{d}t\right)$$

$$
\begin{aligned}
&+\frac{\beta_3}{cI_2}\sin\nu t_0\int_{-\infty}^{\infty}\omega_3(t;c)\cos\nu t\,\mathrm{d}t\\
=&\beta_2\left(D_1(c)\cos\nu t_0-\frac{2}{\nu I_2}\right)\pm\beta_3 D_2(c)\sin\nu t_0
\end{aligned}
\tag{5.8}
$$

for $\omega=\omega_\pm^{\mathrm{h}}$, and

$$
\begin{aligned}
\tilde{D}_\pm(t_0;c)=&\frac{2\beta_2}{cI_2}\left(\cos\nu t_0\int_0^{\infty}\Delta_+\omega_2(t;c)\sin\nu t\,\mathrm{d}t+c\int_{-t_0}^{0}\sin\nu(t+t_0)\mathrm{d}t\right)\\
&+\frac{\beta_3}{cI_2}\sin\nu t_0\int_{-\infty}^{\infty}\omega_3(t;c)\cos\nu t\,\mathrm{d}t\\
=&-\beta_2\left(D_1(c)\cos\nu t_0-\frac{2}{\nu I_2}\right)\mp\beta_3 D_2(c)\sin\nu t_0
\end{aligned}
\tag{5.9}
$$

for $\omega=\tilde{\omega}_\pm^{\mathrm{h}}$, where

$$
D_1(c)=\frac{\pi}{kcI_2}\operatorname{cosech}\left(\frac{\pi\nu}{2kc}\right),\quad D_2(c)=\frac{\pi}{kc}\sqrt{\frac{I_2-I_1}{I_2I_3(I_3-I_1)}}\operatorname{sech}\left(\frac{\pi\nu}{2kc}\right).
$$

Since by (5.2) and (5.3) $M_\pm(\theta;c)$ and $\tilde{M}_\pm(\theta;c)$ have a simple zero at $\theta=0,\pi$ for $\alpha=0$ and at $\theta=\pi/2,3\pi/2$ for $\beta_3=0$, we obtain the following result.

Theorem 4. *Assume that $\alpha=0$ or $\beta_3=0$. Then $\gamma_\varepsilon^-(c)$ and $\gamma_\varepsilon^+(c+\varepsilon\Delta c)$ have a heteroclinic cycle, where*

$$
\Delta c=\beta_2\left(\pm D_1(c)-\frac{2}{\nu I_2}\right)+O(\varepsilon)
$$

for $\alpha=0$ and

$$
\Delta c=-\frac{2\beta_2}{\nu I_2}+O(\varepsilon)
$$

for $\beta_3=0$. See Fig. 7 for the dependence of Δc on c given in the above formulas.

Proof. Applying Theorem 2 and using (5.8), we see that $W^{\mathrm{u}}\big(\gamma_\varepsilon^-(c)\big)$ intersects $W^{\mathrm{s}}\big(\gamma_\varepsilon^+(c+\varepsilon\Delta c)\big)$, i. e., there exists a heteroclinic orbit from $\gamma_\varepsilon^-(c)$ to $\gamma_\varepsilon^+(c+\varepsilon\Delta c)$. Hence, by Lemma 4, there also exists a heteroclinic orbit from $\gamma_\varepsilon^+(c+\varepsilon\Delta c)$ to $\gamma_\varepsilon^-(c)$. This completes the proof. □

Remark 3. Applying Theorem 2 and using (5.9), we see that $W^{\mathrm{u}}\big(\gamma_\varepsilon^+(c)\big)$ intersects $W^{\mathrm{s}}\big(\gamma_\varepsilon^-(c+\varepsilon\Delta\tilde{c})\big)$, i. e., there exists a heteroclinic orbit from $\gamma_\varepsilon^-(c)$ to $\gamma_\varepsilon^+(c+\varepsilon\Delta\tilde{c})$, where

$$
\Delta\tilde{c}=\beta_2\left(\mp D_1(c)+\frac{2}{\nu I_2}\right)+O(\varepsilon)
$$

for $\alpha=0$ and

$$
\Delta\tilde{c}=\frac{2\beta_2}{\nu I_2}+O(\varepsilon)
$$

for $\beta_3=0$. Thus, $\Delta\tilde{c}=-\Delta c$ up to $O(1)$, which is consistent with the statement of Theorem 4.

Assume that $\alpha=0$ and $\beta_2,\beta_3\neq 0$ and define

$$
\Delta c_k(c)=\begin{cases}\beta_2\big(2/\nu I_2+D_1(c)\big) & \text{for } k=1;\\ \beta_2\big(2/\nu I_2-D_1(c)\big) & \text{for } k=2.\end{cases}
\tag{5.10}
$$

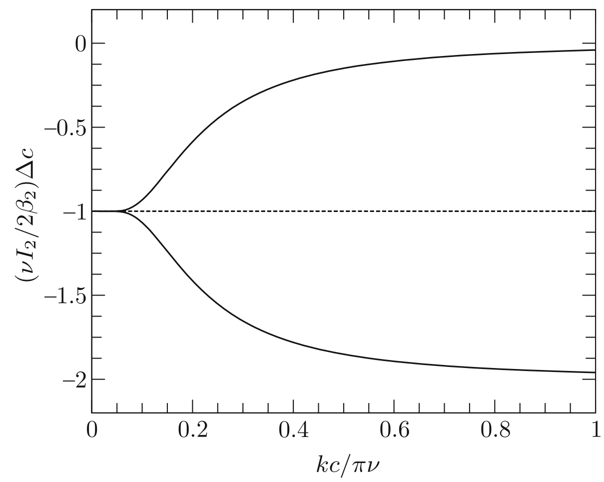

Fig. 7. Dependence of Δc on c given in Theorem 4: The solid and dotted lines represent the results for $\alpha = 0$ and $\beta_3 = 0$, respectively.

For any integer $N > 0$ and sequence $\{k_j \in \{1, 2\}\}_{j=-2N}^{2N}$, let $\{c_j\}_{j=-2N}^{2N}$ be a sequence with the length $4N$ satisfying (4.5). Then by Theorem 4, $W^{\mathrm{u}}\big(\gamma_\varepsilon^-(c_{2j})\big)$ and $W^{\mathrm{u}}\big(\gamma_\varepsilon^+(c_{2j+1})\big)$ intersect $W^{\mathrm{s}}\big(\gamma_\varepsilon^+(c_{2j+1})\big)$ and $W^{\mathrm{s}}\big(\gamma_\varepsilon^-(c_{2j+2})\big)$, respectively, so that there exists an orbit successively visiting neighborhoods of periodic orbits $\gamma_\varepsilon^-(c_{2j})$ and $\gamma_\varepsilon^+(c_{2j+1})$, $j = -N, \ldots, N$. In particular, $c_j = c_{j+2}$ holds including the $O(\varepsilon^2)$ term if $k_{2j} \neq k_{2j+1}$. Thus, transition motions described in Remark 2 occur in (1.5). We note that they may occur even when $\beta_3 = 0$ and $\alpha, \beta_2 \neq 0$, although it is more subtle since Eq. (5.10) is replaced with

$$\Delta c_k(c) = -\frac{2\beta_2}{\nu I_2}, \quad k = 1, 2,$$

in the above arguments.

5.3. Numerical Computations

Finally, we give numerical computations for heteroclinic orbits to periodic orbits in (1.5) for $(\alpha, \beta_3) = (0, 1)$ or $(\alpha, \beta_3) = (1, 0)$. The other parameter values are $I_1 = 0.8$, $I_2 = 1$, $I_3 = 2$, $\beta_2 = 1$ and $\nu = 1$, while ε, c are changed. The computations were carried out by using the computer tool `AUTO` [4] as follows.

Periodic orbits $\bar{\omega}^\pm(t)$ on the invariant manifolds $\mathscr{M}_\varepsilon^\pm$ are computed by applying the argument used in the proof of Proposition 2 based on Lemma 4 and solving the boundary value problem of (1.5) with the boundary conditions

$$\omega_1(0) = \omega_1(T/2) = 0$$

for $\alpha = 0$ and

$$\omega_3(0) = \omega_3(T/2) = 0$$

for $\beta_3 = 0$. So $\mathscr{M}_\varepsilon^\pm$ are obtained from continuations of the solutions of this boundary value problem when $\bar{\omega}_2^\pm(0)$ is changed. Consider the variational equation of (1.5) around the periodic orbits $\bar{\omega}^\pm(t)$,

$$\begin{aligned} \dot{\xi}_1 &= (a_1\bar{\omega}_3^\pm(t) - \varepsilon\bar{\alpha}_1 \sin\nu t)\xi_2 + a_1\bar{\omega}_2^\pm(t)\xi_3, \\ \dot{\xi}_2 &= (a_2\bar{\omega}_3^\pm(t) + \varepsilon\bar{\alpha}_2 \sin\nu t)\xi_1 + a_2\bar{\omega}_1^\pm(t)\xi_3, \\ \dot{\xi}_3 &= a_3\bar{\omega}_2^\pm(t)\xi_1 + a_3\bar{\omega}_1^\pm(t)\xi_2. \end{aligned} \tag{5.11}$$

Heteroclinic orbits from $\bar{\omega}^\mp(t)$ to $\bar{\omega}^\pm(t)$ are computed by solving the boundary value problems of (1.5) with the boundary conditions

$$L_{\mathrm{sc}}^\mp\big(\omega(-NT) - \bar{\omega}^\mp(0)\big) = 0, \quad L_{\mathrm{uc}}^\pm\big(\omega(NT) - \bar{\omega}^\pm(0)\big) = 0,$$

where $N \in \mathbb{N}$ is sufficiently large (in a numerical meaning), $\omega(-NT)$ and $\omega(NT)$ are close to $\bar{\omega}^{\mp}(0)$ and $\bar{\omega}^{\pm}(0)$, respectively, and $L_{\rm sc}^{\pm}$ and $L_{\rm uc}^{\pm}$ are, respectively, 2×3 matrices consisting of row eigenvectors for the monodromy matrices of (5.11) whose eigenvalues have no moduli which are greater and less than one (approximately). Moreover, orbits on the stable and unstable manifolds, $W^{\rm s}\big(\bar{\omega}^{\pm}(t)\big)$ and $W^{\rm u}\big(\bar{\omega}^{\pm}(t)\big)$, are computed by solving the boundary value problems of (1.5) with the boundary conditions

$$L_{\rm uc}^{\pm}\big(\omega(NT) - \bar{\omega}^{\pm}(0)\big) = 0, \quad \omega(0) = \omega_0^{\rm s}$$

and

$$L_{\rm sc}^{\mp}\big(\omega(-NT) - \bar{\omega}^{\mp}(0)\big) = 0, \quad \omega(0) = \omega_0^{\rm u},$$

respectively, where $\omega_0^{\rm s}$ and $\omega_0^{\rm u}$ are points on $W^{\rm s}\big(\bar{\omega}^{\pm}(t)\big)$ and $W^{\rm u}\big(\bar{\omega}^{\pm}(t)\big)$, respectively, and should be determined through the computations. So $W^{\rm s}\big(\bar{\omega}^{\pm}(t)\big)$ and $W^{\rm u}\big(\bar{\omega}^{\pm}(t)\big)$ are, respectively, obtained from continuations of $\omega_0^{\rm s}$ and $\omega_0^{\rm u}$ when $\omega(NT)$ and $\omega(-NT)$ are changed. The unperturbed nonhyperbolic saddles $\omega = (0, \pm c, 0)$ and heteroclinic orbits, $\omega_{\pm}^{\rm h}(t; c)$ and $\tilde{\omega}_{\pm}^{\rm h}(t; c)$, are taken as the starting solutions at $\varepsilon = 0$ for these boundary value problems, and numerically continued by `AUTO` with ε and c. Similar treatments for computations of the stable and unstable manifolds were also used in [20, 22].

Figure 8 shows numerically computed invariant manifolds $\mathscr{M}_{\varepsilon}^{\pm}$ consisting of periodic orbits on the Poincaré section

$$\Sigma = \{(\omega, \theta) \in \mathbb{R}^3 \times \mathbb{S}^1 \mid \theta = 0\}$$

for $(\alpha, \beta_3) = (0, 1)$ when $\varepsilon = 0.01$ and 0.05, where $\theta = \nu t \bmod 2\pi$. Here we take $c_\ell \approx 0.1$ and $c_{\rm r} \approx 1.5$ for the closed interval $I = [c_\ell, c_{\rm r}]$ in (A2).

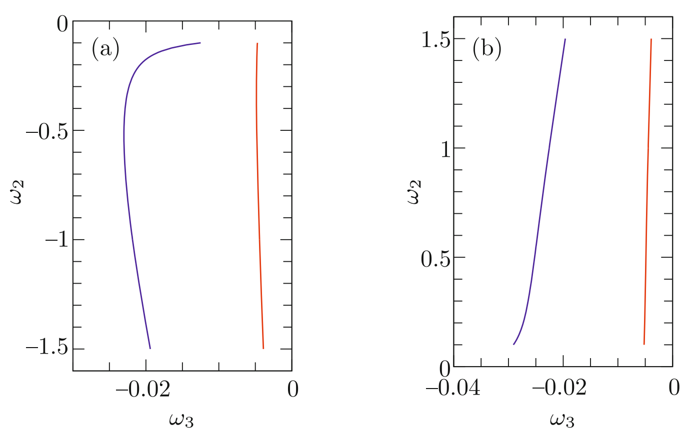

Fig. 8. Numerically computed invariant manifolds on the Poincaré section $\Sigma = \{\theta = 0\}$ for $(\alpha, \beta_3) = (0, 1)$: (a) $\mathscr{M}_{\varepsilon}^{-}$; (b) $\mathscr{M}_{\varepsilon}^{+}$. The red and blue lines represent the manifolds for $\varepsilon = 0.01$ and 0.05, respectively.

Figure 9 shows numerically computed heteroclinic orbits from $\gamma_{\varepsilon}^{-}(c)$ to $\gamma_{\varepsilon}^{+}(c + \varepsilon\Delta c)$ with $c = 1$ for $\varepsilon = 0.01$. Note that heteroclinic orbits exist only for special values of Δc as described in the theory. Actually, only for two such values heteroclinic orbits were continued from the unperturbed one for each value of c in the computations. We see that the arriving periodic orbit is different for the two heteroclinic orbits when $(\alpha, \beta_3) = (0, 1)$, while it is (almost) the same when $(\alpha, \beta_3) = (1, 0)$. Figures 10 and 11 show the dependence of Δc on ε or c for numerically computed heteroclinic orbits when $(\alpha, \beta_3) = (0, 1)$ and $(\alpha, \beta_3) = (1, 0)$, respectively. The red lines represent the theoretical predictions

$$\varepsilon\Delta c = \varepsilon\beta_2 \left(\pm D_1(c) - \frac{2}{\nu I_2}\right)$$

given in Theorem 4 up to $O(\varepsilon)$ in Fig. 10. The theoretical predictions almost completely coincide with the numerically computations and are not plotted in Fig. 11. Thus, excellent agreement between the theoretical predictions and numerical results is found, except when ε is relatively large or c is relatively small in Fig. 10: Heteroclinic bifurcations occur and the heteroclinic orbits disappear there. Note that the numerical observations do not contradict the statement of Theorem 4 since heteroclinic orbits are numerically computed for $\varepsilon > 0$ sufficiently small when the value of $c > 0$ is fixed.

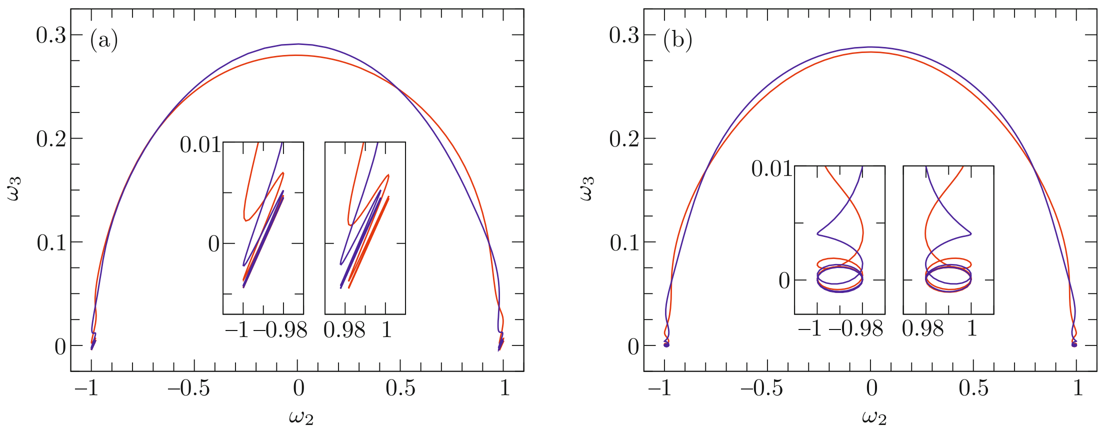

Fig. 9. Numerically computed heteroclinic orbits with $c = 1$ for $\varepsilon = 0.01$: (a) $(\alpha, \beta_3) = (0, 1)$; (b) $(\alpha, \beta_3) = (1, 0)$.

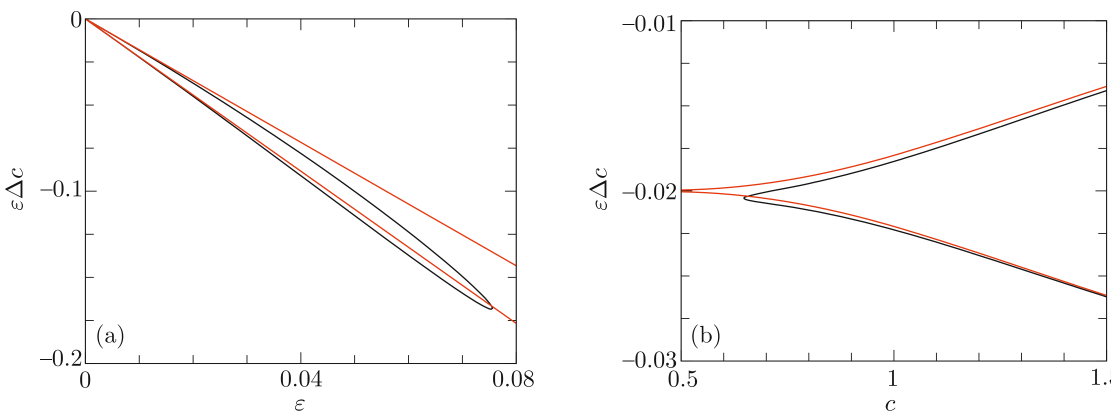

Fig. 10. Numerically computed dependence of $\varepsilon\Delta c$ on ε and c for $(\alpha, \beta_3) = (0, 1)$: (a) $c = 1$; (b) $\varepsilon = 0.01$. The red line represents the theoretical predictions.

Figure 12 shows numerically computed stable and unstable manifolds of periodic orbits with $c_\pm$ on $\mathscr{M}_\varepsilon^\pm$ for $(\alpha, \beta_3) = (0, 1)$ on the Poincaré section Σ, where $c_- = 1$. In Fig. 12a, when $\varepsilon = 0.05$, the stable manifolds of two periodic orbits with $c_+ = 0.885808, 0.899862$, which correspond to the values of Δc in Fig. 10a, on $\mathscr{M}_\varepsilon^+$ intersect the unstable manifold of a periodic orbit with $c_- = 1$ on $\mathscr{M}_\varepsilon^-$. In Fig. 12b, when $\varepsilon = 0.0755865$, the stable manifold of a periodic orbit with $c_+ = 0.832024$, which correspond to the value of Δc near the heteroclinic bifurcation point in Fig. 10a, on $\mathscr{M}_\varepsilon^+$ is almost tangent to the unstable manifold of a periodic orbit with $c_- = 1$ on $\mathscr{M}_\varepsilon^-$. In Fig. 12c, when $\varepsilon = 0.08$, the stable and unstable manifolds of a periodic orbit with $c_- = 1$ on $\mathscr{M}_\varepsilon^-$ intersect, although such heteroclinic orbits as plotted in Fig. 9a already disappear through a heterocliinic bifurcation. Here we remark that by Lemma 4 the unstable manifold of a periodic orbit on $\mathscr{M}_\varepsilon^-$ is symmetric to its stable manifold about the (ω_2, ω_3)-plane on Σ, and in particular, if the unstable manifold intersects the (ω_2, ω_3)-plane on Σ, then it intersects the stable manifold on Σ, as shown in Fig. 12c.

We finally give numerical simulation results: Fig. 13 shows orbits of the Poincaré section Σ and Fig. 14 shows computed variations of $c = F_1(\omega)$ for them. Here a Fortran code called "DOP853" [12]

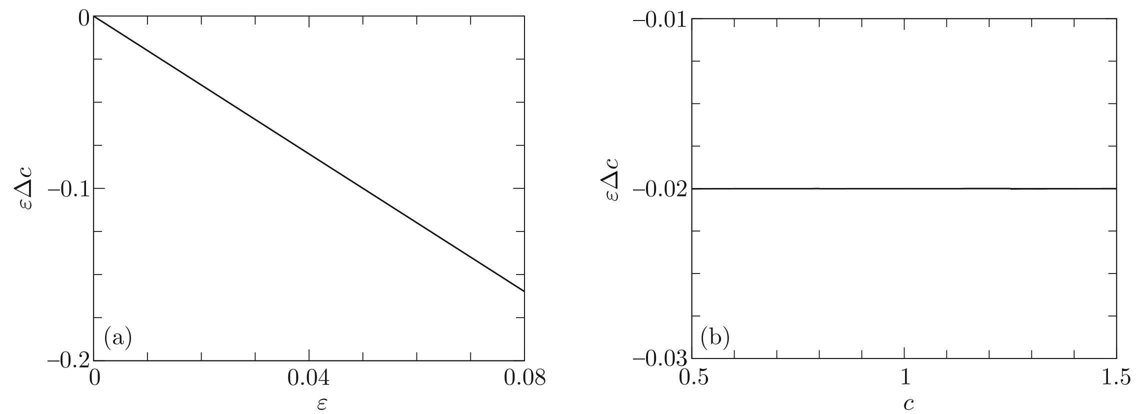

Fig. 11. Numerically computed dependence of $\varepsilon\Delta c$ on ε and c for $(\alpha, \beta_3) = (1, 0)$: (a) $c = 1$; (b) $\varepsilon = 0.01$.

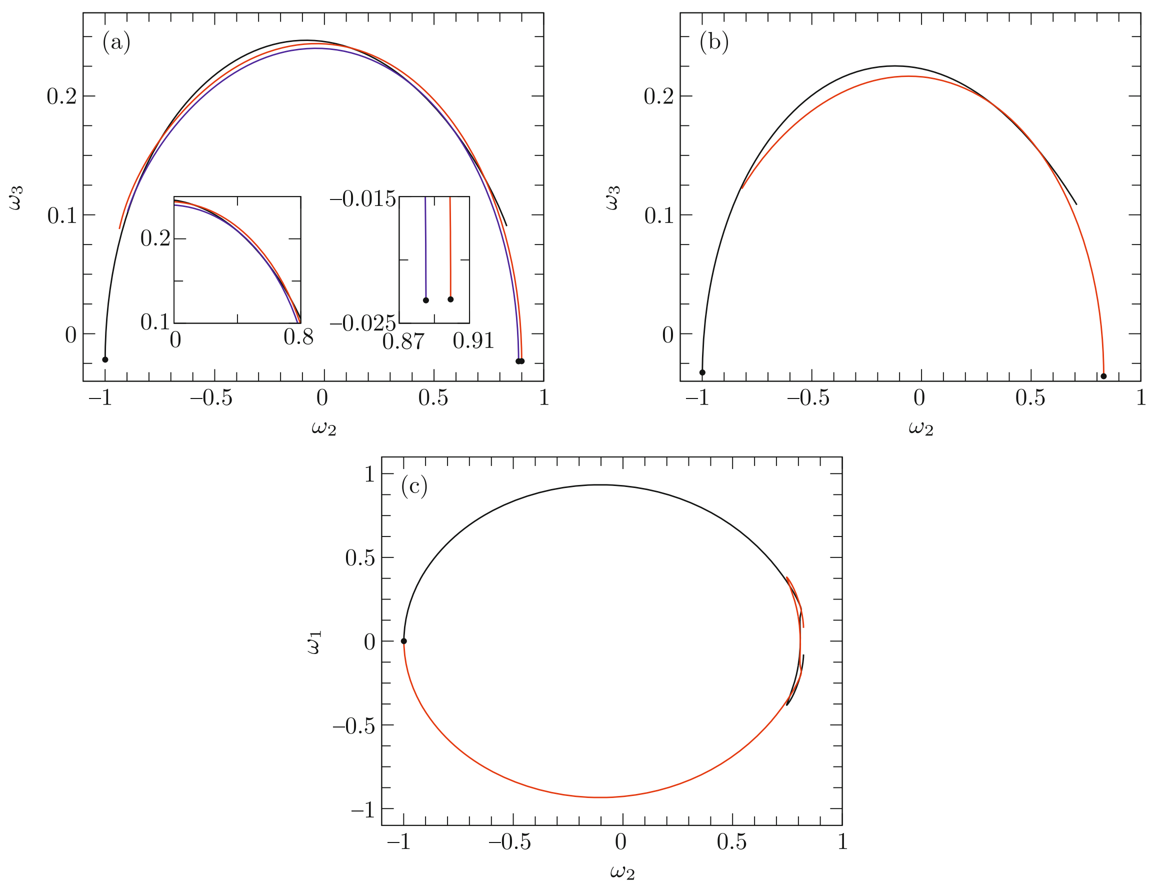

Fig. 12. Numerically computed stable and unstable manifolds of periodic orbits with $c_- = 1$ on $\mathscr{M}_\varepsilon^-$ and with c_+ on $\mathscr{M}_\varepsilon^+$ for $(\alpha, \beta_3) = (0, 1)$ on the Poincaré section Σ: (a) $\varepsilon = 0.05$, $c_+ = 0.885808, 0.899862$; (b) $\varepsilon = 0.0755865$, $c_+ = 0.832024$; (c) $\varepsilon = 0.08$, $c_+; = 1$. The black line represents the unstable manifolds, and the red and blue lines represent the stable manifolds. The point "•" represents periodic orbits.

was used with a tolerance of 10^{-8}. The code is based on the explicit Runge–Kutta method of order 8 and a fifth-order error estimator with third-order correction is utilized. Thus, the numerical results are very accurate. The initial point is chosen at $\omega = (10^{-3}, 1, 10^{-3})$ near the unperturbed nonhyperbolic saddle on Σ for both orbits. We observe that the value of $F_1(\omega)$ varies heavily for $\alpha = 0$, but not for $\beta_3 = 0$. One of the reasons for this observation is thought to be the difference of transition motions described at the end of Section 5.2.

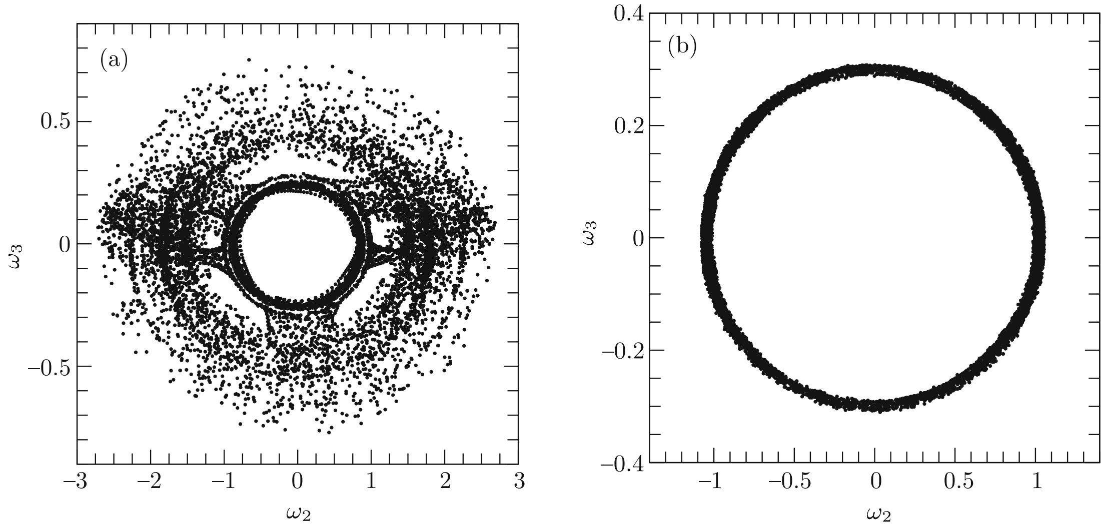

Fig. 13. Numerically computed orbits on the Poincaré section Σ: (a) $(\alpha, \beta_3) = (0, 1)$; (b) $(\alpha, \beta_3) = (1, 0)$.

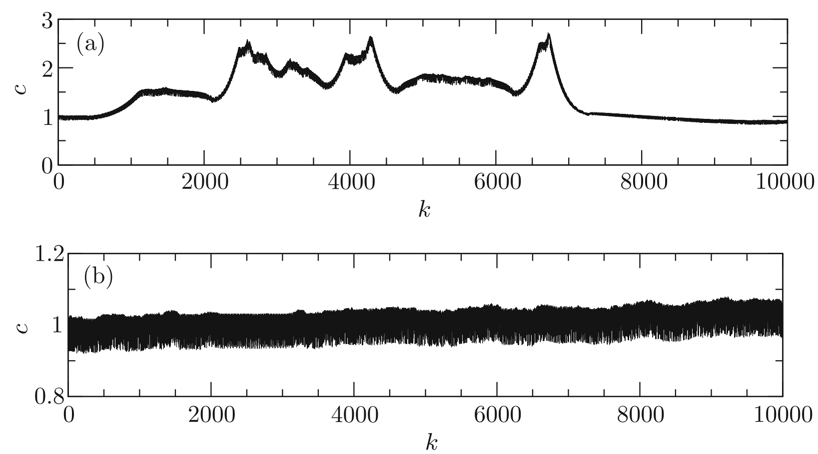

Fig. 14. Numerically computed variations of $F_1(\omega)$ for the orbits shown in Fig. 13: (a) $(\alpha, \beta_3) = (0, 1)$; (b) $(\alpha, \beta_3) = (1, 0)$. The abscissa k represents the iteration number of the Poincaré map.

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