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Linear Hamiltonian Systems: Quadratic Integrals, Singular Subspaces and Stability

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Abstract—A chain of quadratic first integrals of general linear Hamiltonian systems that have not been represented in canonical form is found. Their involutiveness is established and the problem of their functional independence is studied. The key role in the study of a Hamiltonian system is played by an integral cone which is obtained by setting known quadratic first integrals equal to zero. A singular invariant isotropic subspace is shown to pass through each point of the integral cone, and its dimension is found. The maximal dimension of such subspaces estimates from above the degree of instability of the Hamiltonian system. The stability of typical Hamiltonian systems is shown to be equivalent to the degeneracy of the cone to an equilibrium point. General results are applied to the investigation of linear mechanical systems with gyroscopic forces and finite-dimensional quantum systems.

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1. LINEAR HAMILTONIAN SYSTEMS

Consider a linear system of differential equations

$$\Gamma\dot{x} = -Px, \quad x \in \mathbb{R}^r. \tag{1.1}$$

Here, Γ is a skew-symmetric operator and P is a symmetric operator (with respect to the scalar product $(,)$ in $\mathbb{R}^r$). These operators are assumed to be *nondegenerate*. Since $|\Gamma| \neq 0$, r is even. Let $r = 2n$ be the dimension of phase space.

The system (1.1) is Hamiltonian. The symplectic structure is given by the nondegenerate 2-form

$$\Omega(\xi, \eta) = (\Gamma\xi, \eta), \tag{1.2}$$

and the quadratic form

$$H = \frac{1}{2}(Px, x) \tag{1.3}$$

serves as the Hamiltonian. In invariant notation, the Hamiltonian property of the system (1.1) means the following:

$$i_v\Omega = -dH,$$

where i_v is the inner product of the vector field $v(x) = -\Gamma^{-1}Px$ and the 2-form Ω. Indeed,

$$i_v\Omega = (\Gamma v, \eta) = -\big(\Gamma(\Gamma^{-1}Px), \eta\big) = -(Px, \eta) = -dH(\eta).$$

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By a nondegenerate linear transformation the matrix of the operator Γ can be reduced to the symplectic identity matrix

$$\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}, \tag{1.4}$$

where E_n is the identity matrix of the nth order. If $x = (x_1, \ldots, x_{2n})$, then the variables x_k and x_{n+k} $(k \geqslant 1)$ will be canonically conjugate. Conversely, the matrix of the symmetric operator P can be reduced to a diagonal matrix with elements on the diagonal $+1$ or -1.

The system (1.1) admits the nondegenerate quadratic integral (1.3) (energy integral). On the other hand, any *nondegenerate* linear system in $\mathbb{R}^r = \{x\}$ (with the only equilibrium point $x = 0$) which admits an integral in *nondegenerate* quadratic form is Hamiltonian [1].

The spectrum of a linear Hamiltonian system is symmetric with respect to the real and imaginary axes of the complex plane. If i^+ and i^- are the indices of inertia of the quadratic integral (1.3), then for the degree of instability of u (the number of roots of the equation $|\Gamma^{-1}P + \lambda E| = 0$ from the right complex half-plane) we have the inequality

$$u \leqslant \min(i^-, i^+). \tag{1.5}$$

For linear conservative mechanical systems with gyroscopic forces the estimate (1.5) is established in [2]. For general linear systems with a nondegenerate quadratic integral, inequality (1.5) is proved in [3].

Further,

$$u \equiv i^- \pmod 2. \tag{1.6}$$

In particular, if the negative index of inertia of the energy integral is odd, then the equilibrium point $x = 0$ of the system (1.1) is unstable (according to a generalized Kelvin theorem). Since $i^- + i^+ = 2n$, one can replace i^- in (1.6) with i^+.

Hamiltonian systems in general form (1.1) (when the skew-symmetric matrix is not reduced to the symplectic identity matrix (1.4)) are often encountered in applications. As an example, we consider small oscillations of a mechanical system in a potential force field, taking gyroscopic forces into account. The equations of motion linearized near an equilibrium point are brought to the following form:

$$\ddot{x} + G\dot{x} + Wx = 0, \quad x \in \mathbb{R}^n. \tag{1.7}$$

Here, G is the skew-symmetric matrix of gyroscopic forces and the symmetric matrix W generates the potential energy of the system

$$\frac{1}{2}(Wx, x).$$

Equations (1.7) can be represented in the form (1.1):

$$\begin{pmatrix} G & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = - \begin{pmatrix} W & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad y = \dot{x}. \tag{1.8}$$

Here,

$$\Gamma = \begin{pmatrix} G & E \\ -E & 0 \end{pmatrix}, \quad P = \begin{pmatrix} W & 0 \\ 0 & E \end{pmatrix}, \tag{1.9}$$

$H = \frac{1}{2}(y, y) + \frac{1}{2}(Wx, x)$ is the total energy of the mechanical system. The skew-symmetric matrix Γ is obviously nondegenerate, and $|P| = |W|$. Consequently, $|P| \neq 0$ if $x = 0$ is the only equilibrium point of the system (1.7).

We now replace G with NG and introduce a fast time variable $\tau = Nt$ in (1.7), assuming N to be a large parameter. Dividing (1.7) by N^2 and letting N tend to infinity, we obtain a linear equation for the velocity $y = x'$:

$$\frac{dy}{d\tau} + Gy = 0, \quad y \in \mathbb{R}^n. \tag{1.10}$$

If we introduce a slow time variable $\tau = t/N$ and make a formal passage to the limit $N \to \infty$ in (1.7), we obtain a system of the form (1.1):

$$G\frac{dx}{d\tau} = -Wx, \quad x \in \mathbb{R}^n. \tag{1.11}$$

Suppose the matrix of gyroscopic forces G is nondegenerate (then the number of degrees of freedom n is even). In this case, both systems (1.10) and (1.11) are Hamiltonian and all eigenvalues in (1.10) are purely imaginary and different from zero. Moreover, the trivial equilibrium $y = 0$ of the system (1.10) is stable. Thus, the problem of stability of the equilibrium of the initial system (1.7) with asymptotically large gyroscopic forces reduces to that of conditions for stability of the Hamiltonian system (1.11) with half as many degrees of freedom.

A detailed analysis of passages to the limit from the system (1.7) to the linear systems (1.10) and (1.11) is presented, for example, in [4]. As $N \to \infty$, the phase space of the system (1.7) breaks down into a direct sum of two invariant subspaces of dimension n, and the restrictions of the initial system to these subspaces are Hamiltonian systems. It turns out that such a breakdown takes place for sufficiently large values of $\|G\|$ [5].

2. QUADRATIC INTEGRALS

Theorem 1. *The Hamilton equations* (1.1) *admit a family of quadratic first integrals*

$$\Phi_m = \frac{1}{2}\big(P(P^{-1}\Gamma)^m x,\, (P^{-1}\Gamma)^m x\big), \quad m \in \mathbb{Z}, \tag{2.1}$$

where

1) *all* Φ_m *are nondegenerate quadratic forms and their signatures coincide with that of the total energy* $H = \Phi_0$,

2) *the function* (2.1) *are pairwise in involution* (*in the symplectic structure given by the nondegenerate skew-symmetric operator* Γ).

We first prove that $\dot\Phi_m = 0$. Indeed,

$$\begin{aligned} 2\dot\Phi_m &= -\big(P(P^{-1}\Gamma)^m\Gamma^{-1}Px,\, (P^{-1}\Gamma)^m x\big) \\ &\quad - \big(P(P^{-1}\Gamma)^m x,\, (P^{-1}\Gamma)^m\Gamma^{-1}Px\big) \\ &= 2(-1)^{m+1}\big((\Gamma P^{-1})^{2m-1}Px,\, x\big) = 0, \end{aligned}$$

since

$$(\Gamma P^{-1})^{2m-1}P + \big[(\Gamma P^{-1})^{2m-1}P\big]^* = (\Gamma P^{-1})^{2m-1}P - P(P^{-1}\Gamma)^{2m-1} = 0.$$

The symbol $*$ stands for conjugation.

The inverse substitution

$$x \mapsto (P^{-1}\Gamma)^m x$$

transforms the quadratic form Φ_m into the form H. Consequently, their signatures coincide.

In order to calculate the Poisson bracket of the functions Φ_m and Φ_k, we consider a Hamiltonian system of the form (1.1) with the Hamiltonian Φ_m

$$\Gamma x' = -\frac{\partial \Phi_m}{\partial x} = -(-\Gamma P^{-1})^m P (P^{-1}\Gamma)^m x \tag{2.2}$$

and calculate the derivative of the quadratic form Φ_k by virtue of the system (2.2). This is just the Poisson bracket $\{\Phi_k, \Phi_m\}$, which, up to sign, is equal to

$$2\big((\Lambda^*)^k P \Lambda^k \Gamma^{-1} (\Lambda^*)^m P \Lambda^m x,\, x\big), \tag{2.3}$$

where $\Lambda = P^{-1}\Gamma$. It is easy to calculate the product of the operators in (2.3). It is equal to

$$(\Gamma P^{-1})^{2s}\Gamma, \quad s = k + m - 1.$$

Since

$$(\Gamma P^{-1})^{2s}\Gamma + \big[(\Gamma P^{-1})^{2s}\Gamma\big]^* = 0,$$

the quadratic form (2.3) is zero. This proves the theorem.

The question arises: How many functionally independent quadratic forms are there in the chain (2.1)? It is well known from symplectic geometry that there are no more than n of them. On the other hand, if Γ is the symplectic identity matrix (1.4) and $P = E_n$, then all Φ_m coincide and therefore all these integrals reduce to an energy integral. However, in the family of quadratic forms (2.1) there are typically n functionally independent ones.

Theorem 2. *If the spectrum of the linear Hamiltonian system* (1.1) *is simple* (*i. e., there are no multiple eigenvalues*)*, then the first integrals*

$$\Phi_0,\ \Phi_{-1},\ \ldots,\ \Phi_{-n+1}$$

are functionally independent.

We recall that the presence of multiple roots of a characteristic polynomial is due to the fact that its discriminant vanishes. The discriminant in turn polynomially depends on the coefficients of the characteristic polynomial. In particular, the simplicity of the spectrum of the linear Hamiltonian system can be defined without solving the characteristic equation.

Remark. If among the squares of eigenvalues there are q different ones, then the rank of the Jacobi matrix

$$\frac{\partial(\Phi_0, \Phi_{-1}, \ldots, \Phi_{-n+1})}{\partial(x_1, \ldots, x_n, \ldots, x_{2n})}$$

is no less than q.

Theorem 2 is proved using the theory of real normal forms of linear Hamilton equations (which goes back to Williamson [6, 7]; see also [8, 9]). According to Williamson, the phase space $\mathbb{R}^{2n}$ can be decomposed into a direct sum of skew-orthogonal (with respect to the symplectic structure (1.2)) subspaces in such a way that the matrix of the operator Γ has the "canonical" form (1.4) and the Hamiltonian is represented as a sum of quadratic forms (partial Hamiltonians) on these subspaces, and

(a) the real pair $\pm a$ of eigenvalues corresponds to the Hamiltonian

$$-apq$$

(here and in what follows, p and q denote a pair of conjugate canonical variables),

(b) the purely imaginary pair $\pm ib$ corresponds to

$$\frac{b}{2}(p^2 + q^2),$$

(c) the quadruple of eigenvalues $\pm a \pm ib$ corresponds to

$$-a(p_1q_1 + p_2q_2) + b(p_1q_2 - p_2q_1).$$

In these canonical variables the matrix Γ takes the block-diagonal form

$$\operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right],$$

and the matrix of the symmetric operator P also has block-diagonal form, where the diagonal is filled with matrices corresponding to cases (a), (b) and (c), respectively,

$$\begin{pmatrix} 0 & -a \\ -a & 0 \end{pmatrix}, \quad \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -a & 0 & b \\ -a & 0 & -b & 0 \\ 0 & -b & 0 & -a \\ b & 0 & -a & 0 \end{pmatrix}. \tag{2.4}$$

It is easy to calculate that the symmetric matrix of the quadratic form Φ_{-m} (i. e., the matrix

$$(\Gamma^{-1}P)^{*m}P(\Gamma^{-1}P)^m)$$

in these coordinates has the same form as the matrix P, but only in cases (a) and (b) do the multipliers

$$(-1)^{m+1}a^{2m+1} \quad \text{and} \quad b^{2m+1},$$

respectively, appear in front of the blocks on the diagonal. As for case (c), the restriction of Φ_{-m} to the plane with coordinates p_1, q_1, p_2, q_2 takes the form

$$\Re(-1)^{m+1}(a - ib)^{2m+1}(p_1q_1 + p_2q_2) + \Im(-1)^{m+1}(a - ib)^{2m+1}(p_1q_2 - p_2q_1).$$

Note that the number $a - ib$ is one of the four eigenvalues.

As a result, the question of functional independence of the quadratic forms Φ_0, Φ_{-1}, …, Φ_{-n+1} reduces to that of a nonzero value of the determinant of the matrix of the nth order, where in cases (a) and (b) the numbers

$$-a,\ a^3,\ -a^5,\ \dots,\ (-1)^n a^{2n-1} \quad \text{and} \quad b,\ b^3,\ b^5,\ \dots,\ b^{2n-1}$$

are the columns of this matrix, and in case (c) we have two columns

$$\big(-a,\ a^3 - 3ab^2,\ \dots,\ (-1)^n\Re(a - ib)^{2n-1}\big)^*,$$
$$\big(b,\ -3a^2b + b^3,\ \dots,\ (-1)^n\Im(a - ib)^{2n-1}\big)^*.$$

Since

$$(-1)^{m+1}a^{2m+1} = i(ia)^{2m+1},$$
$$(-1)^{m+1}(a - ib)^{2m+1} = i(ia + b)^{2m+1},$$

this determinant is equal (up to a nonzero multiplier) to that of the $(n \times n)$ matrix with columns

$$(\lambda,\ \lambda^3,\ \dots,\ \lambda^{2n-1})^*,$$

with $\lambda = ia$ in case (a), $\lambda = b$ in case (b) and $\lambda = ia + b$, $\lambda = -ia + b$ in case (c). According to Vandermonde, this determinant is equal to the product of (nonzero) values of λ and to the product of differences of the squares of these numbers. Since the spectrum of the Hamiltonian system is simple, all sums and differences of values of λ are nonzero. This proves Theorem 2.

It is well known that a linear Hamiltonian system with n degrees of freedom always admits n independent *quadratic* first integrals in involution. This assertion was first advanced as a hypothesis

by A. Wintner [10] and proved by J. Williamson in [7]. Another proof, different in form from that in [7], is given in [8]. These proofs are based on using the real normal forms of linear Hamiltonian systems. The meaning of Theorems 1 and 2 is that a complete set of quadratic involutive integrals is presented without preliminarily finding the eigenvalues and eigenvectors of the linear system.

Consideration was also given to the problem of polynomial and rational integrals of the linear system of differential equations of general form (see, e. g., [11, 12] and references therein).

Remark. Take the integral Φ_{-m} as a Hamiltonian. The Hamiltonian system

$$\Gamma\dot{x} = -\frac{\partial\Phi_{-m}}{\partial x}$$

has the following explicit form:

$$\dot{x} = (-1)^{m+1}(\Gamma^{-1}P)^{2m+1}x. \tag{2.5}$$

If $\lambda_1, \ldots, \lambda_{2n}$ ($\lambda_{n+j} = -\lambda_j$) are the eigenvalues of the Hamiltonian system (1.1), then

$$(-1)^m\lambda_1^{2m+1}, \ldots, (-1)^m\lambda_{2n}^{2m+1}$$

are the eigenvalues of the linear system (2.5) with the same eigenvectors.

3. INTEGRAL CONES AND SINGULAR SUBSPACES

The set

$$K = \{x \mid \Phi_m(x) = 0,\ m \in \mathbb{Z}\} \tag{3.1}$$

will be a cone in $2n$-dimensional phase space. Let us call it an *integral cone*. It is invariant under the phase flow of the linear Hamiltonian system (1.1).

We recall that the linear subspace Π is called the *singular subspace* of the cone K if Π lies entirely in the cone K. There is the well-known inequality

$$\dim\Pi \leqslant \min(i^-,\, i^+), \tag{3.2}$$

where i^- and i^+ are the indices of inertia of the quadratic form (1.3). Indeed, the right-hand side of (3.2) is the dimension of the maximal completely singular subspace of the cone

$$\{x \in \mathbb{R}^{2n} \mid H(x) = \Phi_0(x) = 0\} \supset K$$

(see, e. g., [13]).

The linear subspace $\Pi \subset \mathbb{R}^{2n}$ is called *isotropic* if

$$(\Gamma\xi,\, \eta) = 0$$

for all $\xi, \eta \in \Pi$. The isotropic subspace of the maximum possible dimension (equal to n) is called Lagrangian subspace.

Theorem 3. *Through each point $a \in K$ there passes a singular subspace $\Pi(a)$ of the cone* (3.1) *such that*

1. $\Pi(a)$ *contains a sequence of points*

$$\Lambda^m a, \quad m \in \mathbb{Z}; \quad \Lambda = P^{-1}\Gamma, \tag{3.3}$$

2.

$$\dim\Pi(a) = r = \operatorname{rank}[\ldots, \Lambda^m a, \ldots],$$

3.

$$\dim\Pi(a) \leqslant \min(i^-,\, i^+), \tag{3.4}$$

4. $\Pi(a)$ *is invariant under the phase flow of the Hamiltonian system* (1.1),
5. $\Pi(a)$ *is an isotropic subspace.*

Remarks. 1°. If $a = 0$, then the rank of the matrix (3.4) is zero and the singular subspace is a zero subspace.

2°. Since m runs over a sequence of *all* integer numbers, the operator Λ in (3.3) can be replaced with the operator

$$M = \Lambda^{-1} = \Gamma^{-1}P.$$

This operator corresponds to the Hamiltonian system (1.1) solved for $\dot{x}$: $\dot{x} = -Mx$. In the formula (3.4) for the dimension of the singular subspace the operator Λ can also be replaced with M.

3°. The rank of the matrix

$$[\ldots, M^m a, \ldots]$$

plays a key role in the *theory of controllability* of linear systems.

4°. Since the matrix Λ satisfies its characteristic equation (of degree $2n$), the infinite matrix in (3.4) can be replaced with the quadratic matrix

$$[a, \Lambda a, \ldots, \Lambda^{2n-1}a] \tag{3.5}$$

of order $2n$. By the way, since the system (1.1) is Hamiltonian, the characteristic polynomials of the matrices Λ and M do not contain odd degrees. In particular, the rank of the matrix (3.5) does not exceed n. This fact also follows from conclusion 3 because $\min(i^-, i^+) \leqslant n$. We also note that in (3.5) the dimension of phase space $2n$ can be replaced with the degree of the *minimal polynomial* of the matrix Λ. Thus, the dimension of the singular subspace from Theorem 3 does not exceed the minimal polynomial of the matrix Λ (or M).

Proof (*of Theorem* 3). 1) If $a \in K$, then obviously $\Lambda^m a \in K$ for all integer m. We show that the linear span $\Pi(a)$ of the set of vectors (3.3) is a singular subspace of the integral cones (3.1). Set

$$a_s = \Lambda^s a, \quad s \in \mathbb{Z} \tag{3.6}$$

and let

$$\langle \xi, \eta \rangle = (P\Lambda^m \xi, \Lambda^m \eta).$$

According to the assumption, $a_p \in K$ and $a_q \in K$ for all integer p, q. We have to show that their linear linear combination

$$\lambda a_p + \mu a_q$$

also belongs to the integral cone K. This means that

$$\langle \lambda a_p + \mu a_q, \lambda a_p + \mu a_q \rangle = 2\lambda\mu \langle a_p, a_q \rangle$$

must be equal to zero.

We have

$$\langle a_p, a_q \rangle = (P\Lambda^{m+p}a, \Lambda^{m+q}a).$$

This is equal, up to sign, to

$$\big((\Gamma P^{-1})^s P (P^{-1}\Gamma)^{s+r} a, a\big), \tag{3.7}$$

where $s = m + q$ and $r = p - q$.

The product of the operators in (3.7) is

$$\Gamma(P^{-1}\Gamma)^l,$$

where $l = 2s + r - 2$. If l is even (and equal to $2k$), then

$$\big(\Gamma(P^{-1}\Gamma)^{2k} a, a\big) = (-1)^k \big(\Gamma(P^{-1}\Gamma)^k a, (P^{-1}\Gamma)^k a\big) = 0,$$

since Γ is a skew-symmetric operator.

Now let $l = 2k + 1$. Then

$$\big(\Gamma(P^{-1}\Gamma)^{2k+1}a,\, a\big) = (-1)^{k+1}\big(P(P^{-1}\Gamma)^{k+1}a,\, (P^{-1}\Gamma)^{k+1}a\big) = 0,$$

because $(P^{-1}\Gamma)^{k+1}a \in K$.

2) Since $\Pi(a)$ is the linear span of the set of vectors (3.3), its dimension is equal to the number of linearly independent vectors from the sequence (3.3). But this number obviously coincides with the rank of the infinite matrix (3.4).

3) Conclusion 3 of the theorem follows from inequality (3.2) for the dimension of maximal singular subspaces.

4) To prove the invariance of $\Pi(a)$, it suffices to show that at points $\Lambda^m a$, $m \in \mathbb{Z}$, the Hamiltonian vector field also lies in $\Pi(a)$. But this field coincides with the vector

$$M(\Lambda^m a) = \Lambda^{m-1}a$$

from the sequence (3.3). This vector belongs to the cone K and hence to the subspace $\Pi(a)$ (according to conclusion 1).

5) The isotropy of the subspace $\Pi(a)$ means that

$$(\Gamma a_p,\, a_q) = 0$$

for all a_p, $a_q \in \Pi(a)$ (defined by (3.6)). Since

$$(\Gamma a_p,\, a_q) = \big(\Gamma(P^{-1}\Gamma)^p a,\, (P^{-1}\Gamma)^q a\big) = (-1)^q\big(\Gamma(P^{-1}\Gamma)^l a,\, a\big),$$

$l = p + q$, this expression is equal to zero (according to the arguments in the proof of conclusion 1). □

Example. Let a be the eigenvector of the Hamilton operator $M = \Gamma^{-1}P$ with the real eigenvalue $\lambda \neq 0$. We show that $a \in K$. Indeed, Eqs. (1.1) admit the solution

$$x(t) = ae^{-\lambda t},$$

which tends to zero as $t \to +\infty$ or $t \to -\infty$. Since Φ_m are first integrals, $\Phi_m\big(x(t)\big) = c = \text{const}$. Passing to the limit as $t \to +\infty$ or $t \to -\infty$, we find that $c = 0$.

In this example, the matrix (3.4) has the form

$$[\ldots,\, \lambda^{-m}a,\, \ldots].$$

Its rank is 1. Hence the invariant isotropic subspace $\Pi(a)$ coincides with the straight line $\{x = \mu a,\ \mu \in \mathbb{R}\}$.

These observations can be generalized. Here is a simple statement which extends the classical Kelvin theorem.

Theorem 4. *If the dimension of* $\Pi(a)$ *is odd for some* $a \in K$*, the equilibrium of the Hamiltonian system is unstable.*

Indeed, let

$$\dot{z} = Az, \quad z \in \Pi$$

be the restriction of the linear system to the invariant subspace Π. Since the initial Hamiltonian system has no nontrivial equilibria, $|A| \neq 0$. Consequently, the characteristic polynomial $|A - \lambda E|$ of odd degree has a nonzero real root λ. Due to the property of being Hamiltonian, the system (1.1) also has the characteristic root $-\lambda$. Since one of the numbers λ, $-\lambda$ is positive, the Hamiltonian system is unstable.

4. ESTIMATE OF THE DEGREE OF INSTABILITY

Let L denote an isotropic singular subspace of the integral cone K which is invariant under the phase flow of the Hamiltonian system (1.1). As we have seen in Section 3, these subspaces form whole families. Let $\mathcal{L}$ denote the set of *all* such subspaces.

The following estimate holds:

$$u \leqslant \max_{L\in\mathcal{L}} \dim L. \tag{4.1}$$

Inequality (1.5) is a particular case of this well-known result, when the cone $\{H(x) = 0\}$ is taken as an integral cone. Inequality (4.1) will become rougher if one assumes that $\mathcal{L}$ contains all isotropic singular subspaces of the cone K (without assuming their invariance). In this case, inequality (4.1) will be purely geometrical.

To prove inequality (4.1), we consider the linear subspace L of solutions of Hamilton equations which tend to an equilibrium point as $t \to -\infty$. This subspace is obviously invariant under the phase flow of the linear system (1.1). We show that it is isotropic. For this purpose, we use the Poincaré theorem:

$$\Omega\big(\xi(t),\, \eta(t)\big) = \big(\Gamma\xi(t),\, \eta(t)\big) = \text{const} \tag{4.2}$$

for any two solutions of $\xi(\cdot)$ and $\eta(\cdot)$ of the Hamilton equations (1.1). Now letting time t tend to $-\infty$, we obtain a zero constant in (4.2). Its singularity is proved in a similar way.

Thus, on the isotropic invariant singular subspace L of the integral cone K the equality $u = \dim L$ is satisfied. Since $L \in \mathcal{L}$, inequality (4.1) follows immediately. This proves the theorem.

Theorem 5. *If the spectrum of the Hamiltonian system* (1.1) *is simple, then*

1. *the cone K is a union of 2^k subspaces of the same dimension, where k is a general number of real pairs and complex quadruples of eigenvalues,*

2. *among these subspaces there is a u-dimensional unstable subspace filled with trajectories of the Hamiltonian system* (1.1)*, which tend to the origin of coordinates as $t \to -\infty$.*

If $k = 0$, then the cone K reduces to the only invariant subspace of zero dimension (i. e., $K = \{0\}$).

Corollary 1. $u = \max\limits_{a\in K} \dim \Pi(a)$.

Corollary 2. *For all $a \in K$*

$$u \geqslant \dim \Pi(a). \tag{4.3}$$

Although inequalities (4.2) and (4.3) are opposite in character, there is no contradiction here, since (by Theorem 3) the subspaces $\Pi(a)$ belong to a set of all possible singular, isotropic and invariant subspaces.

The proof of Theorem 5 uses the normal form method, due to Williamson, which has already been employed in the proof of Theorem 2. According to Section 2, the independent quadratic integrals

$$\Phi_0,\ \Phi_{-1},\ \ldots,\ \Phi_{-n+1} \tag{4.4}$$

are represented in the form of linear combinations of partial Hamiltonians. Moreover, the system of algebraic equations

$$\Phi_0 = \Phi_{-1} = \ldots = \Phi_{-n+1} = 0 \tag{4.5}$$

is equivalent to a system that is obtained by equating to zero (in the notation of Section 2) pq in case (a) (which corresponds to a pair of real eigenvalues), $p^2 + q^2$ in case (b) (a pair of purely imaginary roots) and $p_1q_1 + p_2q_2$, $p_1q_2 - p_2q_1$ in case (c) (complex quadruple of eigenvalues).

Since the integrals (4.4) are pairwise in involution, all the other quadratic integrals Φ_m are linear combinations of the integrals (4.4). Thus, the integral cone K is defined by the system of Eqs. (4.5).

Further, equating to zero the partial Hamiltonian in case (a), we obtain the solutions

$$\{p = 0\} \quad \text{or} \quad \{q = 0\}, \tag{4.6}$$

which yield two invariant straight lines for this partial Hamiltonian. In case (b) we obtain the origin of coordinates $p = q = 0$. In case (c) we have the system of equations

$$p_1q_1 + p_2q_2 = 0, \quad p_1q_2 - p_2q_1 = 0,$$

which (as in case (a)) has two solutions

$$\{p_1 = p_2 = 0\} \quad \text{or} \quad \{q_1 = q_2 = 0\}. \tag{4.7}$$

Each of them is a plane in four-dimensional space which is invariant under the phase flow given by the corresponding partial Hamiltonian. They are both filled with trajectories approaching the origin; one as $t \to +\infty$ and the other as $t \to -\infty$.

The singular isotropic invariant subspaces from Theorem 5 are obtained as direct sums of the straight lines (4.6) or the planes (4.7). The number of different possible combinations is 2^k, where k is the number of real pairs and complex quadruples in the spectrum of the linear Hamiltonian system. All these subspaces have obviously the same dimension, and among them there is exactly one unstable invariant subspace of dimension u. This proves the theorem.

5. STABILITY CONDITIONS

Theorem 6. *If $K = \{0\}$, then the equilibrium $x = 0$ of the Hamiltonian system* (1.1) *is stable. If the equilibrium $x = 0$ is stable and there are no equal numbers among the characteristic numbers of the system* (1.1)*, then $K = \{0\}$.*

As is well known, the absence of multiple roots of a characteristic polynomial can be determined without finding them.

Proof (*of Theorem* 6). Suppose the integral cone K reduces to one point $x = 0$. Then

$$\sum \alpha_m \Phi_m^2, \quad m \in \mathbb{Z}, \tag{5.1}$$

can be taken as a Lyapunov function, where α_m is a sequence of positive numbers which rapidly tends to zero as $|m| \to \infty$ (in order to ensure convergence of the series (5.1)).

Conversely, let the equilibrium $x = 0$ be stable and the spectrum of the Hamiltonian system be simple. In this case, $u = 0$ and hence (by Theorem 5) the dimension of the subspaces forming the integral cone K is zero. But then $K = \{0\}$, which proves Theorem 6. □

As an illustrative example, we consider a Hamiltonian system of the form (1.1) with $n = 4$. Let

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{5.2}$$

and let the Hamiltonian be represented by the diagonal matrix

$$P = \operatorname{diag}(\lambda_1, 1, -\lambda_2, -1). \tag{5.3}$$

Here, λ_1 and λ_2 are positive numbers. Hence the signature of the Hamiltonian as a quadratic form is $++--$. It can be seen from (5.2) that x_1, x_3 and x_2, x_4 serve as canonically conjugate variables.

The characteristic polynomial of the Hamiltonian system $\lambda^4 - (1 + \lambda_1\lambda_2)\lambda^2 + \lambda_1\lambda_2$ has no multiple roots if

$$\lambda_1\lambda_2 \neq 1. \tag{5.4}$$

It is easy to calculate that (by Theorem 1) the second quadratic integral is

$$\Phi = \frac{1}{2}\left(\frac{1}{\lambda_2}x_1^2 + x_2^2 - \frac{1}{\lambda_1}x_3^2 - x_4^2\right).$$

They are independent if condition (5.4) is satisfied (as predicted by Theorem 2).

The integral cone $K = \{H = \Phi = 0\}$ contains the point

$$x_1 = \frac{1}{\sqrt{\lambda_1}}, \quad x_2 = 1, \quad x_3 = \frac{1}{\sqrt{\lambda_2}}, \quad x_4 = 1, \tag{5.5}$$

which we denote (as above) by a. The vector Λa has the components

$$\frac{1}{\lambda_1\sqrt{\lambda_2}}, \quad 1, \quad \frac{1}{\lambda_2\sqrt{\lambda_1}}, \quad 1. \tag{5.6}$$

Comparing (5.5) and (5.6), we see that the rank of the matrix (3.4) is 2 (if condition (5.4) is satisfied). Thus, in a typical situation there is a two-dimensional isotropic invariant plane $\Pi(a)$ passing through the point (5.5). Conversely, if $\lambda_1\lambda_2 = 1$, then the invariant subspace passing through (5.5) is one-dimensional.

Thus, if condition (5.4) is satisfied, the instability of equilibrium $x = 0$ follows from the general theorem (Theorem 6).

By the way, the plane $\Pi(a)$ can be given by the equations

$$x_2 - x_4 = 0, \quad \sqrt{\lambda_1}x_1 - \sqrt{\lambda_2}x_3 = 0.$$

As coordinates on this invariant plane one can choose the linear combinations

$$u = x_2 + x_4, \quad v = \sqrt{\lambda_1}x_1 + \sqrt{\lambda_2}x_3.$$

In terms of these variables the Hamilton equations on $\Pi(a)$ become

$$\dot{u} = -u, \quad \dot{v} = -\sqrt{\lambda_1\lambda_2}v. \tag{5.7}$$

All phase trajectories tend to the origin of coordinates as $t \to +\infty$. From (5.7) we immediately obtain two characteristic roots of the initial Hamiltonian system: -1 and $-\sqrt{\lambda_1\lambda_2}$.

In order to obtain the second invariant Lagrangian singular plane of the Hamiltonian system with trajectories emanating from the origin of coordinates, one should take as the point $x = a$, for example, a point with coordinates

$$x_1 = \frac{1}{\sqrt{\lambda_1}}, \quad x_2 = -1, \quad x_3 = -\frac{1}{\sqrt{\lambda_2}}, \quad x_4 = 1.$$

Now as the matrix Γ we take, instead of (5.2), the skew-symmetric matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and leave the matrix of the Hamiltonian unchanged as in (5.3). Here, the coordinates x_1, x_2 and x_3, x_4 serve as pairs of canonically conjugate variables.

In this case, the integral $2\Phi_1$ from the chain (2.1) takes the form

$$-\frac{1}{\lambda_1}\big(\lambda_1 x_1^2 + x_2^2\big) + \frac{1}{\lambda_2}\big(\lambda_2 x_3^2 + x_4^2\big).$$

If condition (5.4) is satisfied, then the integrals H and Φ_1 are functionally independent (as they should be according to Theorem 2). Now the cone K degenerates to the origin and, according to Theorem 6, the equilibrium point $x = 0$ is stable.

6. APPLICATION TO LINEAR SYSTEMS WITH GYROSCOPIC FORCES

As already noted in Section 1, the equations of motion (1.7) can be represented in the Hamiltonian form (1.1), and the operators Γ and P have the form (1.9). This observation allows us to apply the general results from Sections 2–4.

We briefly discuss the question of quadratic first integrals of the linear equations (1.7). To do so, we should use the formula (2.1) and the following formulae for the inverse operators (1.9):

$$\Gamma^{-1} = \begin{pmatrix} 0 & -E \\ E & G \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} W^{-1} & 0 \\ 0 & E \end{pmatrix}.$$

For $m = 0$, we obtain, of course, an energy integral, and for $m = 1$ the quadratic integral

$$\frac{1}{2}\big(Gx + \dot{x},\, W^{-1}(Gx + \dot{x})\big) + \frac{1}{2}(x,\, x), \tag{6.1}$$

was found previously in [14]. Here, the parentheses $(,)$ denote the Euclidean scalar product in configuration space $\mathbb{R}^n = \{x\}$, which has been induced by the standard form of the kinetic energy of the system.

For $m = -1$ we obtain the following quadratic integral of Eqs. (1.7) from the formula (2.1):

$$\frac{1}{2}(Wx + G\dot{x},\, Wx + G\dot{x}) + \frac{1}{2}(W\dot{x},\, \dot{x}). \tag{6.2}$$

This integral was found in [15]. The integrals (6.1) and (6.2) were used in [14–16] to find conditions for stability of the equilibria of systems with gyroscopic forces.

We present formulae for two new quadratic integrals corresponding to the values $m = 2$ and $m = -2$ from the sequence (2.1). For $m = 2$ we obtain the first integral

$$\frac{1}{2}\Big(W^{-1}\big((GW^{-1}G - E)x + GW^{-1}\dot{x}\big),\, (GW^{-1}G - E)x + GW^{-1}\dot{x}\Big) + \frac{1}{2}\big(W^{-2}(Gx + \dot{x}),\, Gx + \dot{x}\big).$$

The value $m = -2$ corresponds to the quadratic integral

$$\frac{1}{2}\big(W(Wx + G\dot{x}),\, Wx + G\dot{x}\big) + \frac{1}{2}\big(GWx + (-W + G^2)\dot{x},\, GWx + (-W + G^2)\dot{x}\big).$$

Note that all first integrals obtained in this way have the same signature and, moreover, all their standard pairwise Poisson brackets are equal to zero.

The question of their functional independence is more complicated. For example, if $G = 0$, then

$$\Phi_m = \frac{1}{2}\big[(W^m\dot{x},\, \dot{x}) + (W^{m+1}x,\, x)\big], \quad m \in \mathbb{Z}. \tag{6.3}$$

If all eigenvalues of the operator of the potential energy W are different from each other, then the sequence (6.3) has n functionally independent quadratic forms, for example, $\Phi_0 = H, \Phi_1, \ldots, \Phi_{n-1}$.

7. LINEAR SYSTEMS WITH A QUADRATIC INTEGRAL

As mentioned previously in Section 1, nondegenerate linear systems of the general form

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \tag{7.1}$$

which admit a nondegenerate quadratic first integral

$$f = \frac{1}{2}(Px,\, x), \quad \det P \neq 0 \tag{7.2}$$

are reduced to Hamiltonian form. The nondegeneracy of the linear systems means that $\det A \neq 0$. In other words, the linear systems admit no nontrivial equilibrium states.

Let us introduce a closed nondegenerate 2-form

$$\Omega(\xi,\, \eta) = (\Gamma\xi,\, \eta), \quad \Gamma = PA^{-1}. \tag{7.3}$$

As shown in [1], the pair $(\Omega,\, \mathbb{R}^n)$ is a symplectic space. In particular, n is even. The main problem was to prove the skew-symmetry of the operator Γ.

The fact that the linear system (7.1) is Hamiltonian relative to the symplectic structure (7.3) follows from the following obvious formulae:

$$\Omega(\dot{x},\, \cdot) = (\Gamma\dot{x},\, \cdot) = (\Gamma Ax,\, \cdot) = (Px,\, \cdot) = df(\cdot).$$

The quadratic integral (7.2) (taken with reversed sign) plays the role of the Hamiltonian.

Infinite-dimensional versions of this result are discussed in [17].

An immediate corollary of Theorem 1 is

Theorem 7. *Equation* (7.1) *admits a family of quadratic first integrals*

$$\Phi_m = \frac{1}{2}(PA^{-m}x,\, A^{-m}x), \quad m \in \mathbb{Z}, \tag{7.4}$$

where

1) *all* Φ_m *are nondegenerate quadratic forms and their signatures coincide with that of the integral* (7.2)*,*

2) *the functions* (7.4) *are pairwise in involution* (*relative to the symplectic structure* (7.3)).

Since m is any integer, we may omit the sign $-$ in (7.4). For $m = 0$ we obtain the quadratic integral (7.2). The operator Λ from Section 1 coincides with the operator A^{-1}.

By the way, the form of the first integral (7.4) suggests another way of proving the invariance of these quadratic forms. Indeed,

$$\Phi_{-1} = \frac{1}{2}(PAx,\, Ax) = \frac{1}{2}(P\dot{x},\, \dot{x}), \tag{7.5}$$

where the derivative $\dot{x}$ also satisfies the linear differential equation (7.1):

$$(\dot{x})^{\cdot} = A\dot{x}. \tag{7.6}$$

Since Φ_0 is a first integral of the initial system (7.1), it follows that (7.5) is a first integral of the linear system (7.6). Applying this technique sequentially, we obtain a chain of quadratic integrals Φ_{-m}, $m \geqslant 0$. To prove the invariance of the functions Φ_m with negative m, we can use the following interesting fact.

Lemma. *If the nondegenerate quadratic form* (7.2) *is a first integral of the nondegenerate linear system* (7.1)*, then it is a first integral of the linear system*

$$\dot{x} = A^{-1}x, \quad x \in \mathbb{R}^n. \tag{7.7}$$

Indeed, by virtue of the system of differential equations (7.7) the derivative of (7.2) is equal to

$$\frac{1}{2}(PA^{-1}x,\, x) + \frac{1}{2}(Px,\, A^{-1}x).$$

But this sum vanishes, since the operator P is symmetric and PA^{-1} is skew-symmetric.

These simple observations can be generalized. The following simple theorem holds.

Theorem 8. *If $x \mapsto f(x)$ is a first integral of the linear linear system* (7.1)*, then for all integer $m \geqslant 0$ the functions*

$$x \mapsto f(A^m x) \tag{7.8}$$

are first integrals of the same system.

In particular, if f is a homogeneous polynomial, then for all m each of the integrals (7.8) is a homogeneous polynomial of the same degree. The issue of their functional independence is a separate problem. The property of involutiveness of the integrals (7.8) of degree $\geqslant 3$ is worth considering only for linear Hamiltonian systems.

8. FINITE-DIMENSIONAL QUANTUM SYSTEMS

The evolution of a quantum system is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi \tag{8.1}$$

in a complex Hilbert space $\mathbb{H} = \{\psi\}$. The vectors ψ are usually called *wave functions*. They describe the states of the quantum system and obey the normalization condition

$$\langle\psi,\, \psi\rangle = 1, \tag{8.2}$$

where $\langle , \rangle$ is the Hermitian scalar product in $\mathbb{H}$. The linear *Hamilton operator* $\widehat{H}$ is a self-adjoint (Hermitian) operator which maps the Hilbert space $\mathbb{H}$ into itself. The parameter $\hbar$ is the Planck constant; in the atomic system of units, which will be used throughout the rest of this paper, $\hbar = 1$.

The Hermitian property of the operator $\widehat{H}$ implies invariance of the sphere (8.2) under the flow of the Schrödinger equation. The vectors ψ and $-\psi$ specify the same state of the system. Therefore, the projective space that is obtained from the sphere (8.2) by identifying antipodal points is in fact the phase space of the quantum system.

The physical quantities (*observables*) correspond to the Hermitian operators $\widehat{\Lambda}\colon \mathbb{H} \to \mathbb{H}$. The *measurement* of the observable quantity reduces to determining the eigenvalues (the spectrum) of the operator $\widehat{\Lambda}$. A characteristic feature of quantum mechanics is the indeterminacy of measurement results.

Let the spectrum of the operator $\widehat{\Lambda}$ be *discrete* (this is the main case for us) and $\{e_j\}_1^\infty$ be the orthonormal basis in $\mathbb{H}$, consisting of eigenvectors. Also, let λ_j be the corresponding real eigenvalues. Set

$$\psi = \sum \psi_j e_j.$$

The probability that the quantity $\widehat{\Lambda}$ in the state ψ takes the value λ_k is equal to $|\psi_k|^2$. It follows from condition (8.2) that

$$\sum p_k = \sum |\psi_k|^2 = 1.$$

In particular, in the state $\psi = e_k$ the quantity $\widehat{\Lambda}$ takes almost surely the value λ_k.

The *average value* (mathematical expectation) of the quantity $\widehat{\Lambda}$ in the state ψ is equal to

$$\langle\widehat{\Lambda}\rangle_\psi = \sum \lambda_k p_k = \sum \lambda_k |\psi_k|^2 = \langle\widehat{\Lambda}\psi,\, \psi\rangle. \tag{8.3}$$

This value is time invariant if and only if the operators $\widehat{\Lambda}$ and $\widehat{H}$ commute. In this case, the quadratic form (8.3) is a first integral (conservation law) for the Schrödinger equation.

Consider the case where $\mathbb{H}$ is a finite-dimensional complex space with the Hermitian scalar product $\langle , \rangle$. It can be identified with the space $\mathbb{C}^n$ endowed with the standard scalar product

$$\langle \varphi, \psi \rangle = \sum_{k=1}^{n} \varphi_k \overline{\psi}_k.$$

The Schrödinger equation (8.1) is a linear system of ordinary differential equations in $\mathbb{C}^n$. Separating into real and imaginary parts, we arrive at the real space of doubled dimension, and Eq. (8.1) becomes a usual linear Hamiltonian system. For a more detailed discussion of this and related issues, see, e. g., [18, 19].

Set $\psi = x + iy$ $(x, y \in \mathbb{R}^n)$ and $\widehat{H} = A + iB$, where A and B are linear operators in $\mathbb{R}^n$. The self-conjugacy condition of the operator $\widehat{H}$ gives two conditions:

$$A^* = A \quad \text{and} \quad B^* = -B.$$

Here, $*$ is the operation of conjugation relative to the "standard" scalar product in $\mathbb{R}^n$:

$$(x, y) = \sum x_k y_k.$$

Thus, the Schrödinger equation is equivalent to the following linear system in $\mathbb{R}^{2n}$:

$$\dot{x} = Bx + Ay, \quad \dot{y} = -Ax + By. \tag{8.4}$$

It can be represented in the Hamiltonian form (1.1):

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = - \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Here,

$$\Gamma = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

is the symplectic identity matrix and the matrix

$$P = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{8.5}$$

is symmetric. The Hamiltonian

$$H(x, y) = \frac{1}{2}(Ax, x) + (Bx, y) + \frac{1}{2}(Ay, y) \tag{8.6}$$

is half the average value of the Hamilton operator $\langle \widehat{H}\psi, \psi \rangle$ represented in real variables.

In addition to the integral (8.6), the Hamiltonian system (8.4) admits the quadratic integral

$$2f = \langle \psi, \psi \rangle = (x, x) + (y, y).$$

This integral can be taken as a Hamiltonian and the linear equations (8.4) can be represented in Hamiltonian form. But in this case the symplectic structure will be nonstandard (in other words, the variables x and y will not be canonically conjugate variables).

We also assume a nondegeneracy condition of the Hamilton operator:

$$\det \widehat{H} \neq 0. \tag{8.7}$$

This complex inequality $|A + iB| \neq 0$ is equivalent to the nondegeneracy condition of the real operator (8.5). Condition (8.7) implies that, when $\psi(t) = \text{const} \neq 0$, the initial Schrödinger equation has no nontrivial stationary states.

We can apply the general reasoning from Section 2 to the real linear system (8.4) in $\mathbb{R}^{2n}$.

Theorem 9. *The Schrödinger equation admits a family of integrals*

$$\Theta_m = \langle \widehat{H}^m \psi,\, \psi \rangle, \quad m \in \mathbb{Z}, \tag{8.8}$$

and the functions Θ_m with odd m are pairwise in involution (relative to the standard symplectic structure in $\mathbb{R}^{2n}$).

It is clear that the operator $\widehat{H}^m$ is Hermitian and obviously commutes with the Hamilton operator $\widehat{H}$. Therefore, the quadratic forms (8.8) are first integrals of the Schrödinger equation.

The simple fact that the functions (8.8) are invariant can be shown using the reasoning from Section 7. Indeed, the linear Schrödinger equation $\dot{\psi} = -i\widehat{H}\psi$ admits the quadratic integrals

$$\langle \widehat{H}\psi,\, \psi \rangle \quad \text{and} \quad \langle \psi,\, \psi \rangle.$$

Consequently, it admits the integrals

$$\langle \widehat{H}(-i\widehat{H})^k \psi,\, (-i\widehat{H})^k \psi \rangle \quad \text{and} \quad \langle (-i\widehat{H})^l \psi,\, (-i\widehat{H})^l \psi \rangle$$

with integer k and l. The first of them gives the integral (8.8) with odd m; and the second, with even m.

The point of interest in Theorem 9 is the involutiveness of the integrals (8.8) with odd m. For this, one should pass to the "realified" phase space $\mathbb{R}^{2n}$ and use Theorem 1. In the notation of Section 2 the integrals (2.1) have the form

$$\frac{1}{2}\big(P(\Gamma P)^m z,\, (\Gamma P)^m z\big), \quad z = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{8.9}$$

Here, we have used the obvious equality $\Gamma^{-1} = -\Gamma$. The quadratic form (8.9) is equal to

$$\frac{1}{2}(Mz,\, z), \quad M = (-1)^m P(\Gamma P)^{2m}.$$

Further,

$$(\Gamma P)^2 = \begin{pmatrix} B^2 - A^2 & AB + BA \\ -AB - BA & B^2 - A^2 \end{pmatrix} = -P^2.$$

Hence,

$$M = P^{2m+1}. \tag{8.10}$$

Let again $\widehat{H} = A + iB$. Set

$$(A + iB)^k = A_k + iB_k, \quad k \in \mathbb{Z}.$$

By induction on k one can easily prove the equation

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}^k = \begin{pmatrix} A_k & -B_k \\ B_k & A_k \end{pmatrix}. \tag{8.11}$$

If $\psi = x + iy$, then

$$(A + iB)^{2m+1}(x + iy) = (A_{2m+1}x - B_{2m+1}y) + i(A_{2m+1}y + B_{2m+1}x).$$

Consequently, according to (8.10) and (8.11),

$$\langle \widehat{H}^{2m+1}\psi,\, \psi \rangle = \frac{1}{2}(P^{2m+1}z,\, z).$$

After these observations, the involutiveness of the quadratic forms Θ_m with odd numbers follows from conclusion 2 of Theorem 1. This proves the theorem.

We briefly discuss the issue of functional independence of the involutive set of integrals from Theorem 9. Let $\lambda_1, \ldots, \lambda_n$ be the real eigenvalues of the Hermitian operator $\widehat{H}$.

Theorem 10. *If there are no equal numbers among $\lambda_1, \ldots, \lambda_n$, then the first integrals $\Theta_1, \ldots, \Theta_{2n-1}$ are functionally independent.*

This assertion is an analog of the general theorem (Theorem 2). Under the same assumptions the integrals $\Theta_0, \ldots, \Theta_{2n-2}$ are also independent. Moreover, they are expressed in terms of $\Theta_1, \ldots, \Theta_{2n-1}$ and hence are also in involution.

We make a few remarks.

1°. Among quadratic integrals generated by Hermitian operators commuting with a Hamilton operator, there are always n functionally independent integrals whose pairwise Poisson brackets are equal to zero. This simple assertion is proved by reducing the matrix of the Hamilton operator using an appropriate unitary transformation to the diagonal form $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

2°. The linear Schrödinger equation can admit homogeneous polynomials of degree > 2 as first integrals. These integrals are *irreducible*, that is, they are not expressed in terms of integrals of lower degree. For example, let $n = 2$, $\psi = \psi_1 e_1 + \psi_2 e_2$ and let the Hamilton operator have the form

$$\widehat{H} = \operatorname{diag}(\lambda, -2\lambda), \quad \lambda \neq 0.$$

In this case, we have an irreducible integral of degree 3

$$\psi_1^2 \psi_2.$$

The existence of integrals of higher degrees is due to resonances between the eigenvalues of the Hamilton operator.

Conversely, if

$$k_1\lambda_1 + \ldots + k_n\lambda_n \neq 0$$

for all integer k_j such that $\sum k_j^2 \neq 0$, then any integral of the Schrödinger equation is a function of a complete set of quadratic involutive integrals which are generated by commuting Hermitian operators.

3°. In the infinite-dimensional case, the situation is much more complicated. For example, for systems with configuration space in the form of a multidimensional torus (with periodic boundary conditions) the existence of nontrivial polynomial (in derivatives) Hermitian differential operators commuting with the Hamilton operator is a very rare phenomenon [20]. Topological obstructions to the existence of such operators are discussed in [21].

The simple reasoning above may be useful in analyzing the complete integrability of infinite-dimensional quantum systems with a discrete spectrum. However, the notion of a completely integrable quantum system requires a more precise definition (for a discussion of this and related issues, see [22]). Despite the fact that there is no generally accepted reasonable definition of an integrable ("regular") quantum system, there is an extensive literature on various aspects of quantum chaos (devoted to the study of "irregular" quantum systems). See, for example, [23] and references therein.

APPENDIX. ON THE ROOTS OF A CHARACTERISTIC EQUATION

Let

$$f(\lambda) = \det(P + \lambda\Gamma)$$

be a characteristic polynomial of the linear system (1.1). The coefficients of the polynomial depend polynomially on the elements of the matrices of the operators P and Γ. As mentioned previously in Section 1, f is a polynomial of degree $2n$ that contains no odd degrees of λ. In particular, the solution of the characteristic equation $f(\lambda) = 0$ reduces to that of an algebraic equation of degree n

$$g(\mu) = 0, \quad \mu = \lambda^2$$

with the same coefficients: $g(\lambda^2) = f(\lambda)$.

We consider a typical case where the characteristic equation has no multiple roots. In other words, the discriminant of the polynomial f is different from zero.

Let the Hamiltonian system (1.1) be stable: all its solutions are bounded. In terms of the characteristic equation this property means that all roots of the polynomial g are real and negative. In terms of the family of quadratic integrals of the system (1.1) (from Theorems 1 and 2) this means that the integral cone K reduces to one point $x = 0$. To put it differently: for almost all values $f_0, \ldots, f_{n-1}$ the nonempty invariant subset of the phase space

$$\{x \in \mathbb{R}^{2n} \colon \Phi_0(x) = f_0, \ldots, \Phi_{-n+1}(x) = f_{n-1}\} \tag{1}$$

is a union of n-dimensional tori carrying quasi-periodic motions with constant frequencies

$$\omega_1 = \sqrt{-\mu_1}, \ldots, \omega_n = \sqrt{-\mu_n}.$$

The numbers $\pm i\omega_1, \ldots, \pm i\omega_n$ obviously coincide with the roots of the characteristic equation $f = 0$. The assertion about tori is a geometric version of the classical Liouville theorem of completely integrable Hamiltonian systems (see, e. g., [9]).

To calculate the roots, one uses various iteration methods. For an interesting discussion of nuances of this approach, see the book [24].

On the other hand, as is well known, for $n \leqslant 4$ the roots of the characteristic equation are expressed in terms of radicals. And for $n = 5$ the roots of the "reduced" equation $g(\mu) = 0$ are expressed in terms of its coefficients by elliptic functions. A detailed discussion of this and related issues can be found in the book [25] (especially in the Russian edition, which contains valuable additions by J.-P. Serre, V. I. Arnold and A. N. Tyurin). For $n > 6$ the analytical properties of the roots of algebraic equations are still poorly understood (for $n = 6$ there is an advanced theory; see the appendix in [25, pp. 320–324]).

It turns out that the frequencies $\omega_1, \ldots, \omega_n$ (and hence the roots of the characteristic equation) can be found using topological considerations, as well as the operation of integrating (1-forms over closed algebraic cycles on an n-dimensional invariant torus). To do this, it is convenient to pass to *action-angle variables* in a neighborhood of the invariant torus (1) (more precisely, its connected component). As is well known, this requires introducing n homologically independent one-dimensional cycles $\Gamma_1, \ldots, \Gamma_n$ on the n-dimensional invariant torus and integrating over these cycles the 1-form

$$\omega = \sum_{i>j} \Gamma_{ij} x_j \, dx_i.$$

Here, $\|\Gamma_{ij}\|$ is the skew-symmetric matrix of the operator Γ. It is clear that $d\omega = \Omega$ is a symplectic structure in phase space.

More precisely, we define n action variables

$$I_s = \frac{1}{2\pi} \oint_{\Gamma_s} \omega, \quad 1 \leqslant s \leqslant n. \tag{2}$$

These variables are linearly expressed in terms of

$$f_0, \ f_1, \ldots, f_{n-1},$$

which are constant of quadratic involutive integrals $\Phi_0, \Phi_{-1}, \ldots, \Phi_{-n+1}$. As is well known, the map

$$(f_0, \ldots, f_{n-1}) \mapsto (I_1, \ldots, I_n)$$

is invertible. Consequently, by merely performing simple algebraic operations one can be find f_0 as a function of the action variables:

$$f_0 = \sum_{s=1}^{n} c_s I_s.$$

Then, as is well known,

$$\omega_1 = c_1, \ldots, \omega_n = c_n. \tag{3}$$

As a result, we obtain the roots of the characteristic equation:

$$\lambda_{1,2} = \pm i c_1, \ldots, \lambda_{2n-1,n} = \pm i c_n.$$

Remark. A system of independent cycles on the torus can be chosen in different ways. Because of this the uniformly changing angle variables on the invariant torus are defined up to a linear unimodular transformation with integer coefficients. In particular, according to (3), the roots of the characteristic equation are linear combinations of numbers $\pm i\omega_1, \ldots, \pm i\omega_n$ with integer coefficients.

This uncertainty can be eliminated by using continuity considerations. We illustrate this by considering linear systems that are perturbations of a canonical system of Hamiltonian equations with the Hamiltonian function

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \ldots + \frac{\omega_n}{2}(p_n^2 + q_n^2). \tag{4}$$

Here, the skew-symmetric operator Γ coincides with the symplectic unit (1.4). In particular,

$$\omega = \sum_{s=1}^{n} p_s\, dq_s.$$

The discriminant of the characteristic polynomial is different from zero if there are no equal numbers among $\omega_1, \ldots, \omega_n$. We assume this condition to be satisfied.

It is easy to verify that the system of quadratic equations

$$\Phi_0 = f_0, \ldots, \Phi_{-n+1} = f_{n-1}$$

is equivalent to the system

$$p_1^2 + q_1^2 = h_1, \ldots, p_n^2 + q_n^2 = h_n. \tag{5}$$

Hence (according to (4)),

$$f_0 = \frac{\omega_1 h_1}{2} + \ldots + \frac{\omega_n h_n}{2}.$$

We choose the independent cycles Γ_s as follows:

$$p_s = \sqrt{h_s}\cos\varphi, \quad q_s = \sqrt{h_s}\sin\varphi, \quad \varphi \in \mathbb{R}, \tag{6}$$

and treat the other variables p and q as fixed constants. By the formula (2),

$$I_s = \frac{1}{2\pi}\oint_{\Gamma_s} p_s\, dq_s = \frac{h_s}{2\pi}\int_0^{2\pi} \cos^2\varphi\, d\varphi = \frac{h_s}{2}.$$

Consequently, $h_s = 2I_s$ and

$$f_0 = \sum \omega_s I_s.$$

This implies that the roots of the characteristic equation are equal to $\pm i\omega_1, \ldots, \pm i\omega_n$ (in fact, this has been obvious from the very beginning).

Now we shall continuously perturb the Hamiltonian (4). The invariant tori of the perturbed system will be a continuous deformation of the torus (5). We can take the continuous perturbations of the cycles (6) as independent cycles on these tori. The above general formulae will give us correct values of the roots of the characteristic equation (at least until multiple roots of the characteristic equation appear as the Hamiltonian system is perturbed).

We can look at all these problems from a different point of view. The integral torus (1) is an n-dimensional real algebraic manifold, an intersection of n quadrics in $2n$-dimensional phase space $\mathbb{R}^{2n}$. On these tori we can choose n homologically independent cycles in the form of *closed real algebraic curves*.

Bearing in mind the above example of perturbation of the Hamiltonian system with the Hamiltonian function (4), we can take the following algebraic curves as algebraic cycles:

$$\begin{aligned}\Gamma_s = \{x \in \mathbb{R}^{2n}\colon \Phi_0(x) = f_0, \ldots, \Phi_{-n+1}(x) = f_{n-1};\\ q_1 = \ldots = \widehat{q_s} = \ldots = q_n = 0\}, \quad s = 1, \ldots, n,\end{aligned} \tag{7}$$

where the hat denotes omission of the corresponding variable. Here $x = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ are the canonical coordinates. Then the integrals (2) become *complete Abelian* integrals: functions that are rational in the entire phase space are integrated over the closed algebraic curves (7). It is well known that to each real algebraic curve one can assign its Riemannian surface so that this curve is a real cycle on this Riemannian surface.

For quadratic forms $\Phi_0, \ldots, \Phi_{-n+1}$ of the most general form the genus of the Riemannian surface is estimated from above by the number

$$2^{n-1}(n-2)+1. \tag{8}$$

In particular, when $n = 1$, the Riemannian surface is a two-dimensional sphere (and therefore the Abelian integrals are expressed in terms of elementary functions), and when $n = 2$, we obtain either a sphere or a two-dimensional torus. In the case of a torus the Abelian integrals are elliptic integrals. In general, (8) seems to be an overestimate, since it does not take into account the specificity of equations that define an n-dimensional algebraic torus.

Remark. By a suitable linear transformation the system can be reduced to canonical form with the Hamiltonian function (4). After that, it is easy to find algebraic cycles of *genus zero* on the invariant tori. However, we first need to know the roots of the characteristic equation to be solved!

The problem is to obtain in "explicit" form n homologically independent algebraic cycles, such as (7), with the smallest possible genus (of course, without previously finding the roots of the characteristic equation).

This leads us to the following conclusion: the roots of the algebraic characteristic equation are expressed in terms of complete Abelian integrals of rational functions on Riemannian surfaces of algebraic curves. Moreover, the parameters of these Abelian integrals depend rationally (and even polynomially) on the elements of the matrices of the operators Γ and P.

This assertion can be viewed as an extension of the classical results, due to Hermite, Kronecker and Brioschi, on solutions to equations of degrees 5 and 6. It can be made more precise in different directions. In particular, one should obtain a more accurate estimate of the genus of the Riemannian surface associated with the algebraic cycle (7). In addition, one should bear in mind that concrete Abelian integrals on Riemannian surfaces of greater genus can be reduced to Abelian integrals on Riemannian surfaces of smaller genus.

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