ISSN 1560-3547, Regular and Chaotic Dynamics, 2017, Vol. 22, No. 8, pp. 909–936. © Pleiades Publishing, Ltd., 2017.

Maxwell Strata and Cut Locus in the Sub-Riemannian Problem on the Engel Group

Andrei A. Ardentov* **and Yuri L. Sachkov****

Program Systems Institute of RAS,
Pereslavl-Zalessky, Yaroslavl Region, 152020 Russia

Received September 20, 2017; accepted October 21, 2017

Abstract—We consider the nilpotent left-invariant sub-Riemannian structure on the Engel group. This structure gives a fundamental local approximation of a generic rank 2 sub-Riemannian structure on a 4-manifold near a generic point (in particular, of the kinematic models of a car with a trailer). On the other hand, this is the simplest sub-Riemannian structure of step three. We describe the global structure of the cut locus (the set of points where geodesics lose their global optimality), the Maxwell set (the set of points that admit more than one minimizer), and the intersection of the cut locus with the caustic (the set of conjugate points along all geodesics). The group of symmetries of the cut locus is described: it is generated by a one-parameter group of dilations $\mathbb{R}_+$ and a discrete group of reflections $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The cut locus admits a stratification with 6 three-dimensional strata, 12 two-dimensional strata, and 2 one-dimensional strata. Three-dimensional strata of the cut locus are Maxwell strata of multiplicity 2 (for each point there are 2 minimizers). Two-dimensional strata of the cut locus consist of conjugate points. Finally, one-dimensional strata are Maxwell strata of infinite multiplicity, they consist of conjugate points as well. Projections of sub-Riemannian geodesics to the 2-dimensional plane of the distribution are Euler elasticae. For each point of the cut locus, we describe the Euler elasticae corresponding to minimizers coming to this point. Finally, we describe the structure of the optimal synthesis, i. e., the set of minimizers for each terminal point in the Engel group.

MSC2010 numbers: 53C17, 49K15
DOI: 10.1134/S1560354717080020

Keywords: sub-Riemannian geometry, optimal control, Engel group, Maxwell strata, cut locus, mobile robot, Euler's elasticae

1. INTRODUCTION

The Engel group M is a nilpotent four-dimensional Lie group, connected and simply connected, which has Lie algebra $L = \operatorname{span}(X_1, X_2, X_3, X_4)$ with the multiplication table

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = [X_1, X_4] = [X_2, X_4] = 0.$$

This article studies the sub-Riemannian structure [1] on the Lie group M generated by the left-invariant orthonormal frame X_1, X_2. This structure gives a nilpotent approximation [2] to a generic rank two sub-Riemannian structure in four-dimensional space near a generic point.

In certain coordinates (x, y, z, w) on the Engel group $M \cong \mathbb{R}^4$, the nilpotent sub-Riemannian problem is stated as follows:

$$\begin{cases} \dot{x} = \mathtt{u}_1, \qquad \dot{y} = \mathtt{u}_2, \qquad \dot{z} = -\mathtt{u}_1 \dfrac{y}{2} + \mathtt{u}_2 \dfrac{x}{2}, \qquad \dot{w} = \mathtt{u}_2 \dfrac{x^2}{2}, \\ q = (x, y, z, w) \in \mathbb{R}^4, \qquad (\mathtt{u}_1, \mathtt{u}_2) \in \mathbb{R}^2, \\ q(0) = q_0 = (0, 0, 0, 0), \qquad q(t_1) = (x_1, y_1, z_1, w_1), \end{cases}$$

$$l = \int_0^{t_1} \sqrt{\mathtt{u}_1^2 + \mathtt{u}_2^2}\, dt \to \min.$$

*E-mail: aaa@pereslavl.ru
**E-mail: yusachkov@gmail.com

This paper has the following structure. In Section 1.1 we recall results on the problem obtained in previous works [3–5]. In Section 1.2 we prove some simple preliminary results on the cut locus. In Sections 2–6 we describe, respectively, the intersection of the cut locus with the sets $\{x = z = 0\}$, $\{z = 0,\ x > 0\}$, $\{z = 0\}$, $\{x = 0,\ z > 0\}$, and $\{x = 0\}$. In Section 7 we sum up these results by describing a global stratification of the cut locus. In Appendices A–C we prove technical lemmas from Sections 2, 3, and 5.

1.1. Previously Obtained Results

This paper continues a series of works [3–5], where a detailed study of the sub-Riemannian problem on the Engel group was started (in these works, instead of the coordinate w, we used the coordinate $v = w + y^3/6$). First we recall the main results of these works.

For each point $q_1 \in M$ there exists an optimal trajectory (sub-Riemannian minimizer). Sub-Riemannian geodesics are described by the Pontryagin maximum principle. Abnormal trajectories are simultaneously normal, and their endpoints fill two rays

$$\mathcal{A}_\pm = \{q \in M \mid x = z = w = 0, \operatorname{sgn} y = \pm 1\}.$$

Geodesics are parametrized by Jacobi elliptic functions [9]. Projections of geodesics onto the plane (x, y) are Euler's elasticae [6, 7]. Small arcs of geodesics are optimal; however, large arcs are, in general, not optimal. A point at which a geodesic loses its optimality is called a cut point. The union of cut points along all geodesics is called the cut locus. The cut locus is one of the most important characteristics of sub-Riemannian structures [2], and the main goal of this paper is its description for the left-invariant sub-Riemannian problem on the Engel group.

A generic geodesic loses its optimality at a Maxwell point, i. e., a point where several geodesics of the same length meet one another. Maxwell points are fixed points of discrete symmetries ε^1, ε^2, ε^4 acting on the Engel group as follows:

$$\varepsilon^1(x, y, z, w) = (x, y, -z, w - xz), \tag{1.1}$$

$$\varepsilon^2(x, y, z, w) = (-x, y, z, w - xz), \tag{1.2}$$

$$\varepsilon^4(x, y, z, w) = (-x, -y, z, -w). \tag{1.3}$$

The subspaces $M_x = \{q \in M \mid x = 0\}$, $M_z = \{q \in M \mid z = 0\}$ consist of fixed points of symmetries (1.1), (1.2), the cut locus is contained in the union of these subspaces (see Theorem 3 below).

The problem has also continuous symmetries — the one-parameter group of dilations given by the flow of the vector field $X_0 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + 2z\dfrac{\partial}{\partial z} + 3w\dfrac{\partial}{\partial w}$.

The main result of [3–5] is the explicit description of the cut time (see Theorem 1 below) and the proof that there is a unique minimizer for each point $q_1 \in M \cap \{xz \neq 0\}$ (see Theorem 2 below).

We have shown in [3] that the family of all extremal trajectories of the problem is parametrized by the cylinder

$$C = \left\{\lambda \in T^*_{q_0}M \mid H(\lambda) = 1/2\right\} = \left\{\lambda = (\theta, c, \alpha) \mid \theta \in S^1,\ c, \alpha \in \mathbb{R}\right\},$$

where $H(\lambda) = (\langle \lambda, X_1\rangle^2 + \langle \lambda, X_2\rangle^2)/2$, $\lambda \in T^*M$, is the maximized Hamiltonian of the Pontryagin maximum principle; (θ, c, α) are certain natural coordinates on the cylinder C. The parametrization of extremal trajectories is defined by the exponential map

$$\operatorname{Exp}\colon N \to M, \qquad N = C \times \mathbb{R}_+,$$
$$\operatorname{Exp}(\nu) = q_t = (x_t, y_t, z_t, w_t), \qquad \nu = (\lambda, t).$$

The function $E = c^2/2 - \alpha \cos\theta$ is constant along extremal trajectories, and the cylinder C stratifies according to its values:

$$C = \sqcup_{i=1}^7 C_i, \qquad C_4 = \left\{\lambda \in C \mid \alpha \neq 0, E = -|\alpha|\right\},$$
$$C_1 = \left\{\lambda \in C \mid \alpha \neq 0, E \in (-|\alpha|, |\alpha|)\right\}, \qquad C_5 = \left\{\lambda \in C \mid \alpha \neq 0, E = |\alpha|, c = 0\right\},$$

$$C_2 = \{\lambda \in C \mid \alpha \neq 0, E \in (|\alpha|, +\infty)\}, \qquad C_6 = \{\lambda \in C \mid \alpha = 0,\ c \neq 0\},$$
$$C_3 = \{\lambda \in C \mid \alpha \neq 0, E = |\alpha|, c \neq 0\}, \qquad C_7 = \{\lambda \in C \mid \alpha = c = 0\}.$$

Here, and throughout the remainder of the text, $\sqcup$ is a disjoint union. Denote the corresponding subsets in the preimage of the exponential map: $N_i = C_i \times \mathbb{R}_+$.

An arbitrary extremal trajectory of the sub-Riemannian problem on the Engel group projects to the plane (x, y) into an Euler elastica. Each subset C_i, $i = 1, \dots, 7$, corresponds to a certain type of Euler elasticae.

For the parametrization of trajectories for the subsets C_1 (inflectional elasticae), C_2 (noninflectional elasticae) and C_3 (critical elasticae), we introduce a set of elliptic coordinates $\lambda = (\varphi, k, \alpha)$. The parameter k is a reparametrization of the first integral E. On the set C_3 we have $k = 1$, this set separates C_1 and C_2, where $k \in (0, 1)$. The remaining subsets C_i, $i = 4, \dots, 7$, (lines and circles in the plane (x, y)) are parametrized by the coordinates $\lambda = (\theta, c, \alpha)$. Notice that trajectories for the case $\lambda \in C_4 \cup C_5$ are defined by formulas for $\lambda \in C_7$, when $\sin\theta = 0$, thus the cases C_4, C_5 are not considered in this paper.

On the subsets N_1, N_2, N_3, we introduce a set of coordinates (p, τ, σ):

$$\begin{aligned} &\sigma = \operatorname{sgn}\alpha\sqrt{|\alpha|}, \\ &(\lambda, t) \in N_1 \cup N_3 \quad \Rightarrow \quad p = |\sigma| t/2, \quad \tau = |\sigma|(\varphi + t/2), \\ &(\lambda, t) \in N_2 \quad \Rightarrow \quad p = |\sigma| t/(2k), \quad \tau = |\sigma|(\varphi + t/2)/k. \end{aligned}$$

Definition 1. *The cut time $t_{\mathrm{cut}}(\lambda)$ is the time when the extremal trajectory corresponding to the covector λ loses its global optimality:*

$$t_{\mathrm{cut}}(\lambda) = \sup\{t > 0 \mid \operatorname{Exp}(\lambda, s) \ \textit{optimal for } s \in [0, t]\}.$$

The cut locus is the set $\mathrm{Cut} = \{\operatorname{Exp}(\lambda, t) \mid \lambda \in C, t = t_{\mathrm{cut}}(\lambda)\}$.

Remark 1. If $t_{\mathrm{cut}}(\lambda) = \infty$, then the trajectory $\operatorname{Exp}(\lambda, s)$ is optimal on the whole ray $s \in [0, \infty)$. In this case the nonstrict inequality $s \leqslant t_{\mathrm{cut}}(\lambda)$ should be understood as a strict one.

Theorem 1 ([5], Corollary 4.2). *The cut time has the following explicit expression:*

$$\begin{aligned} &\forall \lambda \in C_1 \qquad t_{\mathrm{cut}}(\lambda) = \frac{\min\left(2p_z^1(k), 4K(k)\right)}{|\sigma|} = \begin{cases} 4K(k)/|\sigma|, & k \in (0, k_0], \\ 2p_z^1(k)/|\sigma|, & k \in [k_0, 1), \end{cases} \\ &\forall \lambda \in C_2 \qquad t_{\mathrm{cut}}(\lambda) = \frac{2K(k)k}{|\sigma|}, \\ &\forall \lambda \in C_6 \qquad t_{\mathrm{cut}}(\lambda) = \frac{2\pi}{\sqrt{|c|}}, \\ &\forall \lambda \in C_3 \cup C_7 \quad t_{\mathrm{cut}}(\lambda) = +\infty. \end{aligned}$$

Here $K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$ is the complete elliptic integral of the first kind; $p_z^1(k) \in \left(K(k), 3K(k)\right)$ is the first positive root of the function $f_z(p, k) = \operatorname{dn} p \operatorname{sn} p + \left(p - 2\operatorname{E}(p)\right)\operatorname{cn} p$. The functions $\operatorname{sn} p, \operatorname{cn} p, \operatorname{dn} p$ are Jacobian elliptic functions with modulus k by default (since the modulus k is constant along extremal trajectories), e.g., $\operatorname{sn} p = \operatorname{sn}(p, k)$; also $\operatorname{E}(p) = \displaystyle\int_0^p \operatorname{dn}^2 t\, dt$.

On the subsets N_1, N_2 the coordinates (p, τ) are transformed, respectively, into (u_1, u_2) via the formulas $u_1 = \operatorname{am} p$, $u_2 = \operatorname{am}\tau$, where am is the elliptic amplitude, inverse function to the incomplete elliptic integral of the first kind: $F(\operatorname{am} p) = p$. Summing up, we use the following coordinates for parametrization of the exponential map on subsets:

$$\nu \in N_1 \cup N_2, \quad \nu = (k, u_1, u_2, \sigma), \tag{1.4}$$
$$\nu \in N_3, \quad \nu = (p, \tau, \sigma), \tag{1.5}$$

$$\nu \in N_6, \qquad \nu = (\theta, c, t), \tag{1.6}$$
$$\nu \in N_7, \qquad \nu = (\theta, t). \tag{1.7}$$

Explicit formulas for the exponential map $\mathrm{Exp}(\lambda, t)$ for each subset N_i, $i = 1, \dots, 7$, are given in [5].

In [3] we showed that the vanishing of one of the coordinates x, z on geodesics is related to discrete symmetries ε^1, ε^2 of the exponential map. The group of all discrete symmetries $\{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\}$ is isomorphic to the group of symmetries of parallelepiped $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Explicit expression of the symmetries ε^1, ε^2, ε^4 is presented above in (1.1)–(1.3). The action of the symmetries ε^1, ε^2, ε^4 in the preimage N is expressed as follows:

$$\varepsilon^1(\theta, c, \alpha, t) = (\theta_t, -c_t, \alpha, t), \tag{1.8}$$
$$\varepsilon^2(\theta, c, \alpha, t) = (-\theta_t, c_t, \alpha, t). \tag{1.9}$$
$$\varepsilon^4(\theta, c, \alpha, t) = (\theta + \pi, c, -\alpha, t). \tag{1.10}$$

Conditions of invariance of points in the preimage of the exponential map w.r.t. the symmetries ε^1, ε^2 are given, respectively, by the equalities $c_{t/2} = 0$ and $\sin\theta_{t/2} = 0$. On the subsets $N_i, i = 1, \dots, 7$, in coordinates (1.4)–(1.7), these conditions are described in Table 1.

Table 1. Conditions of invariance of points w.r.t. the action of $\varepsilon^1, \varepsilon^2$ in N.

$\nu \in N_i$	N_1	N_2	N_3	N_6	N_7
$\varepsilon^1(\nu) = \nu$	$\cos u_2 = 0$	$\varnothing$	$\varnothing$	$\varnothing$	N_7
$\varepsilon^2(\nu) = \nu$	$\sin u_2 = 0$	$\sin u_2 \cos u_2 = 0$	$\tau = 0$	$2\theta + ct = \underset{n\in\mathbb{Z}}{2\pi n,}$	$\sin\theta = 0$

In the preimage of the exponential map, optimal trajectories correspond to the set

$$\widehat{N} = \big\{(\lambda, t) \in N \mid t \leqslant t_{\mathrm{cut}}(\lambda)\big\}.$$

After having excluded the initial point q_0, we get the following set of terminal points q_1:

$$\widehat{M} = \big\{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 \neq 0\big\}.$$

Below we mean that $q_1 = (x, y, z, w) \in \widehat{M}$.

According to the action of the main symmetries $\varepsilon^1, \varepsilon^2$, the subsets $\widehat{M}, \widehat{N}$ decompose into the following subsets:

$$\widehat{M} = M' \sqcup \widetilde{M}, \qquad \widehat{N} = N' \sqcup \widetilde{N},$$
$$M' = \big\{q \in M \mid xz = 0, x^2 + y^2 + z^2 + w^2 \neq 0\big\}, \qquad \widetilde{M} = \big\{q \in M \mid xz \neq 0\big\},$$
$$N' = \big\{(\lambda, t) \in N \mid t = t_{\mathrm{cut}}(\lambda) \text{ or } c_{t/2}\sin\theta_{t/2} = 0, t < t_{\mathrm{cut}}(\lambda)\big\},$$
$$\widetilde{N} = \big\{(\lambda, t) \in N \mid t < t_{\mathrm{cut}}(\lambda), c_{t/2}\sin\theta_{t/2} \neq 0\big\}.$$

Theorem 2 ([5], Corollary 3.21). *The map* $\mathrm{Exp}\colon \widetilde{N} \to \widetilde{M}$ *is a diffeomorphism.*

1.2. Preliminary Results on the Cut Locus

The set N' consists of cut points and fixed points of the symmetries $\varepsilon^1, \varepsilon^2$:

$$N' = N_{\mathrm{cut}} \sqcup \mathrm{FIX}, \qquad \mathrm{FIX} = \mathrm{FIX}^1 \sqcup \mathrm{FIX}^2 \sqcup \mathrm{FIX}^{12},$$
$$N_{\mathrm{cut}} = \big\{(\lambda, t) \in N \mid t = t_{\mathrm{cut}}(\lambda)\big\}, \qquad \mathrm{FIX}^{12} = \big\{\nu \in N_7 \mid \sin\theta = 0\big\},$$
$$\mathrm{FIX}^i = \big\{(\lambda, t) \in N \setminus \mathrm{FIX}^{12} \mid t < t_{\mathrm{cut}}(\lambda),\ \varepsilon^i(\lambda, t) = (\lambda, t)\big\}, \qquad i = 1, 2.$$

Obviously, $\mathrm{Cut} = \mathrm{Exp}(N_{\mathrm{cut}})$.

The set M' stratifies as follows:

$$M' = M_{00} \sqcup M_{0+} \sqcup M_{0-} \sqcup M_{+0} \sqcup M_{-0}, \qquad M_{00} = \{q \in M \mid x = z = 0, y^2 + w^2 \neq 0\},$$
$$M_{0\pm} = \{q \in M \mid x = 0, \operatorname{sgn} z = \pm 1\}, \qquad M_{\pm 0} = \{q \in M \mid \operatorname{sgn} x = \pm 1, z = 0\}.$$

Certain simple geometric properties of the cut locus follow immediately from previously obtained results.

Theorem 3. 1) $\mathrm{Cut} \subset M_x \cup M_z$,

2) $\varepsilon^i(\mathrm{Cut}) = \mathrm{Cut}$, $i = 1, \ldots, 7$,

3) $e^{tX_0}(\mathrm{Cut}) = \mathrm{Cut}$, $t \in \mathbb{R}$.

Proof. Inclusion 1) follows from Theorem 1. Equalities 2) and 3) follow from invariance of the cut time w.r.t. reflections ε^i and its homogeneity w.r.t. dilations e^{tX_0}, see item (3) of Corollary 4.2 [5].

Denote $N_- = \big\{(\lambda, t) \in N \mid t < t_{\mathrm{cut}}(\lambda)\big\}$.

Lemma 1. *We have* $\mathrm{Cut} \cap \mathrm{Exp}(N_-) = \varnothing$, *thus* $\mathrm{Cut} \cap \mathrm{Exp}(\mathrm{FIX}) = \varnothing$.

Proof. By contradiction, let $\mathrm{Cut} \cap \mathrm{Exp}(N_-) \ni q_1 = \mathrm{Exp}(\lambda_1, t_1) = \mathrm{Exp}(\lambda_2, t_2)$, where $(\lambda_1, t_1) \in N_{\mathrm{cut}}$, $(\lambda_2, t_2) \in N_-$. One can easily see that $\lambda_1 \neq \lambda_2$ and $q_1(t) = \mathrm{Exp}(\lambda_1, t) \not\equiv \mathrm{Exp}(\lambda_2, t) = q_2(t)$. Both trajectories $q_1(t), t \in [0, t_1]$, and $q_2(t), t \in [0, t_2]$, are optimal, thus $t_1 = t_2$, and $q_1 = q_2(t_2)$ is a Maxwell point. Thus the trajectory $q_2(t)$ is not optimal for $t > t_2$, which contradicts the inequality $t_2 < t_{\mathrm{cut}}(\lambda)$.

The second inequality in the statement of this lemma follows from the inclusion $\mathrm{FIX} \subset N_-$.

Lemma 2. *Let* $\nu \in N$ *and* $\mathrm{Exp}(\nu) = (x, y, z, w)$. *If* $\nu \in N_2 \cup N_3 \cup N_6$, *then* $z \neq 0$. *If* $\nu \in N_7$, *then* $z = 0$.

Proof. The statement follows immediately from the formulas of the exponential map (see [3], Sections 5.5–5.6, 7.4) for the sets C_i, $i = 2, \ldots, 7$.

2. STRUCTURE OF $\mathrm{Cut} \cap \{x = z = 0\}$

Now we study the set $N_{00} = \mathrm{Exp}^{-1}(M_{00}) \cap \widehat{N}$. Recall [3] that the components of geodesics x, z vanish either at the cut time (see Theorem 1) or at a fixed point of a symmetry in the image of the exponential map (see Table 1). Denote the function $u_{1z}(k) = \mathrm{am}\left(p_z^1(k)\right)$, recall also that $k_0 \approx 0.909$ is a unique solution to the equation $2E(k) - K(k) = 0$, where $E(k)$ is the complete elliptic integral of the second kind.

We show below in Lemma 3 that points from N_{00} related to the cut locus belong to the subset N_1 parametrized by the coordinates (k, u_1, u_2, σ). We use these coordinates to define the components $\mathrm{MAX}^{12}_{\pm}, \mathrm{MAX}^{20}_{ij}, \mathrm{MAX}^{10}_{ij}$, $i, j \in \{+, -\}$ from N_{00} related to the cut locus, see Table 2. This table should be read by columns. For example, the first column means that

$$\mathrm{MAX}^{20}_{++} = \left\{(k, u_1, u_2, \sigma) \in N_1 \mid k \in (0, k_0), u_1 = \pi, u_2 = \frac{\pi}{2}, \sigma \in (0, +\infty)\right\}.$$

Let

$$\mathrm{MAX}^{20} = \mathrm{MAX}^{20}_{++} \sqcup \mathrm{MAX}^{20}_{+-} \sqcup \mathrm{MAX}^{20}_{-+} \sqcup \mathrm{MAX}^{20}_{--}, \qquad \mathrm{MAX}^{12} = \mathrm{MAX}^{12}_{+} \sqcup \mathrm{MAX}^{12}_{-},$$
$$\mathrm{MAX}^{10} = \mathrm{MAX}^{10}_{++} \sqcup \mathrm{MAX}^{10}_{+-} \sqcup \mathrm{MAX}^{10}_{-+} \sqcup \mathrm{MAX}^{10}_{--}.$$

Lemma 3. *The following equality holds:*

$$N_{00} = \mathrm{MAX}^{20} \sqcup \mathrm{MAX}^{12} \sqcup \mathrm{MAX}^{10} \sqcup \mathrm{FIX}^{12}. \tag{2.1}$$

Table 2. Components of $N_{00} \cap N_{\text{cut}} \cap N_1$.

MAX	MAX^{20}				MAX^{12}		MAX^{10}			
	MAX^{20}_{++}	MAX^{20}_{+-}	MAX^{20}_{-+}	MAX^{20}_{--}	MAX^{12}_{-}	MAX^{12}_{+}	MAX^{10}_{++}	MAX^{10}_{+-}	MAX^{10}_{-+}	MAX^{10}_{--}
k	$(0, k_0)$				k_0		$(k_0, 1)$			
u_1	π				$u_{1z}(k_0) = \pi$		$u_{1z}(k)$			
u_2	$\pi/2$	$3\pi/2$	$\pi/2$	$3\pi/2$	$[0, 2\pi)$		0	π	0	π
σ	$(0, +\infty)$		$(-\infty, 0)$			$(0, +\infty)$			$(-\infty, 0)$	

Proof. Let $\nu = (\lambda, t) \in N_{00}$ and $t < t_{\text{cut}}(\lambda)$. Then (λ, t) is a fixed point for both symmetries $\varepsilon^1, \varepsilon^2$. One can see from Table 1 that the invariance condition holds for both symmetries only in the case $\nu \in N_7$, when $\sin\theta = 0$. Further, since we have $z = 0$ in the image, Lemma 2 implies the inclusion $N_{00} \subseteq \{(\lambda, t) \in N_1 \mid t = t_{\text{cut}}(\lambda)\} \cup \text{FIX}^{12}$.

Consider further the case $t = t_{\text{cut}}(\lambda)$ for $(\lambda, t) \in N_1$. The following cases are possible:

- $k \in (0, k_0)$. Then $t_{\text{cut}}(\lambda) = 4K(k)$, thus $u_1 = \pi$, which implies $x = 0$. The equality $z = 0$ holds only in the case $\cos u_2 = 0$. These points correspond to Maxwell points for the symmetry ε^2, namely, to the set MAX^{20}.

- $k = k_0$. Then $t_{\text{cut}}(\lambda) = 4K(k_0)$, i. e., $u_1 = u_{1z}(k_0) = \pi$, whence $x = z = 0$. In this case $(\lambda, t) \in \text{MAX}^{12}$, the points are Maxwell points for both symmetries $\varepsilon^1, \varepsilon^2$.

- $k \in (k_0, 1)$. Then $t_{\text{cut}}(\lambda) = 2p_z^1(k)/|\sigma|$, i. e., $u_1 = u_{1z}(k) \in (\pi/2, \pi)$, thus $z = 0$. Moreover, the equality $x = 0$ holds only in the case $\sin u_2 = 0$. These points correspond to Maxwell points for the symmetry ε^1, namely, to the set MAX^{10}.

Equality (2.1) follows.

The set FIX^{12} decomposes into two connected components: $\text{FIX}^{12}_{\pm} = \{\nu \in N_7 \mid \cos\theta = \pm 1\}$.
A stratification of the set M_{00} is shown in Table 3 and in Fig. 1.

Table 3. Stratification of M_{00} in coordinates (y, w).

$M_{0,0}$	$\mathcal{I}^0_{x+}$	$\mathcal{E}_+$	$\mathcal{I}^0_{z+}$	$\mathcal{A}_+$	$\mathcal{A}_-$	$\mathcal{I}^0_{z-}$	$\mathcal{E}_-$	$\mathcal{I}^0_{x-}$
y	$(0, \infty)$	0	$(-\infty, 0)$	$(0, \infty)$	$(-\infty, 0)$	$(0, \infty)$	0	$(-\infty, 0)$
w	$(0, +\infty)$			0		$(-\infty, 0)$		

Lemmas 4, 5 are obvious, and Lemmas 6, 7 are proved in Appendix A.

Lemma 4. *The map* $\text{Exp}\colon \text{FIX}^{12}_{\pm} \to \mathcal{A}_{\pm}$ *is a diffeomorphism.*

Lemma 5. *For any* $u_2^0 \in [0, 2\pi)$ *the map* $\text{Exp}\colon \left(\text{MAX}^{12}_{+} \cap \{u_2 = u_2^0\}\right) \to \mathcal{E}_+$ *is a diffeomorphism.*

Lemma 6. *The map* $\text{Exp}\colon \text{MAX}^{20}_{++} \to \mathcal{I}^0_{x+}$ *is a diffeomorphism.*

Lemma 7. *The map* $\text{Exp}\colon \text{MAX}^{10}_{++} \to \mathcal{I}^0_{z+}$ *is a diffeomorphism.*

Corollary 1. *The following restrictions of the exponential map are diffeomorphisms:*

1. $\text{Exp}\colon \text{FIX}^{12}_{-} \to \mathcal{A}_-$,

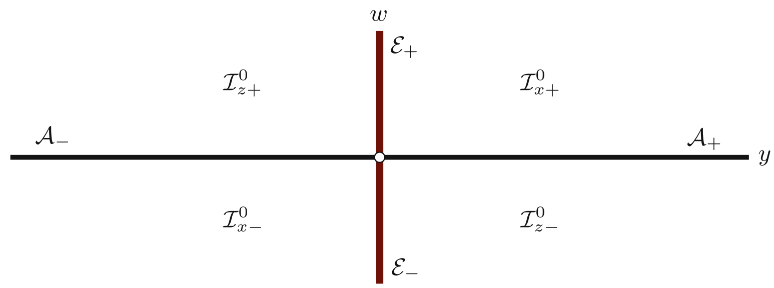

Fig. 1. Stratification of the plane $\{x = z = 0\}$.

2. $\mathrm{Exp}\colon \big(\mathrm{MAX}^{12}_- \cap \{u_2 = u_2^0\}\big) \to \mathcal{E}_-, u_2^0 \in [0, 2\pi)$,

3. $\mathrm{Exp}\colon \mathrm{MAX}^{20}_{+-} \to \mathcal{I}^0_{x+}$, $\mathrm{Exp}\colon \mathrm{MAX}^{20}_{-+} \to \mathcal{I}^0_{x-}$, $\mathrm{Exp}\colon \mathrm{MAX}^{20}_{--} \to \mathcal{I}^0_{x-}$,

4. $\mathrm{Exp}\colon \mathrm{MAX}^{10}_{+-} \to \mathcal{I}^0_{z+}$, $\mathrm{Exp}\colon \mathrm{MAX}^{10}_{-+} \to \mathcal{I}^0_{z-}$, $\mathrm{Exp}\colon \mathrm{MAX}^{10}_{--} \to \mathcal{I}^0_{z-}$.

Proof. After taking the symmetries $\varepsilon^1, \varepsilon^2, \varepsilon^4$ into account, the proof follows immediately from Lemmas 4–7.

We sum up the results of this section in the following statement.

Theorem 4. *There is a stratification*

$$\mathrm{Cut} \cap M_{00} = \{q \in M \mid x = z = 0, w \neq 0\} = \bigsqcup_{i \in \{+,-\}} \big(\mathcal{I}^0_{xi} \sqcup \mathcal{I}^0_{zi} \sqcup \mathcal{E}_i\big).$$

At each point of the quadrants $\mathcal{I}^0_{x\pm}, \mathcal{I}^0_{z\pm}$ there are exactly two sub-Riemannian minimizers, while at any point of the rays $\mathcal{E}_\pm$ there is a one-parametric family of minimizers.

The rays $\mathcal{A}_+ \sqcup \mathcal{A}_- = M_{00} \setminus \mathrm{Cut}$ are filled by abnormal trajectories.

Now we describe minimizers for $q_1 = (0, y_1, 0, w_1) \in M_{00}$:

1. for $q_1 \in \mathcal{A}_\pm$ there is a unique minimizer $\nu \in \mathrm{FIX}^{12}_\pm$.

2. for $q_1 \in \mathcal{I}^0_{x\pm}$ there are two minimizers

$$\nu_1 = (\hat{k}, \pi, \pi/2, \hat{\sigma}) \in \mathrm{MAX}^{20}_{\pm+}, \quad \nu_2 = (\hat{k}, \pi, 3\pi/2, \hat{\sigma}) \in \mathrm{MAX}^{20}_{\pm-}, \quad \hat{k} \in (0, k_0), \quad \pm\hat{\sigma} > 0.$$

Figure 2 shows two minimizers with $\hat{k} = 0.84, \hat{\sigma} = 1$.

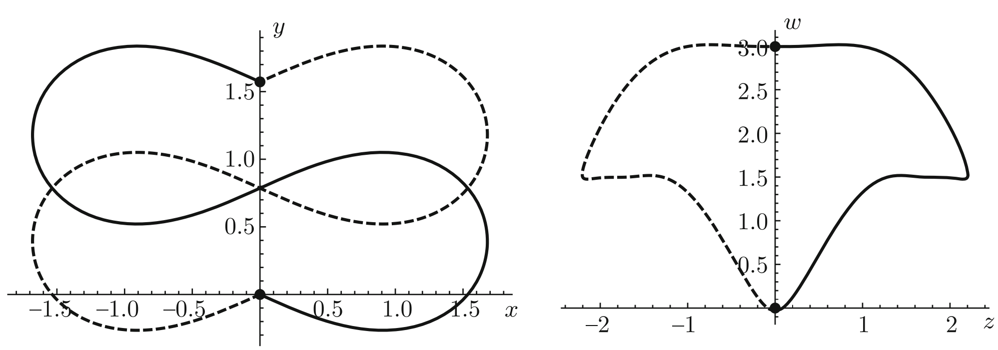

Fig. 2. Example of symmetric trajectories for $x_1 = z_1 = 0, y_1 > 0, w_1 > 0$. Left: projections to the plane (x, y), right: projections to the plane (z, w).

3. for $q_1 \in \mathcal{I}^0_{z\pm}$ there are two minimizers

$$\nu_1 = \left(\hat{k}, u_{1z}(\hat{k}), 0, \hat{\sigma}\right) \in \mathrm{MAX}^{10}_{\pm+}, \quad \nu_2 = \left(\hat{k}, u_{1z}(\hat{k}), \pi, \hat{\sigma}\right) \in \mathrm{MAX}^{10}_{\pm-}, \quad \hat{k} \in (k_0, 1), \quad \pm\hat{\sigma} > 0.$$

Figure 3 shows two optimal trajectories with $\hat{k} = 0.95$, $\hat{\sigma} = 1$.

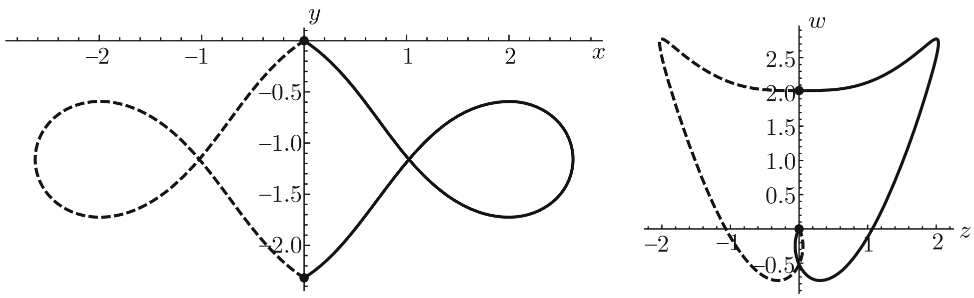

Fig. 3. Example of symmetric trajectories for $x_1 = z_1 = 0, y_1 < 0, w_1 > 0$. Left: projections to the plane (x, y), right: projections to the plane (z, w).

4. for $q_1 \in \mathcal{E}_\pm$ there exists a one-parameter family of minimizers

$$\nu(u_2^0) = (k_0, \pi, u_2^0, \hat{\sigma}) \in \mathrm{MAX}^{12}_{\pm}, \qquad u_2^0 \in [0, 2\pi).$$

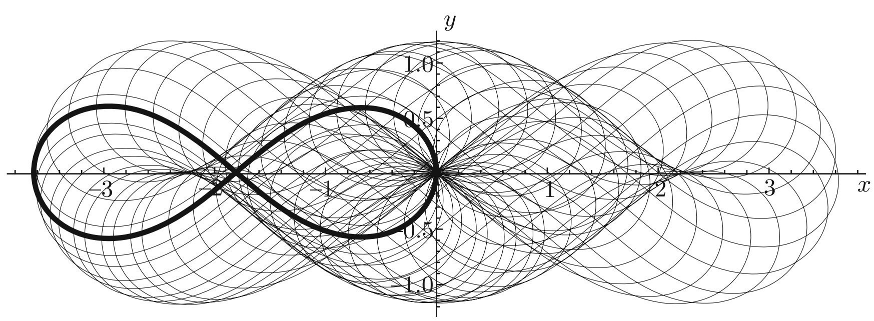

Fig. 4. One-parameter family of "figure-eight" trajectories coming to the same point (projections to (x, y)).

In the plane (x, y) the minimizers project to "figure-eight" elasticae (see Fig. 4, the boldface curve corresponds to $u_2^0 = 0$).

Despite the fact that solutions in the plane (x, y) have the same form and are transformed one to another by parallel translations, there is no continuous symmetry in the space M that transforms these solutions one to another. There are only discrete symmetries that decompose solutions into four-tuples for $u_2^0 \neq \pi n/2, n = 0, \ldots, 3$, and two pairs for $u_2^0 = 0, \pi$ and for $u_2^0 = \pi/2, 3\pi/2$. Figure 5 shows a four-tuple of symmetric solutions.

3. STRUCTURE OF $\mathrm{Cut} \cap \{z = 0,\ x > 0\}$

In this section we study the set $\mathrm{Cut} \cap M_{+0}$. Consider the corresponding set in the preimage of the exponential map $N_{+0} = \mathrm{Exp}^{-1}(M_{+0}) \cap \widehat{N}$ and study its intersection with the cut locus N_{cut}. Notice that the equality $z = 0$, by Lemma 2, implies that $N_{+0} \subset N_1 \cup N_7$. Since $t_{\mathrm{cut}}|_{C_7} = \infty$, we get

$$\begin{aligned} N_{+0} \cap N_{\mathrm{cut}} &= \{(k, u_1, u_2, \sigma) \in N_1 \mid k \in (k_0, 1), u_1 = u_{1z}(k), \sin u_2 \neq 0\} \\ &= \mathrm{MAX}^{1+}_{++} \sqcup \mathrm{MAX}^{1+}_{+-} \sqcup \mathrm{MAX}^{1+}_{-+} \sqcup \mathrm{MAX}^{1+}_{--} \sqcup \mathrm{CMAX}^{1+}_{+} \sqcup \mathrm{CMAX}^{1+}_{-}, \end{aligned}$$

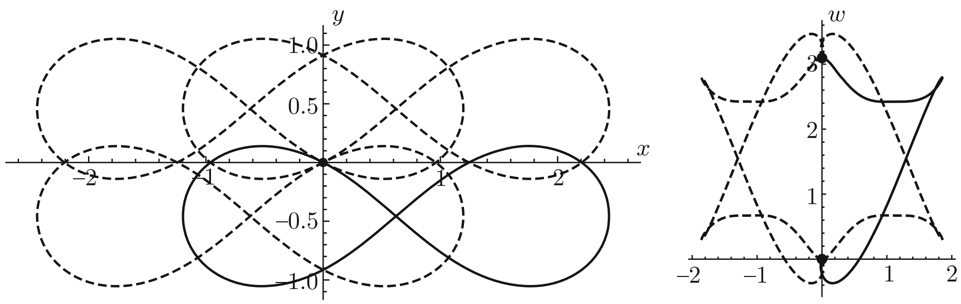

Fig. 5. Example of symmetric trajectories for $x_1 = y_1 = z_1 = 0$. Left: projections to (x, y), right: projections to (z, w).

Table 4. Components of $N_{+0} \cap N_{\mathrm{cut}} \cap N_1$.

	MAX^{1+}_{++}	CMAX^{1+}_{+}	MAX^{1+}_{+-}	MAX^{1+}_{-+}	CMAX^{1+}_{-}	MAX^{1+}_{--}
u_2	$(0, \pi/2)$	$\pi/2$	$(\pi/2, \pi)$	$(\pi, 3\pi/2)$	$3\pi/2$	$(3\pi/2, 2\pi)$
σ	$(-\infty, 0)$			$(0, +\infty)$		
k	$(k_0, 1)$					
u_1	$u_{1z}(k)$					

where the sets $\mathrm{MAX}^{1+}_{+\pm}$,$\mathrm{MAX}^{1+}_{-\pm}$, and $\mathrm{CMAX}^{1+}_{\pm}$ are defined via Table 4.

In order to describe a decomposition of the set M_{+0}, we pass to new coordinates $(Y^1, W^1) = \left(\dfrac{y}{x}, \dfrac{w}{x^3}\right)$ invariant under dilations e^{tX_0} .

Further in the text, we use the following notation:

$$s_i = \sin u_i, \qquad c_i = \cos u_i, \qquad d_i = \sqrt{1 - k^2 s_i^2}, \tag{3.1}$$

$$E_1 = E\big(u_1, k\big), \quad F_1 = F\big(u_1, k\big), \quad \Delta = 1 - k^2 s_1^2 s_2^2. \tag{3.2}$$

In the case $N_{+0} \cap N_{\mathrm{cut}}$ the exponential map is determined by the formulas:

$$Y_1^1(k, u_1, u_2) = -\frac{1 + k^2(s_1^2 - 2)s_2^2}{2k\, c_1 s_2 d_2},$$

$$W_1^1(k, u_1, u_2) = \frac{1}{48k^3 s_1^3 c_1 d_1^3 s_2^3 d_2^3}\Big(- E_1 c_1 \Delta^3 + d_1^3 s_1 \Big(1 - k^2 s_1^2 s_2^2 \big(6 - 3k^2(4 - s_1^2)s_2^2 + 4k^4(2 - s_1^2)s_2^4\big)\Big)\Big).$$

The sets of conjugate points $\mathcal{CI}^+_{z\pm} = \mathrm{Exp}(\mathrm{CMAX}^{1+}_{\pm})$ consist of limit points for Maxwell points, these sets are parametrically defined as follows:

$$\mathcal{CI}^{1+}_{\pm} = \{(Y^1, W^1) = \big(Y_1^1(k, u_1, u_2), W_1^1(k, u_1, u_2)\big) \mid k \in (k_0, 1), u_1 = u_{1z}(k), \sin u_2 = \pm 1\}.$$

Lemmas 8–10 are proved in Appendix B.

Lemma 8. *The map* $\mathrm{Exp}\colon \mathrm{CMAX}^{1+}_{+} \to \mathcal{CI}^+_{z+}$ *is a diffeomorphism.*

Lemma 9. 1) *The curve* $\mathcal{CI}^+_{z+}$ *in the plane* (Y^1, W^1) *is a graph of a smooth function* $W^1_{\mathrm{conj}}(Y^1)$, *increasing from* $-\infty$ *to* ∞ *in the interval* $Y^1 \in (Y^1_0, \infty)$, *where* $Y^1_0 = \dfrac{1 - 2k_0^2}{2k_0\sqrt{1 - k_0^2}} < 0$.

2) *The curve* $W_1 = W^1_{\rm conj}(Y^1)$ *lies below the line* $W^1 = Y_1/6$.

3) $\lim\limits_{Y^1\to+\infty} \dfrac{W^1_{\rm conj}(Y^1)}{Y^1} \in \mathbb{R}\backslash\{0\}$.

Define the following sets in the plane (Y^1, W^1):

$$\mathcal{I}^+_{z+} = \left\{(Y^1, W^1) \mid W^1_{\rm conj}(Y^1) > W^1, Y^1 \in (Y^1_0, \infty)\right\},\ \mathcal{I}^+_{z-} = \left\{(Y^1, W^1) \in \mathbb{R}^2 \mid (-Y^1, -W^1) \in \mathcal{I}^+_{z+}\right\},$$

and show their relation to the cut locus.

Lemma 10. *The map* $\mathrm{Exp}\colon\ \mathrm{MAX}^{1+}_{+-} \to \mathcal{I}^+_{z+}$ *is a diffeomorphism.*

Corollary 2. *The following restrictions of the exponential map are diffeomorphisms:*

$$\mathrm{Exp}\colon\ \mathrm{MAX}^{1+}_{++} \to \mathcal{I}^+_{z+}, \qquad \mathrm{Exp}\colon\ \mathrm{CMAX}^{1+}_{-} \to \mathcal{CI}^+_{z-}, \qquad \mathrm{Exp}\colon\ \mathrm{MAX}^{1+}_{-\pm} \to \mathcal{I}^+_{z-}.$$

Proof. After taking the symmetries ε^1, ε^2, ε^4 into account, the proof follows immediately from Lemmas 8 and 10.

Figure 6 shows a decomposition of the plane (Y^1, W^1).

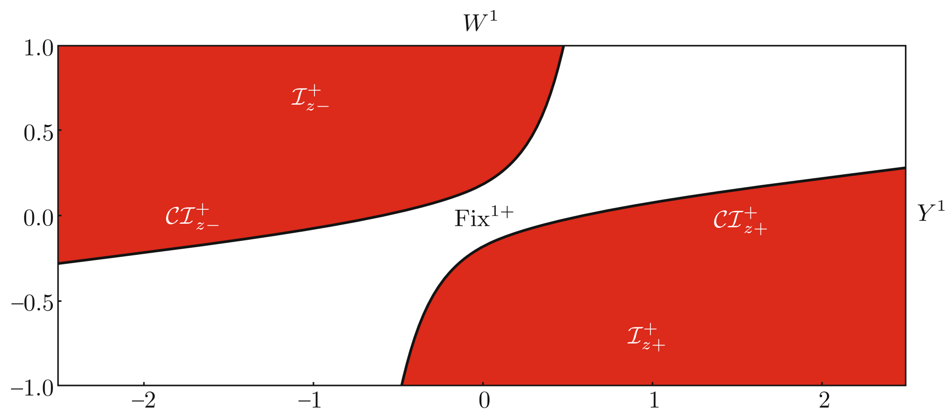

Fig. 6. Decomposition of the set M_{+0}.

Theorem 5. *There is a stratification*

$$\mathrm{Cut} \cap M_{+0} = \mathcal{I}^+_{z+} \sqcup \mathcal{CI}^+_{z+} \sqcup \mathcal{I}^+_{z-} \sqcup \mathcal{CI}^+_{z-}.$$

Moreover,

$$\mathcal{I}^+_{z+} = \left\{q \in M \mid z = 0,\ x > 0,\ y > Y^1_0 x,\ w < W^1_{\rm conj}(y/x)x^3\right\} \cong \mathbb{R}^3, \tag{3.3}$$

$$\mathcal{I}^+_{z-} = \left\{q \in M \mid z = 0,\ x > 0,\ y < -Y^1_0 x,\ w > -W^1_{\rm conj}(-y/x)x^3\right\} \cong \mathbb{R}^3,$$

$$\mathcal{CI}^+_{z+} = \left\{q \in M \mid z = 0,\ x > 0,\ y > Y^1_0 x,\ w = W^1_{\rm conj}(y/x)x^3\right\} \cong \mathbb{R}^2,$$

$$\mathcal{CI}^+_{z-} = \left\{q \in M \mid z = 0,\ x > 0,\ y < -Y^1_0 x,\ w = -W^1_{\rm conj}(-y/x)x^3\right\} \cong \mathbb{R}^2.$$

For each point of the sets $\mathcal{I}^+_{z\pm}$ *there exist two minimizers, and for each point of the remaining part* $M_{+0}\backslash(\mathcal{I}^+_{z+} \sqcup \mathcal{I}^+_{z-})$ *there is a unique minimizer.*

Proof. Follows from Lemmas 8–10 and from Corollary 2.

4. STRUCTURE OF $\mathrm{Cut} \cap \{z = 0\}$

In this section we describe the intersection of the cut locus with the subspace

$$M_z = \{q \in M \mid z = 0\}.$$

Theorem 6. *There is a stratification*

$$\mathrm{Cut} \cap M_z = \bigsqcup_{i \in \{+,-\}} \Big(\mathcal{I}_{zi} \sqcup \big(\sqcup_{j \in \{+,-\}} \mathcal{CI}_{zi}^{j} \big) \sqcup \mathcal{I}_{xi}^{0} \sqcup \mathcal{E}_i \Big), \tag{4.1}$$

where $\mathcal{I}_{zi} = \bigsqcup_{j \in \{+,-,0\}} \mathcal{I}_{zi}^{j}$, $\mathcal{I}_{zi}^{-} = \varepsilon^2(\mathcal{I}_{zi}^{+})$, $\mathcal{CI}_{zi}^{-} = \varepsilon^2(\mathcal{CI}_{zi}^{+})$, $i \in \{+,-\}$. *Moreover,*

$$\mathcal{I}_{z+} = \{q \in M \mid z = 0,\ y > Y_0^1|x|,\ w < G_1(x,y)\} \cong \mathbb{R}^3, \qquad \mathcal{I}_{z-} = \varepsilon^4(\mathcal{I}_{z+}) \cong \mathbb{R}^3, \tag{4.2}$$

where G_1 is a function continuous on the set $\{(x,y) \in \mathbb{R}^2 \mid y > Y_0^1|x|\}$ *and satisfying the properties*

$$G_1(0,y) = 0, \quad G_1(-x,y) = G_1(x,y), \quad G_1(\rho x, \rho y) = \rho^3 G_1(x,y), \quad \rho > 0.$$

Proof. By virtue of the equalities $\varepsilon^2(\mathrm{Cut}) = \mathrm{Cut}, \varepsilon^2(M_{+0}) = M_{-0}$, we get, taking Theorem 5 into account, a decomposition $\mathrm{Cut} \cap M_{-0} = \varepsilon^2(\mathrm{Cut} \cap M_{+0}) = \bigsqcup_{i \in \{+,-\}} (\mathcal{I}_{zi}^{-} \sqcup \mathcal{CI}_{zi}^{-})$. Whence, taking Theorem 4 into account, we get a stratification (4.1).

Representation (4.2) is obtained from (3.3), Table 3, and the equality

$$\mathcal{I}_{z+}^{-} = \{q \in M \mid z = 0,\ x < 0,\ y > Y_0^1|x|,\ w < W_{\mathrm{conj}}^1(y/|x|)|x|^3\}$$

for the function

$$G_1(x,y) = \begin{cases} W_{\mathrm{conj}}^1(y/|x|)|x^3|,\ x \neq 0, y > Y_0^1|x|, \\ 0,\ x = 0, y > 0. \end{cases}$$

Continuity of the function G_1 on the set $\{(x,y) \mid x \neq 0, y > Y_0^1|x|\}$ follows from continuity of the function W_{conj}^1 on the ray $(Y_0^1, +\infty)$, see Lemma 9. In order to prove continuity of the function G_1 on the ray $\{(x,y) \mid x = 0, y > 0\}$, take any sequence (x_n, y_n), $x_n \to +0$, $y_n \to \bar{y} > 0$. Then, taking into account item 3) of Lemma 9, we obtain

$$G_1(x_n, y_n) = W_{\mathrm{conj}}^1(y_n/x_n)x_n^3 = \frac{W_{\mathrm{conj}}^1(y_n/x_n)}{y_n/x_n} y_n x_n^2 \to 0.$$

Thus, the function G_1 is continuous on its whole domain $\{(x,y) \mid y > Y_0^1|x|\}$.

Representation (4.2) implies that the stratum $\mathcal{I}_{z+}$ is homeomorphic to $\mathbb{R}^3$. The theorem is proved.

5. STRUCTURE OF $\mathrm{Cut} \cap \{x = 0,\ z > 0\}$

In this section we study the set $N_{0+} = \mathrm{Exp}^{-1}(M_{0+}) \cap \widehat{N}$ and then describe the intersection $M_{0+} \cap \mathrm{Cut}$. It follows from Theorem 1 that $N_{0+} \cap N_{\mathrm{cut}} \subset N_1 \cup N_2 \cup N_6$.

Now we consider cut points for each of the subsets $N_i, i = 1, 2, 6$.

Table 5. Components of $N_{0+} \cap N_{\mathrm{cut}} \cap N_1$.

	CMAX^{2+}_{1+}	MAX^{2+}_{1++}	MAX^{2+}_{1+-}	CMAX^{2+}_{1-}	MAX^{2+}_{1-+}	MAX^{2+}_{1--}
u_2	0	$(0, \pi/2)$	$(3\pi/2, 2\pi)$	0	$(0, \pi/2)$	$(3\pi/2, 2\pi)$
σ	$(0, \infty)$			$(-\infty, 0)$		
k	$(0, k_0)$					
u_1	π					

5.1. Subcase N_1

We have

$$\begin{aligned} N_{0+} \cap N_{\mathrm{cut}} \cap N_1 &= \{(k, u_1, u_2, \sigma) \in N_1 \mid k \in (0, k_0), u_1 = \pi\} \\ &= \mathrm{CMAX}^{2+}_{1+} \sqcup \mathrm{CMAX}^{2+}_{1-} \sqcup \mathrm{MAX}^{2+}_{1++} \sqcup \mathrm{MAX}^{2+}_{1+-} \sqcup \mathrm{MAX}^{2+}_{1-+} \sqcup \mathrm{MAX}^{2+}_{1--}, \end{aligned}$$

where the sets $\mathrm{CMAX}^{2+}_{1\pm}, \mathrm{MAX}^{2+}_{1+\pm}, \mathrm{MAX}^{2+}_{1-\pm}$ are defined by values of the parameters u_2, σ via Table 5; notice that the inequality $z > 0$ implies $\cos u_2 > 0$.

In order to define a decomposition in the set M_{0+}, we introduce new coordinates $(Y^2, W^2) = \left(\dfrac{y}{\sqrt{z}}, \dfrac{w}{\sqrt{z^3}}\right)$ invariant under dilations e^{tX_0}. Then the exponential map takes the form:

$$Y^2_1(k, u_2) = \sqrt{\frac{2\iota_1(k)}{kc_2}}, \qquad W^2_1(k, u_2) = \frac{\iota_2(k) + k^2\iota_1(k)(1 + 3c_2^2)}{3\big(2k\iota_1(k)c_2\big)^{3/2}},$$

where the functions $\iota_1(k), \iota_2(k)$ are defined in Appendix A, see (A.1).

The sets of conjugate points $\mathcal{CI}^+_{x\pm} = \mathrm{Exp}(\mathrm{CMAX}^{2+}_{1\pm})$ are defined as follows:

$$\mathcal{CI}^+_{x\pm} = \big\{(Y^2, W^2) = (\pm Y^2_1(k, u_2), \pm W^2_1(k, u_2)) \mid k \in (0, k_0), u_2 = 0\big\}. \qquad (5.1)$$

Lemmas 11–19 and 21 are proved in Appendix C. Lemma 20 is obvious.

Lemma 11. *The map* $\mathrm{Exp} : \mathrm{CMAX}^{2+}_{1+} \to \mathcal{CI}^+_{x+}$ *is a diffeomorphism.*

Lemma 12. *The curve* $\mathcal{CI}^+_{x+}$ *in the plane* (Y^2, W^2) *is a graph of a certain smooth function* $W^{21}_{\mathrm{conj}}(Y^2) > 0$, *decreasing from* ∞ *to* 0 *at the interval* $Y^2 \in (0, \infty)$.

Define the following sets in the plane (Y^2, W^2):

$$\begin{aligned} \mathcal{I}^+_{x+} &= \{(Y^2, W^2) \mid W^{21}_{\mathrm{conj}}(Y^2) < W^2,\ Y^2 \in (0, \infty)\}, \\ \mathcal{I}^+_{x-} &= \{(Y^2, W^2) \in \mathbb{R}^2 \mid (-Y^2, -W^2) \in \mathcal{I}^+_{x+}\} \end{aligned} \qquad (5.2)$$

and show their relation to the cut locus.

Lemma 13. *The map* $\mathrm{Exp}\colon \mathrm{MAX}^{2+}_{1++} \to \mathcal{I}^+_{x+}$ *is a diffeomorphism.*

Corollary 3. *The following restrictions of the exponential map are diffeomorphisms:*

$$\mathrm{Exp}\colon \mathrm{MAX}^{2+}_{1+-} \to \mathcal{I}^+_{x+}, \qquad \mathrm{Exp}\colon \mathrm{CMAX}^{2+}_{1-} \to \mathcal{CI}^+_{x-}, \qquad \mathrm{Exp}\colon \mathrm{MAX}^{2+}_{1-\pm} \to \mathcal{I}^+_{x-}.$$

Proof. After taking the symmetries $\varepsilon^1, \varepsilon^2, \varepsilon^4$ into account, the proof follows immediately from Lemmas 11 and 13.

5.2. Subcase N_2

The following representation holds:

$$N_{0+} \cap N_{\mathrm{cut}} \cap N_2 = \{(k, u_1, u_2, \sigma) \in N_2 \mid k \in (0,1), u_1 = \pi/2\} = \mathrm{CMAX}^{2+}_{2++} \sqcup \mathrm{CMAX}^{2+}_{2+-} \sqcup \mathrm{CMAX}^{2+}_{2-+} \sqcup \mathrm{CMAX}^{2+}_{2--} \sqcup \mathrm{MAX}^{2+}_{2++} \sqcup \mathrm{MAX}^{2+}_{2+-} \sqcup \mathrm{MAX}^{2+}_{2-+} \sqcup \mathrm{MAX}^{2+}_{2--},$$

where $\mathrm{CMAX}^{2+}_{2+\pm}, \mathrm{CMAX}^{2+}_{2-\pm}, \mathrm{MAX}^{2+}_{2+\pm}$ and $\mathrm{MAX}^{2+}_{2-\pm}$ are defined by values of the parameters u_2, σ via Table 6. Notice that the inequality $z > 0$ implies that $\operatorname{sgn} c = 1$.

Table 6. Components of $N_{0+} \cap N_{\mathrm{cut}} \cap N_2$.

	CMAX^{2+}_{2++}	MAX^{2+}_{2++}	CMAX^{2+}_{2+-}	MAX^{2+}_{2+-}	CMAX^{2+}_{2-+}	MAX^{2+}_{2-+}	CMAX^{2+}_{2--}	MAX^{2+}_{2--}
u_2	0	$(0, \pi/2)$	$\pi/2$	$(\pi/2, \pi)$	0	$(0, \pi/2)$	$\pi/2$	$(\pi/2, \pi)$
σ	$(0, \infty)$				$(-\infty, 0)$			
k	$(0,1)$							
u_1	$(0, \pi/2)$							

For $\nu \in N_{0+} \cap N_{\mathrm{cut}} \cap N_2$ the exponential map is defined in the coordinates (Y^2, W^2) by the formulas:

$$Y^2_2(k, u_2) = -\sqrt{\frac{\iota_4(k) d_2}{\sqrt{1-k^2}}} < 0, \qquad W^2_2(k, u_2) = \frac{k^4 K(k) d_2^2 - \iota_4(k)\big(8 - 7k^2 - k^2(2-k^2)s_2^2\big)}{12\sqrt{\iota_4^3(k)(1-k^2)^{3/2} d_2}},$$

where the function $\iota_4(k)$ is defined in Appendix C, see (C.8).

The sets of conjugate points $\mathcal{CN}^+_{x\pm+}$, $\mathcal{CN}^+_{x\pm-}$ are defined as follows:

$$\mathcal{CN}^+_{x\pm+} = \Big\{(Y^2, W^2) = \big(\pm Y^2_2(k, 0), \pm W^2_2(k, 0)\big) \mid k \in (0,1)\Big\}, \tag{5.3}$$

$$\mathcal{CN}^+_{x\pm-} = \Big\{(Y^2, W^2) = \big(\pm Y^2_2(k, \pi/2), \pm W^2_2(k, \pi/2)\big) \mid k \in (0,1)\Big\}. \tag{5.4}$$

Lemmas 14–19 are proved in Appendix C.

Lemma 14. *The map* $\mathrm{Exp}: \mathrm{CMAX}^{2+}_{2++} \to \mathcal{CN}^+_{x++}$ *is a diffeomorphism.*

Lemma 15. *The curve* $\mathcal{CN}^+_{x++}$ *in the plane* (Y^2, W^2) *is a graph of a smooth function* $W^{22+}_{\mathrm{conj}}(Y^2)$ *decreasing from* ∞ *to* $1/\sqrt{\pi}$ *at the interval* $Y^2 \in (-\infty, 0)$.

Lemma 16. *The map* $\mathrm{Exp}: \mathrm{CMAX}^{2+}_{2+-} \to \mathcal{CN}^+_{x+-}$ *is a diffeomorphism.*

Lemma 17. *The curve* $\mathcal{CN}^+_{x+-}$ *in the plane* (Y^2, W^2) *is a graph of a smooth function* $W^{22-}_{\mathrm{conj}}(Y^2)$, *decreasing from* 0 *to* $-1/\sqrt{\pi}$ *at the interval* $(-\infty, 0)$.

Corollary 4. *The following restrictions of the exponential map are diffeomorphisms:*

$$\mathrm{Exp}\colon \mathrm{CMAX}^{2+}_{2-+} \to \mathcal{CN}^+_{x-+}, \qquad \mathrm{Exp}\colon \mathrm{CMAX}^{2+}_{2--} \to \mathcal{CN}^+_{x--}.$$

The curves $\mathcal{CN}^+_{x-+}$, $\mathcal{CN}^+_{x--}$ *in the plane* (Y^2, W^2) *are, respectively, graphs of the functions* $-W^{22+}_{\mathrm{conj}}(-Y^2)$, $-W^{22-}_{\mathrm{conj}}(-Y^2)$, $Y \in (0, \infty)$.

Further we study the relative position of the curves $\mathcal{CI}^+_{x\pm}$, $\mathcal{CN}^+_{x\pm+}$, $\mathcal{CN}^+_{x\pm-}$. By virtue of Lemma 15 and the symmetry ε^4 (1.3) it follows that the curves $\mathcal{CN}^+_{x++}$ and $\mathcal{CN}^+_{x-+}$ belong, respectively, to the second and fourth quadrants of the plane (Y^2, W^2).

Lemma 12 and Corollary 4 imply that the curves $\mathcal{CI}^+_{x+}, \mathcal{CN}^+_{x--}$ belong to the first quadrant. We show in the following lemma that they do not intersect each other.

Lemma 18. *The inequality* $-W^{22-}_{\mathrm{conj}}(-Y^2) < W^{21}_{\mathrm{conj}}(Y^2)$ *holds for* $Y^2 > 0$.

Define the following sets in the plane (Y^2, W^2):

$$\mathcal{N}^+_{x+} = \{(Y^2, W^2) \in \mathbb{R}^2 \mid W^{22-}_{\mathrm{conj}}(Y^2) < W^2 < W^{22+}_{\mathrm{conj}}(Y^2),\ Y^2 < 0\},$$
$$\mathcal{N}^+_{x-} = \{(Y^2, W^2) \in \mathbb{R}^2 \mid (-Y^2, -W^2) \in \mathcal{N}^+_{x+}\}.$$

Lemma 19. *The map* $\mathrm{Exp}\colon\ \mathrm{MAX}^{2+}_{2--} \to \mathcal{N}^+_{x-}$ *is a diffeomorphism.*

Corollary 5. *The following restrictions of the exponential map are diffeomorphisms:*

$$\mathrm{Exp}\colon\ \mathrm{MAX}^{2+}_{2-+} \to \mathcal{N}^+_{x-}, \qquad \mathrm{Exp}\colon\ \mathrm{MAX}^{2+}_{2++} \to \mathcal{N}^+_{x+}, \qquad \mathrm{Exp}\colon\ \mathrm{MAX}^{2+}_{2+-} \to \mathcal{N}^+_{x+}.$$

5.3. *Subcase* C_6

The following decomposition holds:

$$N_{0+} \cap N_{\mathrm{cut}} \cap N_6 = \left\{(c, \theta, t) \in N_6 \mid t = 2\pi/|c|, \theta \in S^1, c > 0\right\} = \bigsqcup_{i \in \{+,-\}} \left(\mathrm{CMAX}^{2+}_{6i} \sqcup \mathrm{MAX}^{2+}_{6i}\right),$$

where the sets $\mathrm{CMAX}^{2+}_{6\pm}, \mathrm{MAX}^{2+}_{6\pm}$ are defined by values of the parameter θ via Table 7.

Table 7. Components of $N_{0+} \cap \mathcal{N}_{\mathrm{cut}} \cap N_6$.

	MAX^{2+}_{6+}	CMAX^{2+}_{6-}	MAX^{2+}_{6-}	CMAX^{2+}_{6+}
θ	$(-\pi, 0)$	0	$(0, \pi)$	π
c	$(0, +\infty)$			
t	$2\pi/\lvert c\rvert$			

For $\nu \in N_{0+} \cap N_{\mathrm{cut}} \cap N_6$ the exponential map is defined in the coordinates (Y^2, W^2) by the formulas: $Y^2_6(\theta) = 0$, $W^2_6(\theta) = -\cos\theta/\sqrt{\pi}$.

In the image of the exponential map the sets of conjugate points $\mathcal{CC}^+_{x\pm}$ are defined as follows:

$$\mathcal{CC}^+_{x+} = \left\{(Y^2, W^2) = \left(Y^2_6(\pi), W^2_6(\pi)\right) = (0, 1/\sqrt{\pi})\right\},$$
$$\mathcal{CC}^+_{x-} = \left\{(Y^2, W^2) = \left(Y^2_6(0), W^2_6(0)\right) = (0, -1/\sqrt{\pi})\right\}.$$

Define the following set in the plane (Y^2, W^2):

$$\mathcal{C}^+_x = \{(Y^2, W^2) \in \mathbb{R}^2 \mid Y^2 = 0, |W^2| < 1/\sqrt{\pi}\}.$$

Lemma 20. *The maps* $\mathrm{Exp}(\mathrm{MAX}^{2+}_{6\pm}) \to \mathcal{C}^+_x$ *are diffeomorphisms.*

A decomposition of the plane (Y^2, W^2) is shown in Fig. 7.

Denote the sets

$$\mathcal{N}^+_x = \mathcal{N}^+_{x+} \sqcup \mathcal{C}^+_x \sqcup \mathcal{N}^+_{x-},$$
$$\mathcal{CN}^+_{x+} = \mathcal{CN}^+_{x++} \sqcup \mathcal{CC}^+_{x+} \sqcup \mathcal{CN}^+_{x--}, \qquad \mathcal{CN}^+_{x-} = \mathcal{CN}^+_{x+-} \sqcup \mathcal{CC}^+_{x-} \sqcup \mathcal{CN}^+_{x-+}.$$

Lemma 21. *The set* $\mathcal{CN}_{x+}$ *(resp.* $\mathcal{CN}^+_{x-}$*) forms a smooth curve in the upper (lower) half-plane, which is a graph of a smooth function* $W^{22}_{\mathrm{conj}}(Y^2)$ *(resp.* $-W^{22}_{\mathrm{conj}}(-Y^2)$*), where*

$$W^{22}_{\mathrm{conj}}(Y^2) = \begin{cases} W^{22+}_{\mathrm{conj}}(Y^2), & Y^2 \in (-\infty, 0), \\ -W^{22-}_{\mathrm{conj}}(-Y^2), & Y^2 \in (\infty, 0), \\ 1/\sqrt{\pi}, & Y^2 = 0. \end{cases}$$

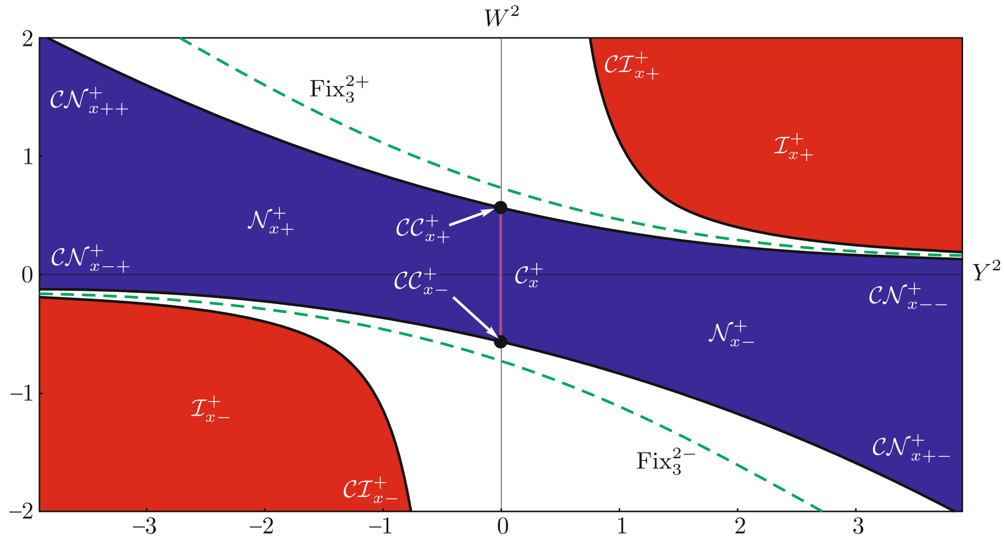

Fig. 7. Decomposition of the set M_{0+}.

We sum up the results of this section as follows.

Theorem 7. *There is a stratification*

$$\mathrm{Cut} \cap M_{0+} = \mathcal{I}^+_{x+} \sqcup \mathcal{I}^+_{x-} \sqcup \mathcal{N}^+_x \sqcup \mathcal{CI}^+_{x+} \sqcup \mathcal{CI}^+_{x-} \sqcup \mathcal{CN}^+_{x+} \sqcup \mathcal{CN}^+_{x-}.$$

Moreover,

$$\mathcal{I}^+_{x+} = \{q \in M \mid x = 0,\ z > 0,\ y > 0,\ w > W^{21}_{\mathrm{conj}}(y/\sqrt{z})\sqrt{z^3}\} \cong \mathbb{R}^3, \tag{5.5}$$
$$\mathcal{I}^+_{x-} = \{q \in M \mid x = 0,\ z > 0,\ y < 0,\ w < -W^{21}_{\mathrm{conj}}(-y/\sqrt{z})\sqrt{z^3}\} \cong \mathbb{R}^3,$$
$$\mathcal{N}^+_x = \{q \in M \mid x = 0,\ z > 0,\ -W^{22}_{\mathrm{conj}}(-y/\sqrt{z})\sqrt{z^3} < w < W^{22}_{\mathrm{conj}}(y/\sqrt{z})\sqrt{z^3}\} \cong \mathbb{R}^3,$$
$$\mathcal{CI}^+_{x\pm} = \{q \in M \mid x = 0,\ z > 0,\ \pm y > 0,\ w = \pm W^{21}_{\mathrm{conj}}(\pm y/\sqrt{z})\sqrt{z^3}\} \cong \mathbb{R}^2,$$
$$\mathcal{CN}^+_{x\pm} = \{q \in M \mid x = 0,\ z > 0,\ w = \pm W^{22}_{\mathrm{conj}}(\pm y/\sqrt{z})\sqrt{z^3}\} \cong \mathbb{R}^2.$$

At each point of the strata $\mathcal{I}^+_{x\pm}$, $\mathcal{N}^+_x$ there are two minimizers, and at each point of the remaining part $M_{0+}/(\mathcal{I}^+_{x+} \sqcup \mathcal{I}^+_{x-} \sqcup \mathcal{N}^+_x)$ there is a unique minimizer.

6. THE STRUCTURE OF $\mathrm{Cut} \cap \{x = 0\}$

Describe the intersection of the cut locus with the subspace $M_x = \{q \in M \mid x = 0\}$.

Theorem 8. *There is a stratification*

$$\mathrm{Cut} \cap M_x = \bigsqcup_{i \in \{+,-\}} \left(\mathcal{I}_{xi} \sqcup \mathcal{N}^i_x \sqcup \bigsqcup_{j \in \{+,-\}} \left(\mathcal{CI}^j_{xi} \sqcup \mathcal{CN}^j_{xi} \right) \sqcup \mathcal{I}^0_{zi} \sqcup \mathcal{E}_i \right), \tag{6.1}$$

where $\mathcal{I}_{xi} = \bigsqcup_{j \in \{+,-,0\}} \mathcal{I}^j_{xi}$, $\mathcal{I}^-_{xi} = \varepsilon^1(\mathcal{I}^+_{xi})$, $\mathcal{CN}^-_{xi} = \varepsilon^1(\mathcal{CI}^+_{xi})$, $\mathcal{CN}^-_{xi} = \varepsilon^1(\mathcal{CN}^+_{xi})$, $i \in \{+,-\}$*;* $\mathcal{N}^-_x = \varepsilon^1(N^+_x)$*. Moreover,*

$$\mathcal{I}_{x+} = \{q \in M \mid x = 0,\ y > 0,\ w > G_2(z, y)\} \cong \mathbb{R}^3, \qquad \mathcal{I}_{x-} = \varepsilon^4(\mathcal{I}_{x+}) \cong \mathbb{R}^3 \tag{6.2}$$
$$\mathcal{N}^\pm_x = \{q \in M \mid x = 0,\ \operatorname{sgn} z = \pm 1,\ -G_3(z, -y) < w < G_3(z, y)\} \cong \mathbb{R}^3, \tag{6.3}$$

where G_2 and G_3 are functions that are continuous in the set $\{(z, y) \in \mathbb{R}^2 \mid y > 0\}$ and satisfy the properties

$$G_2(0, y) = 0, \qquad G_2(-z, y) = G_2(z, y), \qquad G_2(\rho^2 z, \rho y) = \rho^3 G_2(z, y),$$
$$G_3(-z, y) = G_3(z, y), \qquad G_3(\rho^2 z, \rho y) = \rho^3 G_3(z, y), \qquad \rho > 0.$$

Proof. By virtue of the equalities $\varepsilon^1(\mathrm{Cut}) = \mathrm{Cut}$, $\varepsilon^1(M_{0+}) = M_{0-}$, we get, taking Theorem 7 into account, that $\mathrm{Cut} \cap M_{0-} = \varepsilon^1(\mathrm{Cut} \cap M_{0+}) = \bigsqcup_{i \in \{+,-\}} \left(\mathcal{I}^-_{xi} \sqcup \mathcal{CI}^-_{xi} \sqcup \mathcal{CN}^-_{xi}\right)$. Whence, taking Theorem 4 into account, we obtain stratification (6.1). Representation (6.2) is obtained from (5.5), Table 3, and the equality

$$\mathcal{I}^-_{x+} = \varepsilon^1(\mathcal{I}^+_{x+}) = \left\{q \in M \mid x = 0, z < 0, y > 0, w > W^{21}_{\mathrm{conj}}(y/\sqrt{|z|})\sqrt{|z|^3}\right\}$$

for the function

$$G_2(z, y) = \begin{cases} W^{21}_{\mathrm{conj}}(y/\sqrt{|z|})|z|^{3/2}, \ z \neq 0, \\ 0, \ z = 0. \end{cases}$$

Continuity of the function G_2 on the set $\{(z, y) \mid z \neq 0, y > 0\}$ follows from continuity of the function W^{21}_{conj} on the ray $(0, +\infty)$, see Lemma 12. In order to prove continuity of the function G_2 on the ray $\{(z, y) \mid z = 0, y > 0\}$, take any sequence (z_n, y_n), $z_n \to +0$, $y_n \to \bar{y} > 0$. Then, taking Lemma 12 into account, we get $G_2(z_n, y_n) = W^{21}_{\mathrm{conj}}(y_n/\sqrt{z_n})z_n^{3/2} \to 0 = G_2(0, \bar{y})$. Thus, the function G_2 is continuous for $y > 0$.

Representation (6.3) follows from Theorem 7 for the function $G_3(z, y) = W^{22}_{\mathrm{conj}}(y/\sqrt{|z|})|z|^{3/2}$.

Representations (6.2) and (6.3) imply that the strata $\mathcal{I}_{x\pm}, \mathcal{N}^{\pm}_x$ are homeomorphic to $\mathbb{R}^3$. The theorem is proved. □

7. GLOBAL STRATIFICATION OF THE CUT LOCUS

In this section we combine the results of Sections 2, 4, and 6, and provide a global description of the cut locus.

In Fig. 8 we show the contiguity topology of strata of the cut locus in the quotient by dilations X_0. On the left Fig. 9, we show the set $\mathrm{Cut} \cap M_z$ after factorization by dilations X_0; the quotient $M_z/e^{\mathbb{R}X_0}$ is represented by the topological sphere $\{q \in M \mid x^6 + y^6 + w^2 = 1\}$. Similarly, on the right Fig. 9, we show the quotient $(\mathrm{Cut} \cap M_x)/e^{\mathbb{R}X_0}$ on the topological sphere

$$\{q \in M \mid y^6 + |z|^3 + w^2 = 1\}.$$

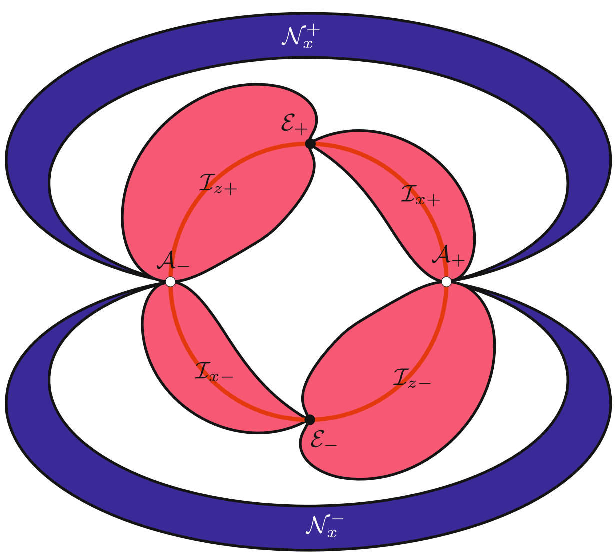

Fig. 8. Stratification of the cut locus: global structure.

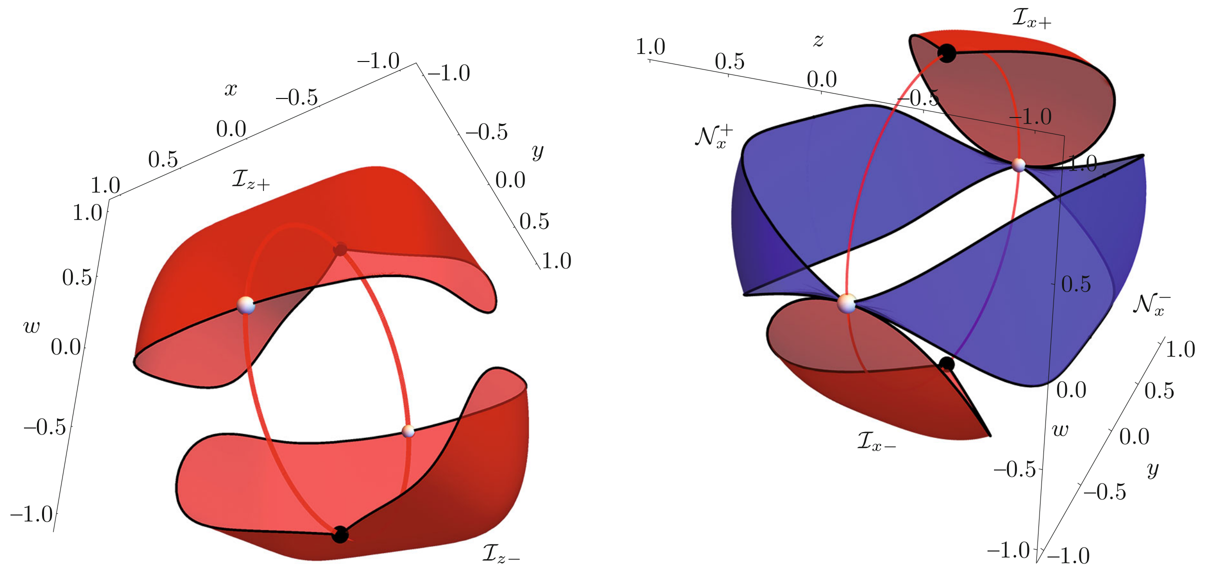

Fig. 9. Stratification of the cut locus: intersections with the subspaces M_z and M_x.

Theorem 9. *The cut locus stratifies as follows:*

$$\mathrm{Cut} = \bigsqcup_{i\in\{+,-\}} \left(\mathcal{I}_{zi} \sqcup \mathcal{I}_{xi} \sqcup \mathcal{N}_x^i \sqcup \left(\bigsqcup_{j\in\{+,-\}} \mathcal{CI}_{zi}^j \sqcup \mathcal{CI}_{xi}^j \sqcup \mathcal{CN}_{xi}^j \right) \sqcup \mathcal{E}_i \right). \tag{7.1}$$

Three-dimensional strata $\mathcal{I}_{zi}$, $\mathcal{I}_{xi}$, $\mathcal{N}_x^i$, $i \in \{+,-\}$, are Maxwell strata, at each point of the strata there are two minimizers. Two-dimensional strata $\mathcal{CI}_{zi}^j$, $\mathcal{CI}_{xi}^j$, $\mathcal{CN}_{xi}^j$, $i, j \in \{+,-\}$, consist of conjugate points that are limit points for Maxwell points; at each point of the strata there is a unique minimizer. One-dimensional strata $\mathcal{E}_i, i \in \{+,-\}$, consist of Maxwell points that are conjugate points; at each point of the strata there is a one-parameter family of minimizers.

The cut locus is not closed since it contains points arbitrarily close to the initial point q_0, but does not contain the point itself (this is a general fact of sub-Riemannian geometry). The closure of the cut locus in the sub-Riemannian problem on the Engel group admits the following simple description.

Theorem 10. $\mathrm{cl}(\mathrm{Cut}) = \mathrm{Cut} \sqcup \mathcal{A}_+ \sqcup \mathcal{A}_- \sqcup \{q_0\}$.

Denote by Conj the caustic, i. e., the set of conjugate points along all geodesics starting from the point q_0 [4]; and by Max the Maxwell set [5]. From Theorem 9 we get the following description of the sets $\mathrm{Cut} \cap \mathrm{Conj}$ and $\mathrm{Cut} \cap \mathrm{Max}$.

Theorem 11. *There are stratifications*

$$\mathrm{Cut} \cap \mathrm{Conj} = \bigsqcup_{i\in\{+,-\},\ j\in\{+,-\}} \left(\mathcal{CI}_{zi}^j \sqcup \mathcal{CI}_{xi}^j \sqcup \mathcal{CN}_{xi}^j \right) \sqcup \mathcal{E}_+ \sqcup \mathcal{E}_-,$$

$$\mathrm{Cut} \cap \mathrm{Max} = \bigsqcup_{i\in\{+,-\}} \left(\mathcal{I}_{zi} \sqcup \mathcal{I}_{xi} \sqcup \mathcal{N}_x^i \sqcup \mathcal{E}_i \right).$$

In other words, $\mathrm{Cut} \cap \mathrm{Conj}$ consists of all two-dimensional and one-dimensional strata, while $\mathrm{Cut} \cap \mathrm{Max}$ consists of all three-dimensional and one-dimensional strata of the cut locus.

8. CONCLUSION

This paper continues a series of publications [3–5] which were the first in the literature to study in detail a sub-Riemannian structure of step more than two. A complete description of the cut locus and the structure of optimal synthesis for the nilpotent sub-Riemannian problem on the Engel group is obtained.

Via nilpotent approximation, the results obtained are important for the study and applications of general sub-Riemannian structures of rank 2 in 4-dimensional space. Theoretically, our results open the way to investigation of basic local properties of sub-Riemannian distance for such structures near the initial point. From the applied point of view, these results lead to algorithms and software for solving a path-planning problem in mobile robotics.

Our research is based upon a detailed study of notions and properties introduced and developed by V. I. Arnold [11]: Maxwell strata and singularities of a Lagrange map generated by a variational problem (exponential map of the sub-Riemannian problem). It provides an example of thorough theoretical research related to important applications (Euler's elasticae and mobile robots).

APPENDIX A

In this section we present proofs of the diffeomorphic property for the restriction of the exponential map to Maxwell sets for the case $x = z = 0$ considered in Section 2.

Denote the functions

$$\iota_1(k) = 2E(k) - K(k), \qquad \iota_2(k) = K(k) - E(k), \qquad k \in (0,1). \tag{A.1}$$

Remark 2. The inequalities $\iota_1(k) > 0$ and $\iota_2(k) > 0$ hold, respectively, for $k \in (0, k_0)$ and $k \in (0,1)$.

Introduce an equivalence relation useful for our proofs.

Definition 2. *Let X be a topological space. Let $f_1, f_2 \colon X \to \mathbb{R}, \{\nu_n\} \subset X$.*

We write $f_1 \approx f_2$ if $\lim\limits_{n\to\infty} \dfrac{f_1(\nu_n)}{f_2(\nu_n)} \in \mathbb{R}\backslash\{0\}$.

Below in the proofs of the diffeomorphic property of maps we apply the following Hadamard global diffeomorphism theorem.

Theorem 12 ([10]). *Let $f \colon X \to Y$ be a smooth map between manifolds of equal dimension. Let the following conditions hold:*

(1) *X is connected,*
(2) *Y is connected and simply connected,*
(3) *f is nondegenerate,*
(4) *f is proper* $\big(f^{-1}(K) \subset X$ *is compact for a compact* $K \subset Y\big)$.

Then f is a diffeomorphism.

Definition 3. *A sequence $\{x_n\}$ in a topological space X tends to the boundary of X if there is no compact in X that contains this sequence.*

Notation: $x_n \to \partial X$.

It is easy to see that a continuous map $f : X \to Y$ between topological spaces is proper iff for any sequence $\{x_n\} \subset X$ the implication $x_n \to \partial X \Rightarrow f(x_n) \to \partial Y$ holds.

Proof (*of Lemma* 6). We apply Theorem 12. It follows from definitions of the sets MAX^{20}_{++}, $\mathcal{I}^0_{x+}$ that MAX^{20}_{++} is connected, $\mathcal{I}^0_{x+}$ is connected and simply connected, i. e., conditions (1) and (2) hold for the restriction of Exp under consideration. It was shown in [4] that in the case $u_1 = \pi$ the exponential map is nondegenerate for $\sin u_2 \neq 0$, thus condition (3) of Theorem 12 holds as well.

Let $\nu = (k, u_1, u_2, \sigma) \in \mathrm{MAX}^{20}_{++}$, then $\mathrm{Exp}(\nu) = \left(0, \dfrac{4\iota_1(k)}{\sigma},\ 0,\ \dfrac{8}{3\sigma^3}\Big(k^2\iota_1(k) + \iota_2(k)\Big)\right)$. It follows from Remark 2 that $\mathrm{Exp}(\nu) \in \mathcal{I}^0_{x+}$, thus $\mathrm{Exp}(\mathrm{MAX}^{20}_{++}) \subseteq \mathcal{I}^0_{x+}$.

In order to prove condition (4), consider any sequence $\big\{\nu_n = (k_n, \pi, \pi/2, \sigma_n)\big\}$, $n = 1, 2, 3, \ldots$, tending to the boundary of the set MAX^{20}_{++} as $n \to \infty$. Denote $\mathrm{Exp}(\nu_n) = (0, y_n, 0, w_n)$. We show that $\big\{\mathrm{Exp}(\nu_n)\big\}$ tends to the boundary of $\mathcal{I}^0_{x+}$. Let us study the possible cases as $n \to \infty$. After passing to a subsequence, only the following cases are possible:

1. $k_n \to 0$. Then $y_n \approx 1/\sigma_n$, whence $w_n \approx k_n^2 y_n^3$. Thus, $w_n \to 0$ or $y_n \to \infty$, may be, on a subsequence, in both cases $\mathrm{Exp}(\nu_n) \to \partial I^0_{x+}$. Below, for brevity, in similar arguments we omit such a phrase about subsequence.
2. $k_n \to k_0$. Then $y_n \to 0$ or $w_n \to \infty$.
3. $\sigma_n \to \infty, k_n \to \bar{k} \in (0, k_0)$. Then $y_n \to 0, w_n \to 0$.
4. $\sigma_n \to 0, k \to \bar{k} \in (0, k_0)$. Then $y_n \to \infty, w_n \to \infty$.

Thus, $\mathrm{Exp}\colon \mathrm{MAX}^{20}_{++} \to \mathcal{I}^0_{x+}$ is a proper map and hence a diffeomorphism by Theorem 12.

Proof (*of Lemma* 7). It follows from definition of the sets MAX^{10}_{++}, $\mathcal{I}^0_{z+}$ that the set MAX^{10}_{++} is connected, while $\mathcal{I}^0_{z+}$ is connected and simply connected, i. e., conditions (1) and (2) of Theorem 12 for the restriction of Exp under consideration hold.

Let $\nu = (k, u_1, u_2, \sigma) \in \mathrm{MAX}^{10}_{++}$, then $\mathrm{Exp}(\nu) = \left(0,\ \dfrac{2(2E_1 - F_1)}{\sigma},\ 0,\ \dfrac{4E_1c_1 - d_1^3 s_1}{3\sigma^3 c_1}\right)$, where we used the equality $F_1 = 2E_1 - \dfrac{s_1 d_1}{c_1}$ equivalent to $f_z(u_1, k) = 0$. Since $u_1 = u_{1z}(k) \in (\pi/2, \pi)$, then $s_1 > 0, c_1 < 0,\ E_1 > 0,\ 2E_1 - F_1 < 0$, whence $\mathrm{Exp}(\nu) \in \mathcal{I}^0_{z+}$, thus $\mathrm{Exp}(\mathrm{MAX}^{10}_{++}) \subseteq \mathcal{I}^0_{z+}$. It was shown in [4] that in the case $u_1 = u_{1z}(k)$ the exponential map is nondegenerate for $\cos u_2 \neq 0$, thus, condition (3) of Theorem 12 holds.

For the proof of condition (4) consider any sequence $\big\{\nu_n = (k_n, u_{1z}(k_n), 0, \sigma_n)\big\}$, $n = 1, 2, 3, \ldots$, tending to the boundary of the set MAX^{10}_{++} as $n \to \infty$. Denote $\mathrm{Exp}(\nu_n) = (0, y_n, 0, w_n)$. We show that $\big\{\mathrm{Exp}(\nu_n)\big\}$ tends to the boundary of $\mathcal{I}^0_{z+}$. Consider the possible cases as $n \to \infty$:

1. $k_n \to k_0$. Then $u_{1z}(k_n) \to \pi$. Whence $y_n \to 0$ or $\sigma_n \to 0$ and $w_n \to \infty$.
2. $k_n \to 1$. Then $u_{1z}(k_n) \to \pi/2$. Whence $y_n \to -\infty$ or $\sigma_n \to \infty$ and $w_n \to 0$.
3. $\sigma_n \to \infty, k_n \to \bar{k} \in (k_0, 1)$. Then $y_n \to 0, w_n \to 0$.
4. $\sigma_n \to 0, k_n \to \bar{k} \in (k_0, 1)$. Then $y_n \to -\infty, w \to +\infty$.

Thus, $\mathrm{Exp}\colon \mathrm{MAX}^{10}_{++} \to \mathcal{I}^0_{z+}$ is a proper map (condition (4) holds) and hence a diffeomorphism by Theorem 12.

APPENDIX B

In this section we prove some lemmas from Section 3.

Proof (*of Lemma* 8). We differentiate the equality $f_z\Big(F\big(u_{1z}(k)\big), k\Big) = 0$ by the variable k and get the following expression for the derivative:

$$u'_{1z}(k) = \frac{c_1\Big(1 - \frac{E_1 c_1}{s_1 d_1}\Big)}{k(1-k^2)s_1}. \tag{B.1}$$

Further, we use this expression to compute the derivative of each coordinate of the restricted exponential map:

$$\frac{\mathrm{d}\, Y_1^1\big(k, u_{1z}(k), \pi/2\big)}{\mathrm{d}\, k} = \frac{E_1 d_1}{2k^2(1-k^2)^{3/2}s_1} > 0, \tag{B.2}$$

$$\frac{\mathrm{d}\, W_1^1\big(k, u_{1z}(k), \pi/2\big)}{\mathrm{d}\, k} = -\frac{(E_1c_1 - s_1 d_1^3)(E_1 c_1^3 - s_1 d_1^3)}{16k^4(1-k^2)^{5/2}s_1^6 c_1} > 0, \tag{B.3}$$

since $u_1 = u_{1z}(k) \in (\pi/2, \pi)$ and $E_1 > 0, s_1 > 0, c_1 < 0, d_1 > 0$. So the map $\mathrm{Exp}\colon\ \mathrm{CMAX}_+^{1+} \to \mathcal{CI}_{z+}^+$ is nondegenerate, thus condition (3) of Theorem 12 holds. Conditions (1) and (2) are also obviously satisfied.

For the proof of condition (4) consider a sequence $k_n,\ n = 1, 2, 3, \ldots$, tending to the boundary of the set CMAX_+^{1+} and show that one of the functions $Y_1^1\big(k_n, u_{1z}(k_n), \pi/2\big), W_1^1\big(k_n, u_{1z}(k_n), \pi/2\big)$ tends to infinity:

1. $k_n \to k_0$, then $u_{1z}(k_n) \to \pi$. We have

$$Y_1^1\big(k_n, u_{1z}(k_n), \pi/2\big) \to \frac{1 - 2k_0^2}{2k_0\sqrt{1-k_0^2}}, \tag{B.4}$$

$$W_1^1\big(k_n, u_{1z}(k_n), \pi/2\big) \approx -\frac{E(k_0^2)}{24k_0^3(1-k_0^2)^{3/2}s_1^3} \to -\infty. \tag{B.5}$$

2. $k_n \to 1$, then $u_{1z}(k_n) \to \pi/2$.

$$Y_1^1\big(k_n, u_{1z}(k_n), \pi/2\big) = \frac{k_n^2 c_1^2 - 1 + k_n^2}{2k_n\sqrt{1-k_n^2}c_1} \approx \frac{k_n^2 c_1}{\sqrt{1-k_n^2}} - \frac{\sqrt{1-k_n^2}}{c_1}. \tag{B.6}$$

It follows from $f_z(F_1, k_n) = 0$ that $c_1^2 = \dfrac{s_1^2(1-k_n^2 s_1^2)}{(F_1 - 2E_1)^2}$, compute further

$$\frac{1 - k_n^2 s_1^2}{1-k_n^2} = 1 + \frac{k_n^2 c_1^2}{1-k_n^2} = 1 + \frac{k_n^2 s_1^2 (1 - k_n^2 s_1^2)}{(1-k_n^2)(F_1 - 2E_1)^2}.$$

Since $F_1 - 2E_1 \to \infty$, we get in the limit

$$\frac{1 - k_n^2 s_1^2}{1-k_n^2} = \frac{1}{1 - \frac{k_n^2 s_1^2}{(F_1 - 2E_1)^2}} \to 1, \tag{B.7}$$

consequently,

$$\frac{c_1^2}{1-k_n^2} = \frac{s_1^2(1-k_n^2 s_1^2)}{(1-k_n^2)(F_1-2E_1)^2} \approx \frac{1}{(F_1 - 2E_1)^2} \to 0. \tag{B.8}$$

From (B.6) we get

$$Y_1^1\big(k_n, u_{1z}(k_n), \pi/2\big) \approx \frac{\sqrt{1-k_n^2}}{c_1} \to \infty. \tag{B.9}$$

Consider the second coordinate:

$$W_1^1\big(k_n, u_{1z}(k_n), \pi/2\big) = -\frac{d_1^3 E_1}{48k_n^3(1-k_n^2)^{3/2}s_1^3} - \frac{\sqrt{1-k_n^2}(-1+4k_n^2)}{48k_n^3 c_1} + \\ + \frac{c_1\big(c_1^2 + (1-k_n^2)(1+k_n^2+4k_n^4)s_1^2\big)}{48k_n^3(1-k_n^2)^{3/2}s_1^2}. \tag{B.10}$$

Further consider (B.10) term by term, using (B.7) and (B.8):

$$-\frac{d_1^3 E_1}{48k_n^3(1-k_n^2)^{3/2}s_1^3} \to -\frac{1}{48}, \qquad -\frac{\sqrt{1-k_n^2}(-1+4k_n^2)}{48k_n^3 c_1} \to \infty,$$

$$\frac{c_1^3}{48k_n^3(1-k_n^2)^{3/2}s_1^2} \to 0, \qquad \frac{c_1(1+k_n^2+4k_n^4)}{48k_n^3\sqrt{1-k_n^2}} \to 0,$$

whence it follows that

$$W_1^1\big(k_n, u_{1z}(k_n), \pi/2\big) \approx \frac{\sqrt{1-k_n^2}}{c_1} \to \infty. \tag{B.11}$$

Thus, $\mathrm{Exp}\colon \mathrm{CMAX}_+^{1+} \to \mathcal{CI}_{z+}^+$ is a proper map and hence a diffeomorphism by Theorem 12.

Proof (*of Lemma* 9). 1) follows from equalities (B.2)–(B.5), (B.9), (B.11).

2) follows from the inequality $Y_1^1(k, u_1, \pi/2) - 6W_1^1(k, u_1, \pi/2) = \dfrac{d_1^3(c_1E_1 - d_1s_1)}{8k^3(1-k^2)^{3/2}s_1^3c_1} > 0$ under the condition $k \in (k_0, 1)$, $u_1 \in (\pi/2, \pi)$.

3) follows from (B.9), (B.11).

Now we prove a lemma we will use in the sequel for localization of two-dimensional parametrically defined sets.

Lemma 22. *Consider a parametrically defined set in the plane* $(X_1, X_2) \in \mathbb{R}^2$

$$\Omega = \Big\{\big(X_1, X_2\big) = \big(f_1(x_1, x_2), f_2(x_1, x_2)\big) \mid x_1 \in (x_1^0, x_1^1), x_2 \in (x_2^0, x_2^1)\Big\},$$

where $f_1, f_2 \in C^1\big((x_1^0, x_1^1) \times [x_2^0, x_2^1)\big)$. *Consider also a curve*

$$\gamma_0 = \Big\{\big(X_1, X_2\big) = \big(f_1(x_1, x_2^0), f_2(x_1, x_2^0)\big) \mid x_1 \in (x_1^0, x_1^1)\Big\}.$$

Let the curve γ_0 *divide the plane* (X_1, X_2) *into two connected components, and let the following conditions hold for* $x_1 \in (x_1^0, x_1^1)$:

$$\frac{\partial f_1}{\partial x_1}(x_1, x_2) > 0, \qquad x_2 \in \big[x_2^0, x_2^1\big), \tag{B.12}$$

$$\frac{\partial f_1}{\partial x_2}(x_1, x_2) > 0, \qquad x_2 \in \big(x_2^0, x_2^1\big), \tag{B.13}$$

$$\frac{\partial f_1}{\partial x_2}(x_1, x_2^0) \geqslant 0, \tag{B.14}$$

$$\nabla(x_1, x_2) = \left(\frac{\partial f_2/\partial x_1}{\partial f_1/\partial x_1} - \frac{\partial f_2/\partial x_2}{\partial f_1/\partial x_2}\right)(x_1, x_2) > 0, \qquad x_2 \in \big[x_2^0, x_2^1\big), \tag{B.15}$$

then condition (B.12) *allows us to invert the function* $X_1 = f_1(x_1, x_2^0)$ *in the interval* (x_1^0, x_1^1): $x_1 = \mathbf{h}_0(X_1)$, *which allows us to define the function* γ_0 *as a graph:* $X_2 = \mathbf{g}_0(X_1)$, *where* $\mathbf{g}_0(X_1) = f_2(\mathbf{h}_0(X_1), x_2^0)$.

Then the following condition holds:

$$f_2(x_1, x_2) < \mathbf{g}_0\big(f_1(x_1, x_2)\big), \quad x_1 \in \big(x_1^0, x_1^1\big), \quad x_2 \in \big(x_2^0, x_2^1\big), \tag{B.16}$$

i. e., the set Ω *lies below the curve* γ *in the plane* (X_1, X_2).

Proof. The set Ω is a union of the curves $\gamma_a = \Big\{\big(X_1, X_2\big) = \big(f_1(x_1, x_2^a), f_2(x_1, x_2^a)\big) \mid x_1 \in \big(x_1^0, x_1^1\big)\Big\}$, where $x_2^a = x_2^0 + (x_2^1 - x_2^0)a$ for $a \in (0, 1)$. Condition (B.12) allows us to define as a graph not only the curve γ_0, but each curve from the family $\{\gamma_a \mid a \in (0, 1)\}$ as well:

$$X_2 = \mathbf{g}_a(X_1), \qquad X_1 \in \big(f_1(x_1^0, x_2^a), f_1(x_1^1, x_2^a)\big),$$

where $\mathbf{g}_a(X_1) = f_2(\mathbf{h}_a(X_1), x_2^a)$, and $x_1 = \mathbf{h}_a(X_1)$ is the inverse function to $X_1 = f_1(x_1, x_2^a)$.

On the other hand, by fixing x_1 the set $\Omega \cup \gamma_0$ becomes a union of curves defined on a half-interval:

$$\beta_k = \Big\{\big(X_1, X_2\big) = \big(f_1(x_1^k, x_2), f_2(x_1^k, x_2)\big) \mid x_2 \in [x_2^0, x_2^1)\Big\},$$

where $x_1^k = x_1^0 + (x_1^1 - x_1^0)k$ for $k \in (0, 1)$. Moreover, conditions (B.13) and (B.14) allow us to define as a graph each curve of the family $\{\beta_k \mid k \in (0, 1)\}$:

$$X_2 = \mathbf{v}_k(X_1), \qquad X_1 \in \big[f_1(x_1^k, x_2^0), f_1(x_1^k, x_2^1)\big),$$

where $\mathbf{v}_k(X_1) = f_2(x_1^k, \mathbf{r}_k(X_1))$, and $x_2 = \mathbf{r}_k(X_1)$ is the inverse function to $X_1 = f_1(x_1^k, x_2)$.

Notice that since $f_1, f_2 \in C^1\big((x_1^0, x_1^1) \times [x_2^0, x_2^1)\big)$, by the inverse function theorem conditions (B.12)–(B.15) guarantee that $\mathbf{g}_a \in C^1\Big(\big(\bar{f}_1(x_1^0, x_2^a), \bar{f}_1(x_1^1, x_2^a)\big)\Big)$, $\mathbf{v}_k \in C^1\Big(\big[f_1(x_1^k, x_2^0), \bar{f}_1(x_1^k, x_2^1)\big)\Big)$, moreover, condition (B.12) implies existence of a limit $\lim\limits_{x_1 \to x_1^0} f_1(x_1, x_2^a) =: \bar{f}_1(x_1^0, x_2^a) \in [-\infty, +\infty]$, we define similarly $\bar{f}_1(x_1^1, x_2^a), \bar{f}_1(x_1^k, x_2^1)$.

Let $\check{X}_1 = f_1(x_1^k, x_2^a)$, then we get $\nabla(x_1^k, x_2^a) = \mathbf{g}_a'(\check{X}_1) - \mathbf{v}_k'(\check{X}_1) > 0$, whence it follows that

$$\forall x_1^k \in (x_1^0, x_1^1)\ x_2^a \in [x_2^0, x_2^1) \ \ \exists \epsilon > 0 \ \ \forall X_1 \in (\check{X}_1, \check{X}_1 + \epsilon) \quad \mathbf{g}_a(X_1) > \mathbf{v}_k(X_1).$$

If $a = 0$, then definitions of the functions $\mathbf{g}_a, \mathbf{v}_k$ imply that

$$\forall x_1^k \in (x_1^0, x_1^1) \ \ \exists \delta > 0 \ \ \forall x_2 \in (x_2^0, x_2^0 + \delta) \qquad \mathbf{g}_0\big(f_1(x_1^k, x_2)\big) > f_2(x_1^k, x_2).$$

Suppose that condition (B.16) is violated, i. e.,

$$\exists \hat{x}_1 \in (x_1^0, x_1^1) \ \ \exists \hat{x}_2 \in (x_2^0, x_2^1) \qquad \mathbf{g}_0\big(f_1(\hat{x}_1, \hat{x}_2)\big) = f_2(\hat{x}_1, \hat{x}_2). \tag{B.17}$$

Introduce the notation $\hat{X}_i = f_i(\hat{x}_1, \hat{x}_2), i = 1, 2$, and $\check{x}_1 = \mathbf{h}_0(\hat{X}_1)$. Notice that by definition of $\mathbf{g}_0$ we have $f_i(\check{x}_1, x_2^0) = \hat{X}_i, i = 1, 2$. It follows from (B.13), (B.14) that $f_1(\hat{x}_1, x_2^0) < f_1(\hat{x}_1, \hat{x}_2) = \hat{X}_1 = f_1(\check{x}_1, x_2^0)$, thus condition (B.12) implies that $\hat{x}_1 < \check{x}_1$.

Below we define a function $x_2 = \omega(x_1)$ on the segment $x_1 \in [\hat{x}_1, \check{x}_1]$, which satisfies the condition

$$f_1\big(x_1, \omega(x_1)\big) = \hat{X}_1. \tag{B.18}$$

In view of condition (B.13), if a function w is defined, then it is unique. At the endpoints of the segment $[\hat{x}_1, \check{x}_1]$ the function is defined: $\omega(\hat{x}_1) = \hat{x}_2$, $\omega(\check{x}_1) = x_1^0$, moreover, we have

$$f_2\big(\hat{x}_1, \omega(\hat{x}_1)\big) = f_2\big(\check{x}_1, \omega(\check{x}_1)\big) = \hat{X}_2. \tag{B.19}$$

Notice that inequality (B.12) implies that for $x_1 \in (\hat{x}_1, \check{x}_1)$ we have

$$f_1(x_1, x_2^0) < f_1(\check{x}_1, x_2^0) = \hat{X}_1 = f_1(\hat{x}_1, \hat{x}_2) < f_1(x_1, \hat{x}_2),$$

which, together with continuity of the function f_1, implies that the function ω is defined at the segment $x_1 \in [\hat{x}_1, \check{x}_1]$. If $x_1^k \in [\hat{x}_1, \check{x}_1]$, then $\omega(x_1^k) = \mathbf{r}_k(\hat{X}_1)$. In other words, $x_2 = \omega(x_1^k)$ is a uniquely defined function inverse to $\hat{X}_1 = f_1(x_1^k, x_2)$ at the segment $x_1 \in [\hat{x}_1, \check{x}_1]$. By the inverse function theorem, $\omega \in C^1([\hat{x}_1, \check{x}_1])$.

Denote the functions $\hat{f}_i(x_1) = f_i\big(x_1, \omega(x_1)\big)$, $i = 1, 2$, and compute their derivatives:

$$\frac{\mathrm{d}\,\hat{f}_i}{\mathrm{d}\,x_1}(x_1) = \frac{\partial f_i}{\partial x_1}\big(x_1, \omega(x_1)\big) + \frac{\partial f_i}{\partial x_2}\big(x_1, \omega(x_1)\big)\frac{\mathrm{d}\,w}{\mathrm{d}\,x_1}(x_1), \qquad i = 1, 2.$$

It follows from (B.18) that $\dfrac{\mathrm{d}\,\hat f_1}{\mathrm{d}\,x_1} = 0$, whence

$$\frac{\mathrm{d}\,\hat f_2}{\mathrm{d}\,x_1}(x_1) = \Big(\frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big(- \frac{\partial f_1/\partial x_1}{\partial f_1/\partial x_2}\Big)\Big)\big(x_1, \omega(x_1)\big) = \Big(\frac{\partial f_1}{\partial x_1}\nabla\Big)\big(x_1, \omega(x_1)\big) > 0,$$

where $x_1 \in [\hat x_1, \check x_1)$, thus the function $\hat f_2(x_1)$ increases at the segment $[\hat x_1, \check x_1)$, whence by continuity of the function we come to a contradiction with condition (B.19).Thus assumption (B.17) is violated, q.e.d. ☐

Lemma 23. *The inclusion* $\mathrm{Exp}(\mathrm{MAX}^{1+}_{+-}) \subseteq \mathcal{I}^+_{z+}$ *holds.*

Proof. Notice that for $k \in (k_0, 1), u_2 \in (\pi/2, \pi)$ we have

$$J^1_{11}(k, u_2) = \frac{\partial Y^1_1\big(k, u_{1z}(k), u_2\big)}{\partial u_2} = \frac{\Delta c_2}{2kc_1s_2^2d_2^3} > 0, \tag{B.20}$$

$$J^1_{21}(k, u_2) = \frac{\partial Y^1_1\big(k, u_{1z}(k), u_2\big)}{\partial k} = \frac{\Delta(E_1c_1d_2^2 - k^2s_1d_1c_2^2)}{2k^2(1-k^2)s_1c_1d_1s_2d_2^3} > 0, \tag{B.21}$$

$$\begin{aligned}\nabla^1(k, u_2) &= \frac{\partial W^1_1\big(k, u_{1z}(k), u_2\big)/(\partial k)}{\partial Y^1_1\big(k, u_{1z}(k), u_2\big)/(\partial k)} - \frac{\partial W^1_1\big(k, u_{1z}(k), u_2\big)/(\partial u_2)}{\partial Y^1_1\big(k, u_{1z}(k), u_2\big)/(\partial u_2)}\\ &= \frac{\Delta^2\Big(- 2E_1s_1c_1d_1^5 + s_1^2d_1^6 + E_1^2c_1^2\big((1-ks_1^2)^2 + 2k(1-k)s_1^2\big)\Big)}{8k^2s_1^5d_1^5s_2^2(k^2s_1d_1c_2^2 - E_1c_1d_2^2)} > 0.\end{aligned} \tag{B.22}$$

Moreover, for all $k \in (k_0, 1)$ there exists $\lim\limits_{u_2 \to \pi/2} \nabla^1(k, u_2) \in (0, +\infty)$, and the inequality $J^1_{21}(k, \pi/2) > 0$ holds. Hence, given Lemma 9, it follows that the map $\mathrm{Exp}|_{\mathrm{MAX}^{1+}_{+-}}$ satisfies the conditions of Lemma 22, consequently, $\mathrm{Exp}(\mathrm{MAX}^{1+}_{+-}) \subseteq \mathcal{I}^+_{z+}$.

Proof (*of Lemma* 10). Conditions (1) and (2) of Theorem 12 are obviously satisfied, and the results of [4] imply that condition (3) holds as well. It remains to check the validity of condition (4).

Consider arbitrary sequences $(k_n, u_{1z}(k_n), u_2^n)$, $n = 1, 2, 3, \dots$, in the image of Exp tending to the boundary of the set MAX^{1+}_{+-} and show that either $\mathrm{Exp}\big(k_n, u_{1z}(k_n), u_2^n\big) \to \mathcal{CI}^+_{z+}$ or one of the coordinates in the image tends to infinity:

1. $u_2^n \to \pi/2$, by definition of $\mathcal{CI}^+_{z+}$ we have $\mathrm{Exp}(k_n, u_{1z}(k_n), u_2^n) \to \mathcal{CI}^+_{z+}$.
2. $u_2^n \to \pi$, then $Y^1_1\big(k_n, u_{1z}(k_n), u_2^n\big) \approx -\dfrac{1}{2k_nc_1s_2} \to \infty$.
3. $k_n \to k_0, u_2^n \to \hat u_2 \in (\pi/2, \pi)$, then $u_{1z}(k_n) \to \pi$, thus $s_1 \to 0, c_1 \to -1, d_1 \to 1, \Delta \to 1$. We get $W^1_1\big(k_n, u_{1z}(k_n), u_2^n\big) \approx -\dfrac{E(k_0)}{24k_0^3s_1^3s_2^3d_2^3} \to -\infty$.
4. $k_n \to 1, u_2^n \to \hat u_2 \in (\pi/2, \pi)$, then $u_{1z}(k_n) \to \pi/2$. By virtue of $c_1 \to 0, c_2 \neq 0, s_2 \neq 0$, we get $Y^1_1\big(k_n, u_{1z}(k_n), u_2^n\big) \approx -\dfrac{c_2}{2c_1s_2} \to \infty$.

Consequently, given Lemma 23 $\mathrm{Exp}\colon \mathrm{MAX}^{1+}_{+-} \to \mathcal{I}^+_{z+}$ is proper. Hence by Theorem 12 the restriction of the exponential map is a diffeomorphism.

APPENDIX C

In this Appendix we prove some technical lemmas from Section 5.

Proof (*of Lemma* 11). We apply Theorem 12. Notice first that conditions (1) and (2) hold. Further compute the derivative of each coordinate by k:

$$\frac{\mathrm{d}\, Y_1^2(k,0)}{\mathrm{d}\, k} = -\frac{E(k)}{(1-k^2)\sqrt{2k^3\iota_1(k)}} < 0, \tag{C.1}$$

$$\frac{\mathrm{d}\, W_1^2(k,0)}{\mathrm{d}\, k} = \frac{\iota_3(k)}{\big(2k\iota_1(k)\big)^{5/2}}, \tag{C.2}$$

where $\iota_3(k) = \dfrac{3E^2(k) - (5-4k^2)E(k)K(k) + 2(1-k^2)K^2(k)}{1-k^2} > 0$ for $k \in (0, k_0)$, since $\iota_3(0) = 0$, and $\iota_3'(k) = \dfrac{\iota_2^2(k) + k^2\iota_1(k)\big(2E(k)+K(k)\big)}{k(1-k^2)^2} > 0$. Thus, for $k \in (0, k_0)$ we have

$$\frac{\mathrm{d}\, W_1^2(k,0)}{\mathrm{d}\, k} > 0. \tag{C.3}$$

Thus, the map $\mathrm{Exp} : \mathrm{CMAX}_{1+}^{2+} \to \mathcal{CI}_{x+}^{+}$ is nondegenerate, and condition (3) of Theorem 12 holds.

In order to prove condition (4), consider an arbitrary sequence $k_n, n = 1, 2, 3, \dots$, tending to the boundary of CMAX_{1+}^{2+} and show that one of the coordinates tends to infinity:

1. $k_n \to 0$, then
$$Y_1^2(k_n, 0) \to \infty, \qquad W_1^2(k_n, 0) \to 0. \tag{C.4}$$

2. $k_n \to k_0$, then
$$Y_1^2(k_n, 0) \to 0, \qquad W_1^2(k_n, 0) \to \infty. \tag{C.5}$$

Thus, $\mathrm{Exp} : \mathrm{CMAX}_{1+}^{2+} \to \mathcal{CI}_{x+}^{+}$ is a proper map and hence a diffeomorphism by Theorem 12.

Proof (*of Lemma* 12). Follows from Lemma 11 and expressions (C.1)–(C.5).

Lemma 24. *The inclusion* $\mathrm{Exp}(\mathrm{MAX}_{1++}^{2+}) \subset \mathcal{I}_{x+}^{+}$ *holds.*

Proof. Notice the inequalities

$$J_{11}^{21} = \frac{\partial Y_1^2(k, u_2)}{\partial u_2} = \sqrt{\frac{\iota_1(k)}{2kc_2^3}}\, s_2 > 0, \tag{C.6}$$

$$J_{12}^{21} = \frac{\partial W_1^2(k, u_2)}{\partial u_2} = \frac{\big(\iota_2(k) + k^2 s_2^2 \iota_1(k)\big)s_2}{2c_2\big(2k\iota_1(k)c_2\big)^{3/2}} > 0. \tag{C.7}$$

Thus, if u_2 grows, then both coordinates Y^2 and W^2 grow as well. Thus, for a fixed $k = \hat{k}$ the curve $\big(Y^2(\hat{k}, u_2), W^2(\hat{k}, u_2)\big), u_2 \in (0, \pi/2)$, lies above the curve $W^2 = W_{\mathrm{conj}}^{21}(Y^2)$, $Y_2 > Y^2(\hat{k}, 0)$, see Lemma 12. This proves this lemma by definition (5.2).

Proof (*of Lemma* 13). Apply Theorem 12. Conditions (1) and (2) follow from definitions of the sets MAX_{1++}^{2+}, $\mathcal{I}_{x+}^{+}$. Condition (3) follows from results of [4]. Given Lemma 24, in order to prove condition (4), it suffices to consider arbitrary sequences (k_n, u_2^n) in the image of Exp tending to the boundary of the set MAX_{1++}^{2+}, and to show that the sequence $\big(Y_1^2(k_n, u_2^n), W_1^2(k_n, u_2^n)\big)$ tends to the boundary of the set $\mathcal{I}_{x+}^{+}$:

1. $u_2^n \to 0$, by definition of $\mathcal{CI}_{x+}^+$ we have $\big(Y_1^2(k_n, u_2^n), W_1^2(k_n, u_2^n)\big) \to \mathcal{CI}_{x+}^+$.

2. $u_2^n \to \pi/2, k_n \to \hat{k} \in [0, k_0)$, or $k_n \to 0$, then $Y_1^2(k_n, u_2^n) \approx \dfrac{1}{\sqrt{k_n c_2}} \to \infty$.

3. $u_2^n \to \pi/2, k_n \to k_0$, then $\iota_1(k_n) \to 0$, thus $W_1^2(k_n, u_2^n) \approx \dfrac{E(k_0)}{\big(\iota_1(k)c_2\big)^{3/2}} \to \infty$.

4. $k_n \to k_0, u_2^n \to \hat{u}_2 \in (0, \pi/2)$, then $\iota_1(k_n) \to 0$, thus $Y_1^2(k_n, u_2^n) \approx \sqrt{\iota_1(k_n)} \to 0$.

Thus, Exp: $\mathrm{MAX}_{1++}^{2+} \to \mathcal{I}_{x+}^+$ is a proper map and hence a diffeomorphism by Theorem 12.

Introduce the functions

$$\begin{aligned}
\iota_4(k) &= (2-k^2)K(k) - 2E(k), \\
\iota_5(k) &= (2-k^2)E(k)K(k) + (1-k^2)K^2(k) - 3E^2(k), \\
\iota_6(k) &= E(k) + (-1+k^2)K(k) > 0.
\end{aligned} \tag{C.8}$$

Lemma 25. *Let $k \in (0,1)$. Then inequalities $\iota_i(k) > 0,\ i = 4, 5, 6$ hold.*

Proof. Notice that $\iota_i(0) = 0, i = 4, 5, 6$. Further, $\left(\dfrac{\iota_4(k)}{\sqrt{1-k^2}}\right)' = \dfrac{k\iota_2(k)}{(1-k^2)^{3/2}} > 0$ for $k \in (0,1)$; since $\big((2-k^2)E(k) - 2(1-k^2)K(k)\big)' = 3k\iota_2(k) > 0$, we have

$$\left(\frac{\iota_5(k)}{\sqrt{1-k^2}}\right)' = \frac{\iota_4(k)\big((2-k^2)E(k) - 2(1-k^2)K(k)\big)}{k(1-k^2)^{3/2}} > 0;$$

finally, $\iota_6'(k) = kK(k) > 0$. Hence, $\iota_i(k) > 0, i = 4, 5, 6$, for $k \in (0,1)$.

Proof (*of Lemma* 14). Apply Theorem 12. Notice that conditions (1) and (2) hold. Compute further the derivative of each coordinate in k:

$$\frac{\partial Y_2^2(k,0)}{\partial k} = -\frac{k\iota_2(k)}{2(1-k^2)^{5/4}\sqrt{\iota_4(k)}} < 0, \tag{C.9}$$

$$\frac{\partial W_2^2(k,0)}{\partial k} = \frac{k^3\iota_5(k)}{4(1-k^2)^{7/4}\iota_4^{5/2}(k)} > 0. \tag{C.10}$$

By virtue of sign-definiteness of the derivatives, the map is nondegenerate, thus condition (3) of Theorem 12 holds.

In order to prove condition (4), consider arbitrary sequences $k_n, n = 1, 2, 3, \ldots$, tending to the boundary of the set CMAX_{2++}^{2+}:

1. $k_n \to 0$, then
$$Y_2^2(k_n, 0) \to 0, \qquad W_2^2(k_n, 0) \to 1/\sqrt{\pi}. \tag{C.11}$$

2. $k_n \to 1$, then
$$Y_2^2(k_n, 0) \to -\infty, \qquad W_2^2(k_n, 0) \to \infty. \tag{C.12}$$

Thus, Exp : $\mathrm{CMAX}_{2++}^{2+} \to \mathcal{CN}_{x++}^+$ is a proper map and hence a diffeomorphism by Theorem 12.

Proof (*of Lemma* 15). Follows from Lemma 14 and expressions (C.9)–(C.12).

Proof (*of Lemma* 16). Apply Theorem 12. Conditions (1) and (2) hold. Compute the derivative of each coordinate in k:

$$\frac{\partial Y_2^2(k,\pi/2)}{\partial k} = -\frac{k\iota_6(k)}{2(1-k^2)\sqrt{\iota_4(k)}} < 0, \tag{C.13}$$

$$\frac{\partial W_2^2(k,\pi/2)}{\partial k} = \frac{k^3\iota_5(k)}{4(1-k^2)\iota_4^{5/2}(k)} > 0, \tag{C.14}$$

thus, the map $\mathrm{Exp}|_{\mathrm{CMAX}_{2+-}^{2+}}$ is nondegenerate.

Consider arbitrary sequences $k_n, n = 1, 2, 3, \ldots$, tending to the boundary of the set CMAX_{2+-}^{2+}:

1. $k \to 0$, then

$$Y_2^2(k_n,\pi/2) \to 0, \qquad W_2^2(k_n,\pi/2) \to -1/\sqrt{\pi}. \tag{C.15}$$

2. $k \to 1$, then

$$Y_2^2(k_n,\pi/2) \to -\infty, \qquad W_2^2(k_n,\pi/2) \to 0. \tag{C.16}$$

Thus, $\mathrm{Exp} : \mathrm{CMAX}_{2+-}^{2+} \to \mathcal{CN}_{x+-}^{+}$ is proper and hence a diffeomorphism by Theorem 12.

Proof (*of Lemma* 17). Follows from Lemma 16 and expressions (C.13)–(C.16).

Proof (*of Lemma* 18). To prove the lemma, consider the following fixed points of the symmetry ε^2 for $\lambda \in C_3$: $\mathrm{FIX}_{3+}^{2+} = \Big\{(\sigma, p, \tau) \in C_{3+}^{+} \times \mathbb{R}_+ \mid \tau = 0\Big\} \subset \mathrm{FIX}^2$. The exponential map transforms the set FIX_3^{2+} into the set

$$\mathrm{Fix}_{3+}^{2+} = \Big\{(Y^2, W^2) = \big(Y_3^2(p), W_3^2(p)\big) \mid p \in (0,\infty)\Big\},$$

$$\big(Y_3^2(p), W_3^2(p)\big) = \left(\frac{2\sinh p - p\cosh p}{\sqrt{p\cosh p - \sinh p}}, \frac{9\sinh p - 12p\cosh p + \sinh(3p)}{24(p\cosh p - \sinh p)^{\frac{3}{2}}}\right).$$

Since $\big(2\sinh p - p\cosh p\big)' = \cosh p - p\sinh p$, we have

$$\frac{\mathrm{d}\, Y_3^2(p)}{\mathrm{d}\, p} < 0, \tag{C.17}$$

moreover,

$$\lim_{p\to 0} Y_3^2(p) = +\infty, \qquad \lim_{p\to\infty} Y_3^2(p) = -\infty, \tag{C.18}$$

thus, the curve Fix_{3+}^{2+} is a graph of a smooth function $W_{\mathrm{fix}}^3(Y^2)$, $Y^2 \in (-\infty, +\infty)$.

Let $p = p_3 > 0$ be the positive root of the equation $p = 2\tanh p$, then

$$Y_3^2(p_3) = 0, \ W_3^2(p_3) = \frac{\sinh^2 p_3 - 3}{6\sqrt{\sinh p_3}}.$$

Let $\sinh p_0 = 3$, then $p_0 = \ln(3 + \sqrt{10})$. It is known that the inequalities $\ln 2 < 0.7$, $\ln 3 < 1.1$ hold. Now we estimate p_0:

$$\ln(6 + \sqrt{10} - 3) < \ln 6 + \frac{\sqrt{10} - 3}{6} < 0.7 + 1.1 + \frac{\sqrt{10}}{6} - \frac{1}{2} < \frac{13\sqrt{10}}{30} + \frac{\sqrt{10}}{6} = 2\tanh p_0,$$

whence $p_0 < 2\tanh p_0$, then $p_3 > p_0$, thus the inequality $\sinh p_3 > 3$ holds. It follows that $W_3^2(p_3) > W_3^2(p_0) = \dfrac{1}{\sqrt{3}} > \dfrac{1}{\sqrt{\pi}}$, i.e., $W_{\mathrm{fix}}^3(0) > \dfrac{1}{\sqrt{\pi}}$. Whence it follows that

$$\exists \epsilon > 0 \ \ \forall 0 < Y^2 < \epsilon \qquad -W_{\mathrm{conj}}^{22-}(-Y^2) < W_{\mathrm{fix}}^3(Y^2) < W_{\mathrm{conj}}^{21}(Y^2), \tag{C.19}$$

i.e., the statement of the lemma holds in a neighborhood of zero.

Now assume that there exists a point where this condition is false, i. e., there exists $Y_\epsilon^2 > 0$ such that $-W_{\text{conj}}^{22-}(-Y_\epsilon^2) = W_{\text{conj}}^{21}(Y_\epsilon^2)$. Then by continuity there is a point $(Y^2, W^2) \in \text{Fix}_{3+}^{2+} \cap (\mathcal{CI}_{x+}^{+} \cup \mathcal{CN}_{x--}^{+})$, which contradicts to Lemma 1. Consequently, our assumption is false. □

Lemma 26. *The inclusion* $\text{Exp}(\text{MAX}_{2--}^{2+}) \subset \mathcal{N}_{x-}^{+}$ *holds.*

Proof. $\text{Exp}(\text{MAX}_{2--}^{2+})$ lies in the right half-plane of the plane (Y^2, W^2).

Notice that for $k \in (0,1), u_2 \in (\pi/2, \pi)$ the following conditions hold:

$$J_{11}^{22}(k, u_2) = -\frac{\partial Y_2^2(k, u_2)}{\partial u_2} = -\frac{\sqrt{\iota_4(k)} k^2 c_2 s_2}{2\sqrt{d_2^3 \sqrt{1-k^2}}} > 0, \tag{C.20}$$

$$J_{21}^{22}(k, u_2) = -\frac{\partial Y_2^2(k, u_2)}{\partial k} = \frac{k\Big((1-k^2)\iota_6(k) + c_2^2\big(\iota_4(k) + k^2\iota_6(k)\big)\Big)}{2\sqrt{d_2^3 \iota_4(k)\sqrt{(1-k^2)^5}}} > 0, \tag{C.21}$$

$$\begin{aligned}\nabla^{22}(k, u_2) &= \frac{\partial W_2^2\big(k, u_2\big)/\partial k}{\partial Y_2^2\big(k, u_2\big)/\partial k} - \frac{\partial W_2^2\big(k, u_2\big)/\partial u_2}{\partial Y_2^2\big(k, u_2\big)/\partial u_2} \\ &= \frac{d_2^3 k^2 \big(E^2(k) - (1-k^2)K^2(k)\big)}{\iota_4^2(k)\sqrt{1-k^2}\Big((1-k^2)\iota_6(k) + c_2^2\big(\iota_4(k) + k^2\iota_6(k)\big)\Big)} > 0,\end{aligned} \tag{C.22}$$

moreover, $J_{11}^{22}(k, \pi/2) = 0$, $J_{21}^{22}(k, \pi/2) > 0$, and $\nabla^{22}(k, \pi/2) > 0$. Hence, it follows from Lemma 22 and the definition of $\mathcal{N}_{z-}^{+}$ that the set $\text{Exp}(\text{MAX}_{--}^{1+})$ lies below the curve $-W_{\text{conj}}^{22-}(-Y^2), Y^2 \in (0, \infty)$, which defines the upper boundary of the set $\mathcal{N}_{x-}^{+}$.

Now we should show that the set $\text{Exp}(\text{MAX}_{2--}^{2+})$ lies above the curve $-W_{\text{conj}}^{22+}(-Y^2), Y^2 \in (0, \infty)$. To this end we consider in the preimage of the exponential map a symmetric set MAX_{2++}^{2+} with a symmetric curve $W_{\text{conj}}^{22+}(Y^2), Y^2 \in (-\infty, 0)$, with the inverted parameter $\tilde{k} = 1 - k$, and $u_2 \in (0, \pi/2)$. In this case the hypotheses of Lemma 22 hold as well, thus the set MAX_{2++}^{2+} lies below the curve $W_{\text{conj}}^{22+}(Y^2), Y^2 \in (-\infty, 0)$. So the set $\text{Exp}(\text{MAX}_{2--}^{2+})$ lies above the curve $-W_{\text{conj}}^{22+}(-Y^2), Y^2 \in (0, \infty)$. Consequently, $\text{Exp}(\text{MAX}_{2--}^{2+}) \subset \mathcal{N}_{x-}^{+}$.

Proof (*of Lemma* 19). Apply Theorem 12. Conditions (1) and (2) follow from definitions of the sets in the image and the preimage of the exponential map. Condition (3) follows from results of [4]. By virtue of Lemma 26, to prove condition (4) we consider arbitrary sequences (k_n, u_2^n) in the image of Exp tending to the boundary of the set MAX_{2--}^{2+}, and show that such sequences, under the action of the exponential map, tend to the boundary of $\mathcal{N}_{z-}^{+}$:

1. $k \to 0$, then $Y^2(k_n, u_2^n) \to 0$.
2. $k \to 1$, then $Y^2(k_n, u_2^n) \to \infty$.
3. $u_2^n \to \pi/2$, then by definition we have $\big(Y^2(k_n, u_2^n), W^2(k_n, u_2^n)\big) \to \text{CMAX}_{2--}^{2+}$.
4. $u_2^n \to \pi$, then by definition we have $\big(Y^2(k_n, u_2^n), W^2(k_n, u_2^n)\big) \to \text{CMAX}_{2--}^{2+}$.

Hence, by Theorem 12 the map $\text{Exp}(\text{MAX}_{2--}^{2+}) \to \mathcal{N}_{x-}^{+}$ is a diffeomorphism.

Proof (*of Lemma* 21). Given the symmetry ε^4, it suffices to consider the case $\mathcal{CN}^+_{x+}$.

Lemma 15 and Corollary 4 imply that $\mathcal{CN}^+_{x+}$ form a decreasing continuous curve. Since the curves $\mathcal{CN}^+_{x++}$ and $\mathcal{CN}^+_{x--}$ are continuous, in order to prove smoothness of the union of these curves at the point $\mathcal{CC}^+_{x+}$, it suffices to show that the limits of the corresponding derivatives at this point coincide one with another. To this end we evaluate the corresponding Taylor polynomials at the point $k = 0$:

$$\big(Y_2^2(k,0), W_2^2(k,0)\big) = \left(-\frac{\sqrt{\pi}k^2}{4} + o(k^3), \frac{1}{\sqrt{\pi}} + \frac{3k^2}{16\sqrt{\pi}} + o(k^3)\right),$$

$$\big(-Y_2^2(k,\pi/2), -W_2^2(k,\pi/2)\big) = \left(\frac{\sqrt{\pi}k^2}{4} + o(k^3), \frac{1}{\sqrt{\pi}} - \frac{3k^2}{16\sqrt{\pi}} + o(k^3)\right).$$

Now it follows that the curves $\mathcal{CN}^+_{x++}$ and $\mathcal{CN}^+_{x--}$ join one another smoothly at the point $\mathcal{CC}^+_{x+}$.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation under grant 17-11-01387 and performed at the Ailamazyan Program Systems Institute of the Russian Academy of Sciences.

REFERENCES

1. Agrachev, A. A. and Sachkov, Yu. L., *Control Theory from the Geometric Viewpoint,* Encyclopaedia Math. Sci., vol. 87, Berlin: Springer, 2004.
2. Agrachev, A. A., Barilari, D., and Boscain, U., Introduction to Riemannian and Sub-Riemannian Geometry: *Preprint SISSA 09/2012/M,* `https://webusers.imj-prg.fr/~davide.barilari/2017-11-17-ABB.pdf` (Version Nov 17, 2017).
3. Ardentov, A. A. and Sachkov, Yu. L., Extremal Trajectories in the Nilpotent Sub-Riemannian Problem on the Engel Group, *Sb. Math.,* 2011, vol. 202, nos. 11–12, pp. 1593–1615; see also: *Mat. Sb.,* 2011, vol. 202, no. 11, pp. 31–54.
4. Ardentov, A. A. and Sachkov, Yu. L., Conjugate Points in Nilpotent Sub-Riemannian Problem on the Engel Group, *J. Math. Sci. (N. Y.),* 2013, vol. 195, no. 3, pp. 369–390.
5. Ardentov, A. A. and Sachkov, Yu. L., Cut Time in Sub-Riemannian Problem on Engel Group, *ESAIM Control Optim. Calc. Var.,* 2015, vol. 21, no. 4, pp. 958–988.
6. Euler, L., *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimitrici latissimo sensu accepti,* Lausanne: Bousquet, 1744.
7. Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity,* 4th ed., New York: Dover, 1944.
8. Sachkov, Yu. L., Complete Description of Maxwell Strata in the Generalized Dido Problem, *Sb. Math.,* 2006, vol. 197, nos. 5–6, pp. 901–950; see also: *Mat. Sb.,* 2006, vol. 197, no. 6, pp. 111–160.
9. Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions: With an Account of the Principal Transcendental Functions,* 4th ed., New York: Cambridge Univ. Press, 1962.
10. Krantz, S. G. and Parks, H. R., *The Implicit Function Theorem: History, Theory, and Applications,* Basel: Birkhäuser, 2003.
11. Arnold, V. I., Gusein-Zade, S. M., and Varchenko, A. N., *Singularities of Differentiable Maps: Vol. 1. Classification of Critical Points, Caustics and Wave Fronts,* Modern Birkhäuser Classics, New York: Birkhäuser/Springer, 2012.
    Arnold, V. I., Gusein-Zade, S. M., and Varchenko, A. N., *Singularities of Differentiable Maps: Vol. 2. Monodromy and Asymptotics of Integrals,* Modern Birkhäuser Classics, New York: Birkhäuser/Springer, 2012.