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Connecting Orbits near the Adiabatic Limit of Lagrangian Systems with Turning Points

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Abstract—We consider a natural Lagrangian system defined on a complete Riemannian manifold being subjected to action of a time-periodic force field with potential $U(q,t,\varepsilon) = f(\varepsilon t)V(q)$ depending slowly on time. It is assumed that the factor $f(\tau)$ is periodic and vanishes at least at one point on the period. Let X_c denote a set of isolated critical points of $V(x)$ at which $V(x)$ distinguishes its maximum or minimum. In the adiabatic limit $\varepsilon \to 0$ we prove the existence of a set $\mathcal{E}_h$ such that the system possesses a rich class of doubly asymptotic trajectories connecting points of X_c for $\varepsilon \in \mathcal{E}_h$.

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1. INTRODUCTION

We study a Lagrangian system on a complete Riemannian manifold $\mathcal{M}$ with Lagrangian

$$\mathcal{L}(q,\dot{q},t,\varepsilon) = \frac{1}{2}|\dot{q}|^2 - U(q,t,\varepsilon), \tag{1.1}$$

where the potential $U(q,t,\varepsilon) \in C^2(\mathcal{M}\times\mathbb{R},\mathbb{R})$ has the representation

$$U(q,t,\varepsilon) = f(\varepsilon t)V(q) \tag{1.2}$$

with some periodic function f. The parameter ε is assumed to be small and the Lagrangian depends slowly on time. The limit $\varepsilon \to 0$ is called the adiabatic limit [1].

Introducing the slow time $\tau = \varepsilon t$, we obtain a singular perturbed system:

$$\mathcal{L}(q,\varepsilon q',\tau) = \frac{\varepsilon^2}{2}|q'|^2 - f(\tau)V(q),\ q' = \frac{\mathrm{d}q}{\mathrm{d}\tau} \tag{1.3}$$

and the limit $\varepsilon \to 0$ can also be regarded as the anti-integrable limit [5, 6].

We note here that all results stated below are also valid for a system with Lagrangian $\mathcal{L}(q,\dot{q},t,\varepsilon) = K(q,\dot{q}) - U(q,t,\varepsilon)$, where the kinetic energy K is a positive definite quadratic form in $\dot{q}$ and $U(q,t,\varepsilon)$ satisfies (1.2). However, to simplify exposition, we prefer to consider the case $K = \dfrac{1}{2}|\dot{q}|^2$. We will also assume that the manifold $\mathcal{M}$ is noncontractible to get a multiplicity result.

The study of transversal homoclinic and heteroclinic intersections for systems of type (1.1) or (1.3) is the subject of many papers. In [8, 21] under assumptions that the system $q'' = D_qV$ has a pair of hyperbolic equilibria x_0, x_1 connected by a heteroclinic (homoclinic if $x_0 = x_1$) orbit, while the factor f is positive and has a nondegenerate critical point, the authors proved that for small $\varepsilon \neq 0$ the system (1.3) has a transverse heteroclinic (homoclinic) orbit connecting x_0 and x_1. The

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main tool used in [8, 21] was the theory of exponential dichotomies. Due to the Birkhoff–Smale theory this result also implies the existence of an infinite number of multibump trajectories close to chains of heteroclinic orbits. Using the variational approach allowed constructing similar multibump trajectories without the transversality assumption [5]. In [15, 24] the authors applied variational arguments to prove the existence of infinitely many homoclinic orbits for systems of type (1.1) without the smallness assumption on the parameter ε. Another approach to establish homoclinic (heteroclinic) intersections is due to singular perturbation theory [13]. Systems of type (1.1) give the simplest example of the slow-fast systems. Indeed, the Hamilton equations associated to (1.1) and considered in the extended phase space $T\mathcal{M} \times \mathbb{R}$ can be written as

$$\dot{q} = p, \quad \dot{p} = -f(\tau) D_q V(q, \tau), \quad \dot{\tau} = \varepsilon.$$

Thus, (q, p) can be considered as fast variables and τ as a slow one. Setting $\varepsilon = 0$, one gets the so-called "frozen" system which is a one-parameter family with respect to the parameter τ. Under the assumption that the frozen system possesses for all values of τ a hyperbolic equilibrium which depends smoothly on τ one may prove (see, e. g., [29]) the existence of a periodic hyperbolic orbit of the system (1.1) for sufficiently small $\varepsilon \neq 0$. Besides, if some Melnikov-type condition holds, the separatrices of such a hyperbolic periodic orbit intersect transversally [13, 30]. It is to be noted that the nonvanishing condition on the factor f is crucial for all mentioned results. The aim of this paper is to prove the existence of a rich set of connecting orbits for the systems (1.3) in the case when the factor f has at least one zero on its period. To formulate the main result, we introduce some notations.

Due to compactness of the manifold $\mathcal{M}$, the function V distinguishes maximum and minimum on $\mathcal{M}$. Let $V_{\max}$ ($V_{\min}$) stand for the maximum (minimum) value of V. We introduce $X_{\max} = V^{-1}(V_{\max})$ and $X_{\min} = V^{-1}(V_{\min})$. Then in accordance with the behavior of the factor f one may define a set X_c as follows:

$$\mathcal{T}_+ = \{\tau \in \mathbb{R} : f(\tau) > 0\}, \qquad \mathcal{T}_- = \{\tau \in \mathbb{R} : f(\tau) < 0\},$$

$$X_c = \begin{cases} X_{\max}, & \mathcal{T}_- = \emptyset, \\ X_{\max} \cup X_{\min}, & \mathcal{T}_\pm \neq \emptyset, \\ X_{\min}, & \mathcal{T}_+ = \emptyset. \end{cases} \tag{1.4}$$

We assume that

(A_0) X_c consists of isolated nondegenerate critical points of V.

In addition to (A_0), we also assume that f satisfies the following conditions:

(A_1) there exist L different solutions $\tau_l \in \mathbb{T}^1 = \mathbb{R}/(T\mathbb{Z}), l = 1, \ldots, L$ of the equation $f(\tau) = 0$;

(A_2) for each $l = 1, \ldots, L$ there exists a neighborhood of τ_l where the function f can be represented as $f(\tau) = (\tau - \tau_l)^{\varkappa_l} g_l(\tau)$ with $\varkappa_l \in \mathbb{N}$ and some C^1-function g_l such that $g_l(\tau_l) \neq 0$.

In the theory of singular perturbed equations τ_l is called the turning point and $\varkappa_l$ is the order of the turning point τ_l [28]. Thus, (A_1) and (A_2) are equivalent to an assumption that the system (1.1) possesses L different turning points of finite order.

We will say that a solution $q : \mathbb{R} \to \mathcal{M}$ is a *heteroclinic* (*homoclinic*) solution if there exist $x_1, x_2 \in \mathcal{M}$ (for the homoclinic solution $x_2 = x_1$) such that q joins x_1 to x_2, i. e., $\lim\limits_{t \to -\infty} q(t) = x_1$, $\lim\limits_{t \to +\infty} q(t) = x_2$ and $\lim\limits_{t \to \pm\infty} \dot{q}(t) = 0$.

Let T denote the period of the function f. It follows from the periodicity of a Lagrangian that if $q(t)$ is a solution of the system (1.1), then $q(t + jT/\varepsilon)$ is also a solution for all $j \in \mathbb{Z}$. Thus, connecting (i. e., homoclinic or heteroclinic) solutions are defined up to a translation.

Then we may formulate the main theorem.

Theorem 1. *Under assumptions (A_0)–(A_2) for any $x_1, x_2 \in X_c$ there exist a positive ε_0 and a subset $\mathcal{E}_h \subset (0, \varepsilon_0)$ such that for any $\varepsilon \in \mathcal{E}_h$ the system (1.1) possesses a rich class of heteroclinic (homoclinic) trajectories emanating from x_1 and terminating at x_2.*

Remarks. 1. To prove this theorem, we construct connecting orbits of the system (1.1) which stay most of the time in a small neighborhood of the set X_c and leave this neighborhood when τ approaches some of the turning points. In particular, for any $x_1, x_2 \in X_c$ we prove the existence of connecting orbits with a minimal number of bumps, the so-called one- or two-bump trajectories. One-bump trajectories stay in small neighborhoods of the points x_1, x_2 and jump from one to another when the parameter τ is in a small (with respect to ε) vicinity of a turning point τ_l. Two-bump trajectories connect x_1 and x_2 via some intermediate critical point x_{mid}. They stay near the points x_1, x_{mid}, x_2 and jump from one to another as τ gets close to turning points τ_{l_1}, τ_{l_2}.

2. If $\mathcal{T}_+ = \emptyset$ or $\mathcal{T}_- = \emptyset$, the subset $\mathcal{E}_h$ coincides with $(0, \varepsilon_0)$. It is also to be noted that in this case the existence of heteroclinic (homoclinic) and multibump trajectories can be obtained by variational methods for all $\varepsilon > 0$ (see, e.g., [4, 25]). However, the origin of such trajectories is not related to the existence of turning points in contrast to the connecting orbits from Theorem 1.

3. In the case $\mathcal{T}_\pm \neq \emptyset$ the subset $\mathcal{E}_h$ has the structure $\mathcal{E}_h = (0, \varepsilon_0) \setminus \bigcup\limits_{j=1}^{\infty} [a_j, b_j]$, where positive constants a_j, b_j are such that $a_j = O(j^{-1})$, $b_j - a_j = O\left(\mathrm{e}^{-Cj}\right)$ as $j \to +\infty$ for some constant $C > 0$.

2. STABILITY OF EQUILIBRIA

We begin with the observation that due to the special form of the potential (1.2) any critical point $x_0 \in X_c$ is an equilibrium of the system (1.1). In this section we analyze the stability of an equilibrium x_0 for sufficiently small ε. Consider a smooth embedding of the manifold $\mathcal{M}$ into $\mathbb{R}^N$ for $N = 2n + 1$ with $n = \dim \mathcal{M}$ and denote by $\langle \cdot, \cdot \rangle$ the Euclidean structure in $\mathbb{R}^N$ together with its restriction to $\mathcal{M}$. Let ∇ stand for the gradient operator with respect to the variable x. We fix a critical point x_0 of the function V and assume without loss of generality that the embedding of $\mathcal{M}$ into $\mathbb{R}^N$ is such that a small neighborhood of x_0 lies in a linear subspace $\mathbb{R}^n \subset \mathbb{R}^N$. Taking $r > 0$ to be sufficiently small, we may always assume that this neighborhood coincides with the ball $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$. To analyze the stability of the equilibrium, it is more convenient to introduce the fast time τ and consider the system (1.3). Then the equations of motion take the form

$$\varepsilon^2 D_\tau q' + f(\tau) \nabla V(q) = 0, \tag{2.1}$$

where D_τ is the covariant derivative (with respect to the Riemannian structure of $\mathcal{M}$). Using local coordinates in a vicinity of x_0, we substitute $q(\tau) = x_0 + v(\tau)$ into (2.1) and make linearization around $q = x_0$ to get the variational equations

$$\varepsilon^2 \frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2} + f(\tau) H^V(x_0) v = 0, \tag{2.2}$$

where H^V stands for the Hessian of V. Since H^V is symmetric, we may perform a change of coordinates to diagonalize $H^V(x_0) = \mathrm{diag}\{\Lambda_1^2, \ldots, \Lambda_n^2\}$ and represent (2.2) as

$$\varepsilon^2 \frac{\mathrm{d}^2 v_k}{\mathrm{d}\tau^2} + f(\tau) \Lambda_k^2 v_k = 0, \qquad k = 1, \ldots, n. \tag{2.3}$$

Further we will refer to such a local coordinate system near the point x_0 as $LC(x_0)$.

To analyze the stability of x_0, we rewrite the kth equation of (2.3) in the Hamiltonian form

$$\mu_k^{-1} v_k' = p_k, \qquad \mu_k^{-1} p_k' = -f(\tau) v_k, \qquad \mu_k = \varepsilon^{-1} \Lambda_k \tag{2.4}$$

and consider the Poincaré map Φ_k, which is the period map for the equation of (2.4), i.e.

$$\Phi_k(\tau, \mu_k) : \begin{pmatrix} v_k(\tau) \\ p_k(\tau) \end{pmatrix} \to \begin{pmatrix} v_k(\tau + T) \\ p_k(\tau + T) \end{pmatrix}. \tag{2.5}$$

Then x_0 becomes a fixed point of Φ_k. Since the Poincaré map is an area-preserving diffeomorphism, $\det \Phi_k = 1$ and the point x_0 is hyperbolic if

$$|\mathrm{Tr} \Phi_k(\tau, \mu_k)| > 2.$$

Note that $\mathrm{Tr} \Phi_k(\tau, \mu_k)$ is independent of τ.

We use WKB-method to obtain an asymptotics of Φ_k when $\mu_k \gg 1$ (see e.g. [8, 12, 18] and literature therein). For simplicity we will omit the subscript k when it does not lead to misunderstanding. Let $\delta > 0$ be sufficiently small. Then one may divide $\mathbb{T}^1$ into intervals of the form $\Delta'_l = [\tau_l + \delta, \tau_{l+1} - \delta]$ and $\Delta''_l = [\tau_l - \delta, \tau_l + \delta]$. According to WKB-theory the general solution of (2.3) has different representation depending on how close the parameter τ is to a turninig point. In particular, on the interval Δ'_l (i.e. far from the turning points) it has the following asymptotics [12]:

$$v(\tau, l) = \begin{cases} \dfrac{1}{|f(\tau)|^{1/4}} \left(A_l e^{-\mu S_l^{(1)}(\tau)} + B_l e^{\mu S_l^{(1)}(\tau)} \right) \left(1 + O(\mu^{-1})\right), \quad f(\tau) < 0, \\ \dfrac{1}{|f(\tau)|^{1/4}} \Bigg(A_l \sin\left(\mu S_l^{(1)}(\tau) + \dfrac{\pi}{4} \right) \\ \qquad + B_l \cos\left(\mu S_l^{(1)}(\tau) + \dfrac{\pi}{4} \right) \Bigg) (1 + O(\mu^{-1})), \quad f(\tau) > 0, \end{cases} \tag{2.6}$$

where A_l, B_l are arbitrary constants and

$$S_l^{(1)}(\tau) = \int\limits_{\tau_l + \delta}^{\tau} |f(s)|^{1/2} \mathrm{d}s.$$

On the interval Δ''_l (i.e. in a vicinity of a turning point τ_l) the general solution of (2.3) can be represented [11] as

$$v(\tau, l) = \frac{1}{(\hat{f}(\tau))^{1/4}} \left(C_l^{(1)} V_l^{(1)} \left(\mu^{\frac{2}{m_l}} S_l^{(2)}(\tau) \right) + C_l^{(2)} V_l^{(2)} \left(\mu^{\frac{2}{m_l}} S_l^{(2)}(\tau) \right) \right) \left(1 + O(\mu^{-1})\right), \tag{2.7}$$

where $C_l^{(1)}, C_l^{(2)}$ are arbitrary constants and

$$S_l^{(2)}(\tau) = \left(\int\limits_{\tau_l}^{\tau} |f(s)|^{1/2} \mathrm{d}s \right)^{\frac{2}{m_l}} \mathrm{sign}(\tau - \tau_l), \quad \hat{f}(\tau) = \frac{4}{m_l^2} |f(\tau)| |S_l^{(2)}(\tau)|^{2 - m_l}, \quad m_l = \varkappa_l + 2,$$

while the functions $\{V_l^{(1)}, V_l^{(2)}\}$ denote a fundamental system of solutions of *the model equation*

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} = \alpha_l \frac{m_l^2}{4} \xi^{m_l - 2} w, \quad \alpha_l = \pm 1. \tag{2.8}$$

The model Eq. (2.8) possesses (see e.g. [11]) the following fundamental system:
Case 1: m_l is even, $\alpha_l = 1$

$$V_l^{(1)}(\xi) = \begin{cases} \sqrt{\frac{2\xi}{\pi}} K_{1/m_l}(\xi^{m_l/2}), & \xi > 0, \\ \sqrt{\frac{2|\xi|}{\pi}} \left[\pi \csc \frac{\pi}{m_l} I_{1/m_l}(|\xi|^{m_l/2}) + K_{1/m_l}(|\xi|^{m_l/2}) \right], & \xi < 0, \end{cases}$$

where $I_{1/m}$ and $K_{1/m}$ stand for the modified Bessel functions of the first and second kind, respectively.

The second function $V_l^{(2)}$ is defined as $V_l^{(2)}(\xi) = V_l^{(1)}(-\xi)$ and the Wronskian W of these solutions is

$$W(V_l^{(1)}, V_l^{(2)}) = m_l \csc \frac{\pi}{m_l}.$$

The function $V_l^{(1)}$ has the following asymptotics as $|\xi| \to \infty$:

$$V_l^{(1)}(\xi) = \begin{cases} \xi^{\frac{2 - m_l}{4}} \mathrm{e}^{-\xi^{m_l/2}} \left(1 + O\left(\xi^{-m_l/2}\right)\right), & \xi \to +\infty, \\ \csc \frac{\pi}{m_l} |\xi|^{\frac{2 - m_l}{4}} \mathrm{e}^{|\xi|^{m_l/2}} \left(1 + O\left(|\xi|^{-m_l/2}\right)\right), & \xi \to -\infty. \end{cases} \tag{2.9}$$

Case 2: m_l is even, $\alpha_l = -1$

$$V_l^{(1)}(\xi) = \begin{cases} -\sqrt{\frac{\pi\xi}{2}}\left[\tan\frac{\pi}{2m_l}J_{1/m_l}(\xi^{m_l/2}) + Y_{1/m_l}(\xi^{m_l/2})\right], & \xi > 0, \\ \sqrt{\frac{\pi|\xi|}{2}}\left[\cot\frac{\pi}{2m_l}J_{1/m_l}(|\xi|^{m_l/2}) - Y_{1/m_l}(|\xi|^{m_l/2})\right], & \xi < 0, \end{cases}$$

where $J_{1/m}$ and $Y_{1/m}$ denote the Bessel functions of the first and second kind, respectively.

As in the previous case, the second solution can be defined as $V_l^{(2)}(\xi) = V_l^{(1)}(-\xi)$, while the Wronskian takes the form

$$W(V_l^{(1)}, V_l^{(2)}) = m_l \sec\frac{\pi}{m_l}.$$

In this case the asymptotics of the function $V_l^{(1)}$ for large values of argument is

$$V_l^{(1)}(\xi) = \begin{cases} \sec\frac{\pi}{2m_l}\xi^{\frac{2-m_l}{4}}\left[\cos\left(\xi^{m_l/2} + \frac{\pi}{4}\right) + O\left(\xi^{-m_l/2}\right)\right], & \xi \to +\infty, \\ \csc\frac{\pi}{2m_l}|\xi|^{\frac{2-m_l}{4}}\left[\cos\left(|\xi|^{m_l/2} - \frac{\pi}{4}\right) + O\left(|\xi|^{-m_l/2}\right)\right], & \xi \to -\infty. \end{cases} \tag{2.10}$$

Case 3: m_l is odd, $\alpha_l = 1$

$$V_l^{(1)}(\xi) = \begin{cases} \sqrt{\frac{2\xi}{\pi}}K_{1/m_l}(\xi^{m_l/2}), & \xi > 0, \\ \sqrt{\frac{\pi|\xi|}{2}}\left[\cot\frac{\pi}{2m_l}J_{1/m_l}(|\xi|^{m_l/2}) - Y_{1/m_l}(|\xi|^{m_l/2})\right], & \xi < 0, \end{cases}$$

$$V_l^{(2)}(\xi) = \begin{cases} \sqrt{\frac{2\xi}{\pi}}\left[\pi\csc\frac{\pi}{m_l}I_{1/m_l}(\xi^{m_l/2}) + K_{1/m_l}(\xi^{m_l/2})\right], & \xi > 0, \\ -\sqrt{\frac{\pi|\xi|}{2}}\left[\tan\frac{\pi}{2m_l}J_{1/m_l}(|\xi|^{m_l/2}) + Y_{1/m_l}(|\xi|^{m_l/2})\right], & \xi < 0, \end{cases}$$

with the Wronskian

$$W(V_l^{(1)}, V_l^{(2)}) = m_l \csc\frac{\pi}{m_l}.$$

The asymptotics of $V_l^{(1)}, V_l^{(2)}$ for large $|\xi|$ are

$$V_l^{(1)}(\xi) = \begin{cases} \xi^{\frac{2-m_l}{4}}\mathrm{e}^{-\xi^{m_l/2}}\left(1 + O\left(\xi^{-m_l/2}\right)\right), & \xi \to +\infty, \\ \csc\frac{\pi}{2m_l}|\xi|^{\frac{2-m_l}{4}}\left[\cos\left(|\xi|^{m_l/2} - \frac{\pi}{4}\right) + O\left(|\xi|^{-m_l/2}\right)\right], & \xi \to -\infty, \end{cases} \tag{2.11}$$

$$V_l^{(2)}(\xi) = \begin{cases} \csc\frac{\pi}{m_l}\xi^{\frac{2-m_l}{4}}\mathrm{e}^{\xi^{m_l/2}}\left(1 + O\left(\xi^{-m_l/2}\right)\right), & \xi \to +\infty, \\ \sec\frac{\pi}{2m_l}|\xi|^{\frac{2-m_l}{4}}\left[\cos\left(|\xi|^{m_l/2} + \frac{\pi}{4}\right) + O\left(|\xi|^{-m_l/2}\right)\right], & \xi \to -\infty. \end{cases} \tag{2.12}$$

Case 4: m_l is odd, $\alpha_l = -1$.

One may see that this case can be reduced to the previous one by the change $\xi \to -\xi$.

It has to be noted that all asymptotics for the functions $V_l^{(1)}, V_l^{(2)}$ can be differentiated with respect to ξ.

To obtain an asymptotics of the Poincaré map Φ, we introduce new variables:

$$\begin{pmatrix} \hat{v}(\tau) \\ \hat{p}(\tau) \end{pmatrix} = \Xi(\tau)\begin{pmatrix} v(\tau) \\ p(\tau) \end{pmatrix}, \quad \Xi(\tau) = \begin{pmatrix} |f(\tau)|^{1/4} & 0 \\ 0 & |f(\tau)|^{-1/4} \end{pmatrix}, \quad \tau \neq \tau_l. \tag{2.13}$$

Due to periodicity of f the map $\Phi(\tau_1-\delta)$ can be represented as a composition

$$\Phi(\tau_1-\delta)=\Xi^{-1}(\tau_1-\delta)\circ\Phi^{(L)}\circ\Phi^{(L-1)}\circ\ldots\circ\Phi^{(1)}\circ\Xi(\tau_1-\delta), \tag{2.14}$$

where

$$\Phi^{(l)}:\begin{pmatrix}\hat v(\tau_l-\delta)\\ \hat p(\tau_l-\delta)\end{pmatrix}\to\begin{pmatrix}\hat v(\tau_{l+1}-\delta)\\ \hat p(\tau_{l+1}-\delta)\end{pmatrix}. \tag{2.15}$$

We also decompose the map $\Phi^{(l)}$ as follows:

$$\Phi^{(l)}=\Theta^{(l)}\circ\Psi^{(l)}, \tag{2.16}$$

where

$$\Psi^{(l)}:\begin{pmatrix}\hat v(\tau_l-\delta)\\ \hat p(\tau_l-\delta)\end{pmatrix}\to\begin{pmatrix}\hat v(\tau_l+\delta)\\ \hat p(\tau_l+\delta)\end{pmatrix}, \tag{2.17}$$

$$\Theta^{(l)}:\begin{pmatrix}\hat v(\tau_l+\delta)\\ \hat p(\tau_l+\delta)\end{pmatrix}\to\begin{pmatrix}\hat v(\tau_{l+1}-\delta)\\ \hat p(\tau_{l+1}-\delta)\end{pmatrix}. \tag{2.18}$$

Introduce the following notations:

$$R(\alpha)=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix},\quad Z(\alpha)=\begin{pmatrix}\mathrm{e}^{-\alpha} & 0\\ 0 & \mathrm{e}^{\alpha}\end{pmatrix}.$$

Then substituting (2.9)–(2.12) into (2.7) and taking into account that

$$\left(\mu^{2/m_l}S_l^{(2)}(\tau)\right)^{\frac{2-m_l}{4}}\left(\hat f_l(\tau)\right)^{-1/4}=\mu^{\frac{2-m_l}{2m_l}}\left(\frac{m_l}{2}\right)^{1/2}|f(\tau)|^{-1/4},$$

one gets the asymptotics of the map $\Psi^{(l)}$ up to the factor $(1+O(\mu^{-1}))$:

$$\Psi^{(l)}=\begin{cases}R\left(\frac{\pi}{4}\right)Z\left(\mu S_l^+ +\mu S_l^- +\ln\left(\csc\frac{\pi}{m_l}\right)\right)R\left(-\frac{\pi}{4}\right), & m_l-\text{ even}, g_l(\tau_l)<0,\\ R\left(\mu S_l^+ +\frac{\pi}{4}\right)Z\left(\ln\left(\csc\frac{\pi}{m_l}\right)\right)R\left(\mu S_l^- -\frac{\pi}{4}\right), & m_l-\text{ even}, g_l(\tau_l)>0,\\ R\left(\frac{\pi}{4}\right)Z\left(\mu S_l^+ +\frac12\ln\left(\csc\frac{\pi}{m_l}\right)+\frac12\ln\left(\cot\frac{\pi}{2m_l}\right)\right)R\left(\mu S_l^- -\frac{\pi}{4}\right), & m_l-\text{ odd}, g_l(\tau_l)<0,\\ R\left(\mu S_l^+ +\frac{\pi}{4}\right)Z\left(\mu S_l^- +\frac12\ln\left(\csc\frac{\pi}{m_l}\right)+\frac12\ln\left(\cot\frac{\pi}{2m_l}\right)\right)R\left(-\frac{\pi}{4}\right), & m_l-\text{ odd}, g_l(\tau_l)>0,\end{cases} \tag{2.19}$$

where

$$S_l^-=\int\limits_{\tau_l-\delta}^{\tau_l}|f(s)|^{1/2}\mathrm{d}s,\quad S_l^+=\int\limits_{\tau_l}^{\tau_l+\delta}|f(s)|^{1/2}\mathrm{d}s.$$

Using (2.6) we obtain

$$\Theta^{(l)}=\begin{cases}R\left(\frac{\pi}{4}\right)Z\left(\mu S_l^0\right)R\left(-\frac{\pi}{4}\right)\left(1+O(\mu^{-1})\right), & g_l(\tau_l)<0,\\ R\left(\mu S_l^0\right)\left(1+O(\mu^{-1})\right), & g_l(\tau_l)>0,\end{cases} \tag{2.20}$$

where

$$S_l^0 = \int\limits_{\tau_l+\delta}^{\tau_{l+1}-\delta} |f(s)|^{1/2} \mathrm{d}s.$$

Note that $R(\alpha_1+\alpha_2) = R(\alpha_1)R(\alpha_2)$, $Z(\alpha_1+\alpha_2) = Z(\alpha_1)Z(\alpha_2)$. Hence, (2.19), (2.20) together with (2.14)–(2.18) yield

$$\mathrm{Tr}\Phi(\tau,\mu) = \mathrm{Tr}\left[R(\mu S_e)Z(\mu S_h+\gamma)\right]\left(1+O(\mu^{-1})\right), \tag{2.21}$$

where

$$S_e = \int\limits_{\mathcal{T}_+} |f(s)|^{1/2}\mathrm{d}s, \quad S_h = \int\limits_{\mathcal{T}_-} |f(s)|^{1/2}\mathrm{d}s, \quad \gamma = \prod_{l=1}^{L}\gamma_l, \tag{2.22}$$

$$\gamma_l = \begin{cases} \ln\left(\csc\frac{\pi}{m_l}\right), & m_l-\text{ even} \\ \frac{1}{2}\ln\left(\csc\frac{\pi}{m_l}\right) + \frac{1}{2}\ln\left(\cot\frac{\pi}{2m_l}\right), & m_l-\text{ odd.} \end{cases} \tag{2.23}$$

Consider an inequality

$$|\mathrm{Tr}\Phi_0| \leqslant 2.$$

Due to (2.21)–(2.23) it is equivalent to

$$|\cos(\mu S_e)|\mathrm{e}^{\mu S_h+\gamma}\left(1+O(\mu^{-1})\right) \leqslant 2. \tag{2.24}$$

Solving (2.24) we arrive at the following lemma.

Lemma 1. *There exists a sufficiently large constant $\mu_0>0$ such that on the interval $(\mu_0,+\infty)$ the inequality (2.24) holds if $\mu \in \bigcup\limits_{j=j_0}^{\infty}\left[\left(\frac{\pi}{2}+\pi j\right)S_e^{-1} - r_j', \left(\frac{\pi}{2}+\pi j\right)S_e^{-1} + r_j''\right]$, where r_j', r_j'' are positive constants of the order $O\left(\mathrm{e}^{-\left(\frac{\pi}{2}+\pi j\right)\frac{S_h}{S_e}}\right)$ and j_0 is the least integer such that $\left(\frac{\pi}{2}+\pi j_0\right)S_e^{-1} > \mu_0$.*

We now point out that the subscript k enumerating degrees of freedom was omitted. Taking this into account and the definition of μ_k (2.4), one obtains the following corollaries to the previous lemma:

Corollary 1. *There exists a sufficiently small $\varepsilon_0>0$ such that for any $k=1,\ldots,n$ and for any positive ε such that*

$$\varepsilon \in (0,\varepsilon_0)\setminus\mathcal{E}_k(x_0), \quad \mathcal{E}_k(x_0) = \bigcup_{j=j_0}^{\infty}\left[\Lambda_k\left(\frac{\pi}{2}+\pi j\right)^{-1}S_e - \epsilon_j', \Lambda_k\left(\frac{\pi}{2}+\pi j\right)^{-1}S_e + \epsilon_j''\right],$$

where $\epsilon_j', \epsilon_j''$ are positive constants of the order $O\left(\mathrm{e}^{-\left(\frac{\pi}{2}+\pi j\right)\frac{S_h}{S_e}}\right)$, the origin is a hyperbolic equilibrium for a system described by $(2.3)_k$.

Denote by $\mathcal{E}_h(x_0)$

$$\mathcal{E}_h(x_0) = (0,\varepsilon_0)\setminus\bigcup_{k=1}^{n}\mathcal{E}_k(x_0).$$

Then due to the Floquet theory the following corollary holds:

Corollary 2. *For any positive $\varepsilon \in \mathcal{E}_h(x_0)$ the system (2.3) has two sets of n linear independent solutions:*

$$v^s_{(k)}(\tau) = \mathrm{e}^{-\lambda_k \tau} w^s_{(k)}(\tau), \quad v^u_{(k)}(\tau) = \mathrm{e}^{\lambda_k \tau} w^u_{(k)}(\tau), \quad k = 1, \ldots, n,$$

where $\lambda_k = \lambda_k(x_0)$ are the Floquet exponents and $w^{s,u}_{(k)}(\tau)$ are T-periodic functions. The solutions $v^{s,u}_{(k)}(\tau)$ span the stable and unstable subspaces $E^{s,u}(x_0, \tau)$ of the system (2.3).

Summarizing all these results, one gets

Corollary 3. *For any positive $\varepsilon \in \mathcal{E}_h(x_0)$ the point x_0 is a hyperbolic equilibrium of the system (2.1) possessing $n+1$-dimensional invariant stable $W^s(x_0)$ and unstable $W^u(x_0)$ manifolds in $T\mathcal{M} \times \mathbb{R}$. For any $\varepsilon_1 < \varepsilon_0$ the Lebesgue measure leb of the set $(0, \varepsilon_1) \cap \mathcal{E}_h(x_0)$ can be estimated as*

$$leb((0, \varepsilon_1) \cap \mathcal{E}_h(x_0)) = O\left(\mathrm{e}^{-\frac{S_h}{\varepsilon_1}}\right).$$

For the rest of the paper we introduce the following notations. Consider the modified inequality (2.24)

$$|\cos(\mu_k S_e)| \mathrm{e}^{\mu_k S_h + \gamma}\big(1 + O(\mu_k^{-1})\big) \leqslant 2 + \rho \tag{2.25}$$

for some positive parameter ρ and define a set $\mathcal{E}_k(x_0, \rho)$ such that for $\varepsilon \in \mathcal{E}_k(x_0, \rho)$ (2.25) holds. Then

$$\mathcal{E}_k(x_0, \rho) = \bigcup_{j=j_0}^{\infty} \left[\Lambda_k \left(\frac{\pi}{2} + \pi j\right)^{-1} S_e - \epsilon'_j(\rho), \Lambda_k \left(\frac{\pi}{2} + \pi j\right)^{-1} S_e + \epsilon''_j(\rho)\right]$$

and $\epsilon'_j(\rho), \epsilon''_j(\rho)$ are of the order $O\left(\mathrm{e}^{-\left(\frac{\pi}{2} + \pi j\right)\frac{S_h}{S_e}}\right)$. We also introduce

$$\mathcal{E}_h(x_0, \rho) = (0, \varepsilon_0) \setminus \bigcup_{k=1}^{n} \mathcal{E}_k(x_0, \rho). \tag{2.26}$$

It is to be noted that if $\varepsilon \in \mathcal{E}_h(x_0, \rho)$, the Floquet exponent $\lambda_k(x_0) > \log(1 + \sqrt{\rho})$.

3. DYNAMICS NEAR TURNING POINTS

In this section we fix a turning point $\tau_* = \tau_l$ for some $l = 1, \ldots, L$ and consider the behavior of the system (2.1) in a vicinity of τ_*. Due to assumption (A_2) the factor f is represented in a small neighborhood of τ_* as $f(\tau) = (\tau - \tau_*)^{\varkappa_*} g_*(\tau)$ with some nonvanishing at τ_* C^1-function g_*. If we introduce a new scaled time ζ by the formula $\tau - \tau_* = \varepsilon^{2/m_*}\zeta$ with $m_* = \varkappa_* + 2$, the equations of motion (2.1) in the neighborhood of τ_* take the form

$$D_\zeta q'_\zeta + \zeta^{\varkappa_*} g_*(\tau_* + \varepsilon^{2/m_*}\zeta)\nabla V(q) = 0, \quad q'_\zeta = \frac{\mathrm{d}q}{\mathrm{d}\zeta}. \tag{3.1}$$

Putting $\varepsilon = 0$ in (3.1), we obtain *the reference system*

$$D_\zeta q'_\zeta + \zeta^{\varkappa_*} g_*(\tau_*)\nabla V(q) = 0. \tag{3.2}$$

Note that the reference system is parameter-free and it is expected to be a good approximation for the system (2.1) in a vicinity of the turning point τ_*. The Lagrangian of the system (3.2) can be written as

$$\mathcal{L}(q, q'_\zeta, \zeta) = K(q, q'_\zeta) - a(\zeta)V(q), \quad a(\zeta) = \zeta^{\varkappa_*} g_*(\tau_*).$$

The factor $a(\zeta)$ satisfies the following conditions:

(C_1) $\zeta = 0$ is a unique point $\zeta \in \mathbb{R}$ such that $a(\zeta) = 0$,

(C_2) $|a(\zeta)| \to +\infty$ as $\zeta \to \pm\infty$,

(C_3) there exist constants $C_a, Z_a > 0$ such that $|a(\zeta)| > C_a|a'(\zeta)|$ for all $|\zeta| > Z_a$.

Introduce subsets $X^*_\pm \subset X_c$ defined by the behavior of the factor f near τ_*

$$X^*_+ = \begin{cases} X_{\max}, & a(1) > 0, \\ X_{\min}, & a(1) < 0, \end{cases} \qquad X^*_- = \begin{cases} X_{\max}, & a(-1) > 0, \\ X_{\min}, & a(-1) < 0. \end{cases} \tag{3.3}$$

The following theorem was proved in [17] by variational arguments:

Theorem 2. *For any two points $x_\pm \in X^*_\pm$ there exist infinitely many heteroclinic (homoclinic) solutions of the system (3.2) emanating from x_- and terminating at x_+, i. e., $\lim\limits_{\zeta\to\pm\infty} q(\zeta) = x_\pm$ and $\lim\limits_{\zeta\to\pm\infty} q'_\zeta = 0$.*

We denote the set of heteroclinic (homoclinic) solutions of the reference system (3.2) connecting points $x_\pm \in X_\pm$ by $Q_h(x_-, x_+, \tau_*)$. Then we arrive at the following lemma.

Lemma 2. *For any $q \in Q_h(x_-, x_+, \tau_*)$ there exists $\zeta_0 = \zeta_0(q) > 0$ such that for all $|\zeta| > \zeta_0$*

$$|q(\zeta) - x_\pm| = O\left(\zeta^{\frac{2-m_*}{4}} \mathrm{e}^{-\sigma\zeta^{\frac{m_*}{2}}}\right), \quad \zeta \to \pm\infty,$$

where σ is an arbitrary constant such that

$$0 < \sigma < \sigma_{\min}, \quad \sigma_{\min} = \frac{2|g_*(\tau_*)|^{1/2}\Lambda_{\min}}{m_*}, \quad \Lambda_{\min} = \min_{k=1,\ldots,n} \{\Lambda_k\}.$$

Proof. First we note that due to $\lim\limits_{\zeta\to\pm\infty} q(\zeta) = x_\pm$ there exist constants $\zeta_\pm > 0$ such that $q(\zeta) \in B_r(x_\pm)$ for $\zeta > \zeta_+$ ($\zeta < -\zeta_-$, respectively). Here and in what follows $B_r(x_0)$ stands for the ball $B_r(x_0) = \{x \in \mathcal{M} : |x - x_0| < r\}$. Put $\zeta_0 = \max\{\zeta_+, \zeta_-\}$. Then the lemma follows immediately from the following proposition.

Proposition 1. *Let U be an open subset of the tangent bundle $T\mathcal{M}$ containing the equilibrium $(x_+, 0)$ and ϕ_ζ be the flow of the reference system (3.2). Then there exists an $n+1$-dimensional differentiable manifold $\mathcal{W}^s(x_+) \subset T\mathcal{M} \times \mathbb{R}$ such that for any $\zeta_1 > \zeta_0$ $\phi_{\zeta_1}\big(\mathcal{W}^s(x_+, \zeta_0)\big) \subset \mathcal{W}^s(x_+, \zeta_1)$, where $\mathcal{W}^s(x_+, \zeta_0) = \{(a, b) \in T\mathcal{M} : (a, b, \zeta_0) \in \mathcal{W}^s(x_+)\}$, and for any $(a, b) \in \mathcal{W}^s(x_+, \zeta_0)$*

$$\lim_{\zeta\to+\infty} \phi_\zeta(a, b) = (x_+, 0).$$

Moreover, if $q(\zeta)$ is a solution of (3.2) such that $\big(q(\zeta_0), q'(\zeta_0)\big) \in \mathcal{W}^s(x_+, \zeta_0)$, then

$$|q(\zeta) - x_\pm| = O\left(\zeta^{\frac{2-m_*}{4}} \mathrm{e}^{-\sigma\zeta^{\frac{m_*}{2}}}\right), \quad \zeta \to \pm\infty.$$

Proof. The proof of this proposition is rather straightforward and similar to the proof of the standard Stable Manifold Theorem (see, e. g., [23]). Let $r > 0$ be sufficiently small. We take $\zeta_0 > 0$ such that $q(\zeta) \in B_r(x_+)$ for all $\zeta > \zeta_0$ and consider the reference system on the interval $(\zeta_0, +\infty)$. Then one may rewrite (3.2) as

$$D_\zeta v'_\zeta + \zeta^{\varkappa_*} g_*(\tau_*) H^V(x_+) v = \zeta^{\varkappa_*} g_*(\tau_*) \left(H^V(x_+) v - \nabla V(x_+ + v)\right), \tag{3.4}$$

where $v = q - x_+$. Noting that $-g_*(\tau_*) H^V(x_+)$ is positively defined, we may rewrite (3.4) in the local coordinates $LC(x_+)$ as follows:

$$v''_k - \zeta^{\varkappa_*} |g_*(\tau_*)| \Lambda_k^2 v_k = \zeta^{\varkappa_*} |g_*(\tau_*)| \left(H^V(x_+) v - \nabla V(x_+ + v)\right)_k, \quad k = 1, \ldots, n. \tag{3.5}$$

Introduce a new time

$$\eta = \left(\frac{4|g(\tau_*)|}{m_*^2}\right)^{1/m_*} \zeta.$$

Then (3.5) reads

$$v_k'' - \frac{m_*^2}{4}\eta^{\varkappa_*}\Lambda_k^2 v_k = \frac{m_*^2}{4}\eta^{\varkappa_*}\left(H^V(x_+)v - \nabla V(x_+ + v)\right)_k, \quad k = 1, \ldots, n. \tag{3.6}$$

We supply (3.6) by the initial conditions

$$v_k(\eta_0) = a_k, \quad v_k'(\eta_0) = b_k, \quad \eta_0 = \left(\frac{4|g(\tau_*)|}{m_*^2}\right)^{1/m_*}\zeta_0, \quad k = 1, \ldots, n \tag{3.7}$$

and rewrite the Cauchy problem (3.6), (3.7) as an integral equation:

$$\begin{aligned} v_k(\eta) = \gamma_k^+ v_k^+(\eta) + \gamma_k^- v_k^-(\eta) + v_k^+(\eta)\int\limits_{\eta_0}^{\eta} W_k^{-1} v_k^-(s) h_k(v(s))\mathrm{d}s \\ - v_k^-(\eta)\int\limits_{\eta_0}^{\eta} W_k^{-1} v_k^+(s) h_k(v(s))\mathrm{d}s, \quad k = 1, \ldots, n, \end{aligned} \tag{3.8}$$

where $v_k^{\pm}$ are expressed via solutions of the model Eq. (2.8) with $\alpha = 1$:

$$v_k^+(\eta) = V^{(1)}(\Lambda_k^{2/m_*}\eta), \quad v_k^-(\eta) = \sin\left(\frac{\pi}{m_*}\right)V^{(2)}(\Lambda_k^{2/m_*}\eta), \tag{3.9}$$

$W_k = -m_*\Lambda_k$ is the Wronskian of solutions (v_k^-, v_k^+) and $h_k(v)$ is the right-hand side of Eq. $(3.6)_k$. The constants $\gamma_\pm$ satisfy

$$\gamma_k^+ v_k^+(\eta_0) + \gamma_k^- v_k^-(\eta_0) = a_k, \quad \gamma_k^+ {v'}_k^+(\eta_0) + \gamma_k^- {v'}_k^-(\eta_0) = b_k. \tag{3.10}$$

It follows from (2.9), (2.11), (2.12) that v_k^+, v_k^- have the asymptotics

$$v_k^+(\eta) = \Lambda_k^{\frac{2-m_*}{2m_*}}\eta^{\frac{2-m_*}{4}}\mathrm{e}^{-\Lambda_k\eta^{m_*/2}}\left(1 + O\left(\eta^{-m_*/2}\right)\right), \quad \eta \to +\infty, \tag{3.11}$$

$$v_k^-(\eta) = \Lambda_k^{\frac{2-m_*}{2m_*}}\eta^{\frac{2-m_*}{4}}\mathrm{e}^{\Lambda_k\eta^{m_*/2}}\left(1 + O\left(\eta^{-m_*/2}\right)\right), \quad \eta \to +\infty. \tag{3.12}$$

Note that $\big(v(\eta), v'(\eta)\big)$ should converge to the origin as $\eta \to +\infty$. According to (3.8), (3.9) this holds if

$$\gamma_k^- v_k^-(\eta_0) = \int\limits_{\eta_0}^{+\infty} W_k^{-1} v_k^+(s) h_k\big(v(s)\big)\mathrm{d}s, \quad k = 1, \ldots, n. \tag{3.13}$$

Taking into account (3.13) together with (3.10), we rewrite (3.8) as

$$v_k(\eta) = T_k(v, \eta), \quad k = 1, \ldots, n,$$

where

$$\begin{aligned} T_k(v, \eta) = \Big(b_k v_k^-(\eta_0) - a_k {v'}_k^-(\eta_0)\Big) W_k^{-1} v_k^+(\eta) + v_k^+(\eta)\int\limits_{\eta_0}^{\eta} W_k^{-1} v_k^-(s) h_k\big(v(s)\big)\mathrm{d}s \\ + v_k^-(\eta)\int\limits_{\eta}^{+\infty} W_k^{-1} v_k^+(s) h_k\big(v(s)\big)\mathrm{d}s. \end{aligned} \tag{3.14}$$

One may solve this equation by the method of successive approximations. Define a sequence

$$v_k^{(0)}(\eta) = 0, \quad v_k^{(j+1)}(\eta) = T_k(v^{(j)}, \eta)$$

and introduce

$$\Delta v^{(j+1)}(\eta) = v^{(j+1)}(\eta) - v^{(j)}(\eta).$$

It follows from (3.11), (3.14) that for sufficiently large η_0 and any $0 < \Lambda < \Lambda_{\min}$ there exists a constant $K > 0$ such that for all $\eta > \eta_0$

$$|\Delta v^{(1)}(\eta)| < \frac{K|\gamma^+|\mathrm{e}^{-\Lambda\eta^{m_*/2}}}{\eta^{\frac{m_*-2}{4}}}.$$

Assume that the induction hypothesis

$$|\Delta v^{(j+1)}(\eta)| < \frac{K|\gamma^+|\mathrm{e}^{-\Lambda\eta^{m_*/2}}}{2^j\eta^{\frac{m_*-2}{4}}} \tag{3.15}$$

holds for $j \leqslant m$. Using the Lipschitz estimate satisfied by the function h with some positive constant C_h

$$|h(v_2) - h(v_1)| \leqslant C_h|v_2 - v_1|$$

and taking η_0 large enough for the asymptotics (3.11), (3.12) to be valid, one gets

$$\begin{aligned}
|\Delta v^{(j+1)}(\eta)| &\leqslant \frac{KC_h|\gamma^+|}{2^{j-1}}\max_k\Bigg\{v_k^+(\eta)\int\limits_{\eta_0}^{\eta} W_k^{-1}v_k^-(s)\frac{\mathrm{e}^{-\Lambda s^{m_*/2}}}{s^{\frac{m_*-2}{4}}}\mathrm{d}s \\
&\qquad + v_k^-(\eta)\int\limits_{\eta}^{+\infty} W_k^{-1}v_k^+(s)\frac{\mathrm{e}^{-\Lambda s^{m_*/2}}}{s^{\frac{m_*-2}{4}}}\mathrm{d}s\Bigg\} \\
&\leqslant \frac{KC_hd|\gamma^+|}{2^{j-1}}\max_k\Bigg\{\frac{\mathrm{e}^{-\Lambda_k\eta^{m_*/2}}}{\eta^{\frac{m_*-2}{4}}}\int\limits_{\eta_0}^{\eta} W_k^{-1}\frac{\mathrm{e}^{(\Lambda_k-\Lambda)s^{m_*/2}}}{s^{\frac{m_*-2}{2}}}\mathrm{d}s \\
&\qquad + \frac{\mathrm{e}^{\Lambda_k\eta^{m_*/2}}}{\eta^{\frac{m_*-2}{4}}}\int\limits_{\eta}^{+\infty} W_k^{-1}\frac{\mathrm{e}^{-(\Lambda_k+\Lambda)s^{m_*/2}}}{s^{\frac{m_*-2}{2}}}\mathrm{d}s\Bigg\} \\
&\leqslant \frac{KC_hd|\gamma^+|}{2^{j-2}m_*\eta_0^{m_*-2}}\max_k\Bigg\{\left(\frac{1}{(\Lambda_k-\Lambda)\eta_0^2} + \frac{1}{(\Lambda_k+\Lambda)}\right)\frac{\mathrm{e}^{-\Lambda_k\eta^{m_*/2}}}{\eta^{\frac{m_*-2}{4}}}\Bigg\},
\end{aligned}$$

where $d = 4\max\limits_k\{\Lambda_k^{\frac{2-m_*}{m_*}}\}$.

Hence, for η_0 such that

$$\frac{4C_hd}{m_*\eta_0^{m_*-2}}\max_k\left\{\frac{1}{(\Lambda_k-\Lambda)\eta_0^2} + \frac{1}{(\Lambda_k+\Lambda)}\right\} < 1$$

the estimate (3.15) holds also for $j = m+1$ and consequently for all $j \in \mathbb{N}$. In a similar manner one can prove

$$|\Delta v'^{(j+1)}(\eta)| < 2^{-j}K'|\gamma^+|\eta^{\frac{m_*-2}{4}}\mathrm{e}^{-\Lambda\eta^{m_*/2}}$$

with some positive constant K'.

Thus, for $i > j > N$ and $\eta > \eta_0$

$$|v^{(i)}(\eta) - v^{(j)}(\eta)| \leqslant \sum_{l=N}^{\infty}|v^{(l+1)}(\eta) - v^{(l)}(\eta)| \leqslant \frac{K|\gamma^+|\mathrm{e}^{-\Lambda\eta^{m_*/2}}}{\eta^{\frac{m_*-2}{4}}}\sum_{l=N}^{\infty}\frac{1}{2^l} \leqslant \frac{K|\gamma^+|\mathrm{e}^{-\Lambda\eta^{m_*/2}}}{2^{N-1}\eta^{\frac{m_*-2}{4}}} \tag{3.16}$$

and similarly

$$|{v'}^{(i)}(\eta) - {v'}^{(j)}(\eta)| \leqslant \frac{K'|\gamma^+|\eta^{\frac{m_*-2}{4}}\mathrm{e}^{-\Lambda\eta^{m_*/2}}}{2^{N-1}}.$$

The last two expressions tend to zero as $N \to \infty$ and therefore $\{(v^j, {v'}^j)\}$ is a Cauchy sequence. It is standard to show that for sufficinetly small $|\gamma^+|$ there exists

$$\lim_{j\to\infty} v^{(j)}(\eta) = v_*(\eta, \gamma^+),$$

which is a twice differentiable function of η and γ^+ satisfying (3.8) and therefore (3.6). It also follows from (3.15) that for $\eta > \eta_0$

$$|v_*(\eta, \gamma^+)| < \frac{2K|\gamma^+|\mathrm{e}^{-\Lambda\eta^{m_*/2}}}{\eta^{\frac{m_*-2}{4}}}.$$

Finally, to prove the proposition we note that if one takes γ_k^+ as coordinates, then equations

$$\gamma_k^+ v_k^+(\eta_0) + \int\limits_{\eta_0}^{+\infty} W_k^{-1} v_k^+(s) h_k\big(v_*(s, \gamma^+)\big)\mathrm{d}s = a_k,$$

$$\gamma_k^+ {v'}_k^+(\eta_0) + \frac{{v'}_k^-(\eta_0)}{v_k^-(\eta_0)} \int\limits_{\eta_0}^{+\infty} W_k^{-1} v_k^+(s) h_k\big(v_*(s, \gamma^+)\big)\mathrm{d}s = b_k$$

define an n-dimensional invariant manifold $\mathcal{W}^s(x_+, \eta_0)$. □

4. CONSTRUCTION OF CONNECTING ORBITS

In this section we fix two points $x_\pm \in X_c$ and suppose first that there exists a turning point τ_* such that $x_\pm \in X_\pm^*$ defined by (3.3). Due to Theorem 2 the reference system corresponding to the turning point τ_* possesses a rich set $Q_h(x_-, x_+, \tau_*)$ of connecting orbits. Introduce a subset $Q_h^{tr}(x_-, x_+, \tau_*) \subset Q_h(x_-, x_+, \tau_*)$ consisting of transversal heteroclinic (homoclinic) solutions. In the rest of the paper we will assume

(A_3) the subset $Q_h^{tr}(x_-, x_+, \tau_*)$ is nonempty.

For any $q_0 \in Q_h^{tr}(x_-, x_+, \tau_*)$ define

$$\hat{q}_0(\tau) = q_0\left((\tau - \tau_*)\varepsilon^{-2/m_*}\right) \tag{4.1}$$

and rewrite (2.1) in local coordinates as

$$\begin{aligned} \varepsilon^2 D_\tau \hat{q}'_0 + \varepsilon^2 D_\tau u' + f(\tau) D^2 V(\hat{q}_0) u + f(\tau)\left(\nabla V(\hat{q}_0 + u) - \nabla V(\hat{q}_0) - D^2 V(\hat{q}_0) u\right) \\ +\big(f(\tau) - g_*(\tau_*)(\tau - \tau_*)^{\varkappa_*}\big)\nabla V(\hat{q}_0) + g_*(\tau_*)(\tau - \tau_*)^{\varkappa_*}\nabla V(\hat{q}_0) = 0, \end{aligned} \tag{4.2}$$

where $u = q - \hat{q}_0$. Note that the first and the last terms on the left-hand side of (4.2) vanish since q_0 is a solution of the reference system. Thus, one can rewrite (4.2) as

$$\begin{aligned} \varepsilon^2 D_\tau u' + f(\tau) D^2 V(\hat{q}_0) u = -f(\tau)\left(\nabla V(\hat{q}_0 + u) - \nabla V(\hat{q}_0) - D^2 V(\hat{q}_0) u\right) \\ -\big(f(\tau) - g_*(\tau_*)(\tau - \tau_*)^{\varkappa_*}\big)\nabla V(\hat{q}_0). \end{aligned} \tag{4.3}$$

Now let us consider a linear equation

$$\varepsilon^2 D_\tau u' + f(\tau) D^2 V(\hat{q}_0) u = 0. \tag{4.4}$$

If we set $z = (v, \varepsilon^{-1} v')$, it may be rewritten as the first-order equation

$$D_\tau z = \mathcal{A}(\tau, \varepsilon) z \tag{4.5}$$

with the matrix

$$\mathcal{A}(\tau,\varepsilon)=\varepsilon^{-1}\begin{pmatrix} 0 & I \\ -f(\tau)D^2V(\hat{q}_0(\tau)) & 0 \end{pmatrix}, \tag{4.6}$$

where I stands for the unit matrix in $\mathbb{R}^n$.

Let $\mathcal{A}(\tau)$ be a real $n\times n$ matrix function, piecewise continuous on an interval $\mathcal{J}$. It is said that the system

$$z'=\mathcal{A}(\tau)z \tag{4.7}$$

has an *exponential dichotomy* on the interval $\mathcal{J}$ [9] if there exist a projection P and constants $K\geqslant 1$, $\alpha>0$ such that a fundamental matrix $X(\tau)$ of the system (4.7) satisfies for all $s,\tau\in\mathcal{J}$

$$|X(\tau)PX^{-1}(s)|\leqslant K\mathrm{e}^{-\alpha(\tau-s)}\quad \text{for } s\leqslant\tau, \tag{4.8}$$

$$|X(\tau)(I-P)X^{-1}(s)|\leqslant K\mathrm{e}^{-\alpha(s-\tau)}\quad \text{for } s\geqslant\tau. \tag{4.9}$$

Denote $\mathcal{E}_h^{\pm}(\rho)=\mathcal{E}_h(x_\pm,\rho)$. Then one obtains the following

Lemma 3. *There exists $\varepsilon_1>0$ such that for any $\rho\geqslant 0$ and all $\varepsilon\in(0,\varepsilon_1)\cap\mathcal{E}_h^+(\rho)\cap\mathcal{E}_h^-(\rho)$ Eq. (4.5) possesses an exponential dichotomy on $\mathcal{J}=\mathbb{R}_\pm$ for any constant $\alpha<\min\limits_k\{\lambda_k(x_\pm)\}$, where $\mathbb{R}_+=[0,+\infty)$, $\mathbb{R}_-=(-\infty,0]$ and $\lambda_k(x_\pm)$ stands for the Floquet exponent of the Poincaré map Φ_k corresponding to the point $x_\pm$.*

Proof. Fix $\rho\geqslant 0$. We replace $\hat{q}_0(\tau)$ by x_+ in Definition (4.6) and consider Eq. (4.5) with the matrix

$$\mathcal{A}_+(\tau,\varepsilon)=\varepsilon^{-1}\begin{pmatrix} 0 & I \\ -f(\tau)D^2V(x_+) & 0 \end{pmatrix}.$$

This modified equation will be referred to as $(4.5)_+$. If we use local coordinates $LC(x_+)$ and take $\varepsilon\in\mathcal{E}_h^+(\rho)$, then by Corollary 2 one may define a fundamental matrix $X(\tau)=\{z^s_{(1)},\ldots,z^s_{(n)},z^u_{(1)},\ldots,z^u_{(n)}\}$ and a projection P^s on $E^s(x_+,0)$ along $E^u(x_+,0)$, where $z^{s,u}_{(k)}(\tau)=(v^{s,u}_{(k)}(\tau),\varepsilon^{-1}v'^{s,u}_{(k)}(\tau))$ and $v^{s,u}_{(k)}$ are the Floquet solutions. Note that for such a defined fundamental matrix the projection P^s takes the form

$$P^s=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Taking into account the representation of the Floquet solutions via T-periodic functions $w^{s,u}_{(k)}$ we conclude that Eq. $(4.5)_+$ has an exponential dichotomy on $\mathbb{R}$ with constant $\alpha=\min\limits_k\{\lambda_k(x_+)\}$ and $K=\max\limits_k\|w^{s,u}_{(k)}\|$ with $\|\cdot\|$ standing for the supremum norm. Note that due to Lemma 2 the distance between $\hat{q}_0(\tau)$ and x_+ is of the order

$$|\hat{q}_0(\tau)-x_+|=O\left(\left((\tau-\tau_*)\varepsilon^{-2/m_*}\right)^{\frac{2-m_*}{4}}\mathrm{e}^{-\varepsilon^{-1}\sigma^+(\tau-\tau_*)^{\frac{m_*}{2}}}\right). \tag{4.10}$$

Hence, we may choose $\varepsilon_1^+>0$ such that for all $\varepsilon<\varepsilon_1^+$ and $\tau>\tau_*+\delta$ the point $\hat{q}(\tau)$ lies in the chart $LC(x_+)$. In local coordinates $LC(x_+)$ one may rewrite (4.5) in the form (4.7) and obtain that

$$\|\mathcal{A}(\tau,\varepsilon)-\mathcal{A}_+(\tau,\varepsilon)\|=O(\varepsilon^{-1}|\hat{q}_0(\tau)-x_+|).$$

Then due to (4.10) for any $\Delta>0$ there exists $\tau_+>0$ such that for any $\tau>\tau_+$

$$\|\mathcal{A}(\tau,\varepsilon)-\mathcal{A}_+(\tau,\varepsilon)\|<\Delta.$$

Using the roughness theorem from [9], we conclude that Eq. (4.5) has an exponential dichotomy on the interval $[\tau_+, +\infty)$ with exponent $\tilde{\alpha} = \alpha - 2K\Delta$. Since an exponential dichotomy on the interval $[\tau_+, +\infty)$ implies an exponential dichotomy on $\mathbb{R}_+$ with the same exponent α (see [9]) and due to an arbitrary smallness of Δ we prove that for any $\rho \geqslant 0$ and all $\varepsilon \in (0, \varepsilon_1^+) \cap \mathcal{E}_h^+(\rho)$ Eq. (4.7) has an exponential dichotomy on $\mathbb{R}_+$ for any constant $\alpha < \min\limits_k \{\lambda_k(x_+)\}$.

Arguing in a similar way, one may prove that there exists $\varepsilon_1^- > 0$ such that for any $\rho \geqslant 0$ and all $\varepsilon \in (0, \varepsilon_1^-) \cap \mathcal{E}_h^-(\rho)$ Eq. (4.5) has an exponential dichotomy on $\mathbb{R}_-$ for any constant $\alpha < \min\limits_k \{\lambda_k(x_-)\}$. Combining these results and taking $\varepsilon_1 = \min\{\varepsilon_1^+, \varepsilon_1^-\}$, one finishes the proof.

□

Let $X_\pm(\tau)$ and $P_\pm^s$ stand for the fundamental matrix and the projection corresponding to the exponential dichotomy of (4.5) on $\mathbb{R}_\pm$. Then one may define the stable and unstable subspaces $E_\pm^{s,u}(\tau)$ as

$$E_\pm^s(\tau) = im\big(X_\pm(\tau)P_\pm^s X_\pm^{-1}(\tau)\big), \quad E_\pm^u(\tau) = ker\big(X_\pm(\tau)P_\pm^s X_\pm^{-1}(\tau)\big).$$

Let $\Psi(\tau, \tau_0)$ stand for the evolution operator of Eq. (4.5), i. e., $z : \tau \mapsto \Psi(\tau, \tau_0)(z_0)$ is the unique solution of (4.5) satisfying the initial condition $z(\tau_0) = z_0$. Since (4.5) has an exponential dichotomy on $\mathbb{R}_\pm$, it follows from [9] that Eq. (4.5) has an exponential dichotomy on $\mathbb{R}$ iff $\Psi(\tau_* + \delta, \tau_* - \delta)E_-^u(\tau_* - \delta)$ is transverse to $E_+^s(\tau_* + \delta)$.

To analyze the transversality condition, we consider Eq. (4.4) on the interval $[\tau_* - \delta, \tau_* + \delta]$. If δ is sufficiently small, it can be approximated by

$$D_\zeta v'_\zeta + \zeta^{\varkappa_*} g_*(\tau_*) H^V\big(q_0(\zeta)\big)v = 0, \tag{4.11}$$

where $\zeta \in [-\delta\varepsilon^{-2/m_*}, \delta\varepsilon^{-2/m_*}]$. Let us define the evolution operator $\Psi_0(\zeta, \zeta_0)$ associated with (4.11), i. e., $v : \zeta \mapsto \Psi_0(\zeta, \zeta_0)(v_0, v'_0)$ is the unique solution of (4.11) satisfying $v(\zeta_0) = v_0, v'(\zeta_0) = v'_0$.

Lemma 4. *If $(v_0, v'_0) \notin T_{(q_0(\zeta_0), q'_0(\zeta_0))}\mathcal{W}^s(x_+, \zeta_0)$, then the solution $v(\zeta) = \Psi_0(\zeta, \zeta_0)(v_0, v'_0)$ satisfies*

$$|v(\zeta)| > C(v_0, v'_0)\zeta^{\frac{2-m_*}{4}} \mathrm{e}^{\sigma^+ \zeta^{\frac{m_*}{2}}}, \quad \zeta \to +\infty,$$

where σ^+ is an arbitrary constant such that $0 < \sigma^+ < \sigma^+_{\min}$ and $\sigma^+_{\min}$ is defined in Lemma 2.

Proof. Since $q_0(\zeta) \to x_+$ as $\zeta \to +\infty$, we may take ζ_0 to be sufficiently large and use local coordinates $LC(x_+)$. Then introducing new variables

$$\xi = \frac{2}{m_*}\zeta^{m_*/2}, \quad u = \zeta^{\frac{2-m_*}{4}} v, \tag{4.12}$$

rewrite (4.11) as

$$D_\xi u'_\xi + g_*(\tau_*) H^V\big(\tilde{q}_0(\xi)\big)u + \left(\frac{1}{4} - \frac{1}{m_*^2}\right)\frac{1}{\xi^2}u = 0, \tag{4.13}$$

where $\tilde{q}_0(\xi) = q_0\left(\left(\frac{m_*}{2}\xi\right)^{2/m_*}\right)$.

Letting $w = (u, u'_\xi)$, one may represent (4.13) as

$$w' = \mathcal{B}(\xi)w, \quad \mathcal{B}(\xi) = \mathcal{B}_+ + O(\xi^{-2}) + O(|\tilde{q}_0(\xi) - x_+|), \tag{4.14}$$

$$\mathcal{B}_+ = \begin{pmatrix} 0 & I \\ -g_*(\tau_*)D^2V(x_+) & 0 \end{pmatrix}.$$

Since $\mathcal{B}_+$ is a constant matrix with eigenvalues $\{\pm|g_*|^{1/2}\Lambda_k\}$, the equation $w' = \mathcal{B}_+ w$ has an exponential dichotomy with exponent $\alpha = |g_*|^{1/2}\min_k\{\Lambda_k\}$. Due to (4.14) the difference $\mathcal{B}(\xi) - \mathcal{B}_+$ tends to zero as $\xi \to +\infty$. Hence, one can apply the roughness theorem [9] to show that (4.14) has an exponential dichotomy on $\mathbb{R}_+$ with any exponent $\tilde{\alpha} < |g_*|^{1/2}\min_k\{\Lambda_k\}$. Denote by $W(\xi)$ and $P_{\mathcal{B}}$ a fundamental matrix and an exponential dichotomy projection associated with Eq. (4.14). Then the condition $(v_0, v'_0) \notin T_{(q_0(\zeta_0),q_0'(\zeta_0))}\mathcal{W}^s(x_+,\zeta_0)$ of the lemma guarantees that the initial point $w(\xi_0) \notin im\big(W(\xi_0)P_{\mathcal{B}}W^{-1}(\xi_0)\big)$. Substituting $w(\xi)$ into (4.9) and using (4.12), we get the desired estimate of the lemma. $\square$

Remark. The similar estimate is valid as $\zeta \to -\infty$. Namely, if $(v_0, v'_0) \notin T_{(q_0(\zeta_0),q_0'(\zeta_0))}\mathcal{W}^u(x_-,\zeta_0)$, then the solution $v(\zeta) = \Psi_0(\zeta,\zeta_0)(v_0, v'_0)$ satisfies

$$|v(\zeta)| > C(v_0, v'_0)|\zeta|^{\frac{2-m_*}{4}}\mathrm{e}^{\sigma^-|\zeta|^{\frac{m_*}{2}}}, \quad \zeta \to -\infty,$$

where σ^- is an arbitrary constant such that $0 < \sigma^- < \sigma^-_{\min}$ and $\sigma^-_{\min}$ is defined in Lemma 2.

Denote by $\{\big(v^s_{(k)}(\zeta), {v'}^s_{(k)}(\zeta)\big)\}_{k=1}^n$ and $\{\big(v^u_{(k)}(\zeta), {v'}^u_{(k)}(\zeta)\big)\}_{k=1}^n$ two bases in $T_{(q_0(\zeta),q_0'(\zeta))}\mathcal{W}^s(x_+,\zeta)$ and $T_{(q_0(\zeta),q_0'(\zeta))}\mathcal{W}^u(x_-,\zeta)$, respectively. We also define $n\times n$ matrices

$$\mathcal{V}^s = (v^s_{(1)}, \ldots, v^s_{(n)}), \quad \mathcal{V}^u = (v^u_{(1)}, \ldots, v^u_{(n)}).$$

If the assumption (A_3) holds, then $T_{(q_0(\zeta),q_0'(\zeta))}T\mathcal{M} = T_{(q_0(\zeta),q_0'(\zeta))}\mathcal{W}^s(x_+,\zeta) \oplus T_{(q_0(\zeta),q_0'(\zeta))}\mathcal{W}^u(x_-,\zeta)$ and

$$\begin{aligned}
&\mathcal{V}^s(\zeta) = |\zeta|^{\frac{2-m_*}{4}}\mathrm{e}^{\sigma^-|\zeta|^{m_*/2}}B^{s,-}, && \mathcal{V}'^s(\zeta) = -|\zeta|^{\frac{m_*-2}{4}}\mathrm{e}^{\sigma^-|\zeta|^{m_*/2}}D^{s,-}, && \zeta \to -\infty\\
&\mathcal{V}^u(\zeta) = |\zeta|^{\frac{2-m_*}{4}}\mathrm{e}^{-\sigma^-|\zeta|^{m_*/2}}B^{u,-}, && \mathcal{V}'^u(\zeta) = |\zeta|^{\frac{m_*-2}{4}}\mathrm{e}^{-\sigma^-|\zeta|^{m_*/2}}D^{u,-}, && \zeta \to -\infty\\
&\mathcal{V}^s(\zeta) = |\zeta|^{\frac{2-m_*}{4}}\mathrm{e}^{-\sigma^+|\zeta|^{m_*/2}}B^{s,+}, && \mathcal{V}'^s(\zeta) = -|\zeta|^{\frac{m_*-2}{4}}\mathrm{e}^{-\sigma^+|\zeta|^{m_*/2}}D^{s,+}, && \zeta \to +\infty\\
&\mathcal{V}^u(\zeta) = |\zeta|^{\frac{2-m_*}{4}}\mathrm{e}^{\sigma^+|\zeta|^{m_*/2}}B^{u,+}, && \mathcal{V}'^u(\zeta) = |\zeta|^{\frac{m_*-2}{4}}\mathrm{e}^{\sigma^+|\zeta|^{m_*/2}}D^{u,+}, && \zeta \to +\infty.
\end{aligned}$$

Here $n\times n$ matrices $B^{s,\pm}, B^{u,\pm}, D^{s,\pm}, D^{u,\pm}$ are nondegenerate and satisfy

$$\begin{aligned}
&\|B^{s,-}\| \geqslant \beta^{s,-}, \quad \|B^{u,-}\| \leqslant \beta^{u,-}, \quad \zeta \to -\infty \\
&D^{s,-} = \frac{m_*\sigma^-}{2}B^{s,-}\left(1+O\left(|\zeta|^{-m_*/2}\right)\right), \quad D^{u,-} = \frac{m_*\sigma^-}{2}B^{u,-}\left(1+O\left(|\zeta|^{-m_*/2}\right)\right), \quad \zeta \to -\infty\\
&\|B^{s,+}\| \leqslant \beta^{s,+}, \quad \|B^{u,+}\| \geqslant \beta^{u,+}, \quad \zeta \to +\infty\\
&D^{s,+} = \frac{m_*\sigma^+}{2}B^{s,+}\left(1+O\left(|\zeta|^{-m_*/2}\right)\right), \quad D^{u,+} = \frac{m_*\sigma^+}{2}B^{u,+}\left(1+O\left(|\zeta|^{-m_*/2}\right)\right), \quad \zeta \to +\infty,
\end{aligned} \tag{4.15}$$

where $\beta^{s,\pm}, \beta^{u,\pm}$ are some positive constants independent of ζ.

Represent (4.11) as the first-order equation by setting $y = (v, v'_\zeta)$:

$$y' = \mathcal{A}_0(\zeta)y, \tag{4.16}$$

$$\mathcal{A}_0(\zeta) = \begin{pmatrix} 0 & I \\ -\zeta^{\varkappa_*}g_*(\tau_*)H^V(q_0(\zeta)) & 0 \end{pmatrix}$$

and define a fundamental matrix $Y(\zeta)$ as

$$Y(\zeta) = \begin{pmatrix} \mathcal{V}^s(\zeta) & \mathcal{V}^u(\zeta) \\ \mathcal{V}'^s(\zeta) & \mathcal{V}'^u(\zeta) \end{pmatrix}. \tag{4.17}$$

Consider the evolution operator $\Psi_0(\zeta_0, -\zeta_0) = Y(\zeta_0)Y^{-1}(-\zeta_0) : T_{(q_0(-\zeta_0),q_0'(-\zeta_0))}T\mathcal{M} \to T_{(q_0(\zeta_0),q_0'(\zeta_0))}T\mathcal{M}$, associated with (4.16).

Lemma 5. *In local coordinates $LC(x_\pm)$ the operator $\Psi_0(\zeta_0, -\zeta_0)$ has the following asymptotics as $\zeta_0 \to +\infty$:*

$$\Psi_0(\zeta_0, -\zeta_0) = \mathrm{e}^{(\sigma^+ + \sigma^-)|\zeta_0|^{m_*/2}} \begin{pmatrix} G^u & \frac{2}{m_*\sigma^-}|\zeta_0|^{-\frac{m_*-2}{2}} G^u \\ \frac{m_*\sigma^+}{2}|\zeta_0|^{\frac{m_*-2}{2}} G^u & \frac{\sigma^+}{\sigma^-} G^u \end{pmatrix} \left(1 + O\left(|\zeta_0|^{-m_*/2}\right)\right),$$

where $G^u = B^{u,+}\left(B^{u,-}\right)^{-1}$.

Proof. The lemma immediately follows from (4.15), (4.16) and (4.18). □

Now take sufficiently small $\delta > 0$. In particular, let $\delta = O(\varepsilon^{\gamma/m_*})$ with some $0 < \gamma < 2$. Then we arrive at the following lemma.

Lemma 6. *For any positive constants δ_0 and $\gamma < 2$ define $\delta = \delta_0 \varepsilon^{\gamma/m_*}$. Then the evolution operator $\Psi(\tau_* - \delta, \tau_* + \delta)$ has the following representation as $\varepsilon \to 0$*

$$\Psi(\tau_* - \delta, \tau_* + \delta) = \mathrm{e}^{\frac{\sigma^+ + \sigma^-}{\varepsilon^{1-\gamma/2}}\delta_0^{\frac{m_*}{2}}} \begin{pmatrix} G^u & \frac{2\varepsilon^{1-\gamma\frac{m_*-2}{2m_*}}}{m_*\sigma^-}\delta_0^{-\frac{m_*-2}{2}} G^u \\ \frac{m_*\sigma^+}{2\varepsilon^{1-\gamma\frac{m_*-2}{2m_*}}}\delta_0^{\frac{m_*-2}{2}} G^u & \frac{\sigma^+}{\sigma^-} G^u \end{pmatrix} \left(1 + O\left(\varepsilon^{1-\frac{\gamma}{2}}\right) + O\left(\varepsilon^{\frac{\gamma}{2}}\right)\right).$$

Proof. Due to (A_2) one may rewrite (4.5) in terms of the variable $\zeta = (\tau - \tau_*)\varepsilon^{-2/m_*}$ as

$$D_\zeta u' + \zeta^{\varkappa_*} g_*(\tau_*)\left(1 + O(\zeta\varepsilon^{2/m_*})\right) D^2V(q_0(\zeta))u = 0. \tag{4.18}$$

Hence, if one puts $\zeta_0 = \delta_0 \varepsilon^{-\frac{2-\gamma}{m_*}}$, then on the interval $[-\zeta_0, \zeta_0]$ we get

$$\hat{\Psi}(-\zeta, \zeta) = \Psi_0(-\zeta, \zeta)\left(1 + O\left(\varepsilon^{\frac{\gamma}{m_*}}\right)\right),$$

where $\hat{\Psi}$ denotes the evolution operator of (4.19). Applying Lemma 2 and taking into account that $D_\tau = \varepsilon^{-2/m_*} D_\zeta$ yield the desired estimate. □

In accordance with (2.4) and (2.14), introduce variables

$$z^\pm = (z_1^\pm, z_2^\pm), \quad z_1^\pm = |f(\tau)|^{1/4} v^\pm, \quad z_2^\pm = \varepsilon^{-1}|f(\tau)|^{-1/4}\Lambda_\pm v'^\pm, \tag{4.19}$$

where $v^\pm$ denotes the local coordinates $LC(x_\pm)$ and $\Lambda_\pm = \mathrm{diag}\{\Lambda_1(x_\pm), \ldots, \Lambda_n(x_\pm)\}$.

In these variables the evolution operator Ψ takes the form

$$\Psi(\tau_* - \delta, \tau_* + \delta) = \mathrm{e}^{\frac{\sigma^+ + \sigma^-}{\varepsilon^{1-\gamma/2}}\delta_0^{\frac{m_*}{2}}} \begin{pmatrix} G^u & \frac{2}{m_*\sigma^-} G^u \Lambda_- \\ \frac{m_*\sigma^+}{2}\Lambda_+^{-1} G^u & \frac{\sigma^+}{\sigma^-}\Lambda_+^{-1} G^u \Lambda_- \end{pmatrix} \left(1 + O\left(\varepsilon^{1-\frac{\gamma}{2}}\right) + O\left(\varepsilon^{\frac{\gamma}{2}}\right)\right) \tag{4.20}$$

Since $q_0(\zeta) \to x_\pm$ as $\zeta \to \pm\infty$, the invariant subspaces $E_-^u(\tau_* - \delta)$ and $E_+^s(\tau_* + \delta)$ become exponentially close to $TW^u(x_-; \tau_* - \delta)$ and $TW^s(x_+; \tau_* + \delta)$, respectively. But $TW^u(x_-; \tau_* - \delta)$ and $TW^s(x_+; \tau_* + \delta)$ expressed in terms of $z^\pm$ are the unstable and stable subspaces of the Poincaré map $\Phi_\pm(\tau_* \pm \delta)$ associated with the equilibria $x_\pm$, respectively. We also note that the map $\Phi_\pm$ is of the form

$$\Phi_\pm = \begin{pmatrix} \Phi_1^\pm & 0 & \ldots & 0 \\ 0 & \Phi_2^\pm & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Phi_n^\pm \end{pmatrix},$$

where $\Phi_k^\pm(\tau)$ are defined by (2.5), (2.15).

Let E be a subspace of a Hilbert space V. Denote by $C\mathcal{N}(E,\kappa)$ a cone neighborhood of E, i. e.,

$$CN(E,\kappa) = \{v \in V : |\angle(v,E)| < \kappa\},$$

where $\angle(v,E)$ denotes the angle between v and E.

Define

$$\mathcal{N}_- = ker \begin{pmatrix} I & \frac{2}{m_*\sigma^-}\Lambda_- \\ I & \frac{2}{m_*\sigma^-}\Lambda_- \end{pmatrix}, \quad \mathcal{R}_+ = im \begin{pmatrix} I & -\frac{2}{m_*\sigma^+}\Lambda_+ \\ I & -\frac{2}{m_*\sigma^+}\Lambda_+ \end{pmatrix},$$

Lemma 7. *Assume $TW^u(x_-;\tau_*-\delta)$, $TW^s(x_+;\tau_*+\delta)$ satisfy*

$$z^u \notin CN(\mathcal{N}_-,\kappa), \quad \forall z^u \in TW^u(x_-;\tau_*-\delta),$$

$$z^s \notin CN(\mathcal{R}_+,\kappa), \quad \forall z^s \in TW^s(x_+;\tau_*+\delta),$$

$$\kappa = \kappa_0 \max\{\varepsilon^{1-\frac{\gamma}{2}}, \varepsilon^{\frac{\gamma}{2}}\}$$

for some constant $\kappa_0 > 0$. Then $\Psi(\tau_*-\delta,\tau_*+\delta)E^u_-(\tau_*-\delta)$ is transversal to $E^s_+(\tau_*+\delta)$.

Proof. The first condition of the lemma guarantees that one may use the leading term of (4.20) and $TW^u(x_-;\tau_*-\delta)$ to control the image $\Psi(\tau_*-\delta,\tau_*+\delta)E^u_-(\tau_*-\delta)$. The second condition of the lemma implies the transversality. □

Remark. It has to be noted that Lemma 7 gives sufficient conditions for transversality of the subspaces $\Psi(\tau_*-\delta,\tau_*+\delta)E^u_-(\tau_*-\delta)$ and $E^s_+(\tau_*+\delta)$ for small ε in terms of the stable $E^s(x_+,\tau_*+\delta) = TW^s(x_+;\tau_*+\delta)$ and unstable $E^u(x_-,\tau_*-\delta) = TW^s(x_-;\tau_*-\delta)$ subspaces (see Corollary 2) associated to the equilibria $x_\pm$. They can be analyzed by means of the Poincaré map $\Phi_\pm$ and its representation (2.14). Note that (2.14) contains information on all turning points. It follows from the estimate (2.24) that the invariant subspaces $E^s(x_+,\tau_*+\delta), E^u(x_-,\tau_*-\delta)$ rotate rapidly as ε approaches the boundary of $\mathcal{E}^\pm_h(0)$. In contrary, the subspaces $\mathcal{N}_-, \mathcal{R}_+$ are independent of ε. Hence, one may conclude that the subset $\mathcal{E}_{tr}(x_-,x_+) \subset \mathcal{E}^\pm_h(0)$ of those values of the parameter ε for which the conditions of Lemma 7 hold is nonempty. In Section 5 we discuss the case when the function $f(\tau)$ has two simple turning points. In this case the asymptotics of the Poincaré map $\Phi_\pm$ is constructed using (2.14)–(2.20) and it is shown that the conditions of Lemma 7 are valid for all $\varepsilon \in \mathcal{E}^\pm_h(0)$. We point out that in the general case, i. e., for an arbitrary set of turning points, it seems cumbersome to get the asymptotics of the Poincaré map.

As an immediate consequence of this lemma we get

Corollary 4. *Provided that the conditions of Lemma 7 are fulfilled for some positive κ_0, Eq. (4.5) has an exponential dichotomy on $\mathbb{R}$ with any exponent $\alpha < \min_k\{\lambda_k(x_\pm)\}$.*

We now state the following

Proposition 2. *For any positive constants ρ, κ_0, $\gamma < 2$ and any $q_0 \in Q^{tr}_h(x_-,x_+,\tau_*)$ there exists a positive constant ε_1 such that for all $\varepsilon \in (0,\varepsilon_1) \cap \mathcal{E}^+_h(\rho) \cap \mathcal{E}^-_h(\rho) \cap \mathcal{E}_{tr}(x_-,x_+)$ the system (1.3) possesses a doubly asymptotic trajectory connecting x_- and x_+ which stays in $\varepsilon^{\gamma\frac{m_*-1}{m_*}}$-neighborhood of the curve q_0.*

Proof. To prove Proposition 2, represent (4.3) as

$$D_\tau z = \mathcal{A}(\tau,\varepsilon)z + h(z,\tau), \quad h(z,\tau) = h_1(\tau) + h_2(z,\tau), \tag{4.21}$$

with the matrix (4.6) and

$$h_1(\tau) = \begin{pmatrix} 0 \\ -\left(f(\tau) - g_*(\tau_*)(\tau-\tau_*)^{\varkappa_*}\right)\nabla V(\hat{q}_0) \end{pmatrix}, \tag{4.22}$$

$$h_2(z,\tau) = \begin{pmatrix} 0 \\ -f(\tau)\left(\nabla V(\hat{q}_0 + z_1) - \nabla V(\hat{q}_0) - D^2V(\hat{q}_0)z\right) \end{pmatrix}. \tag{4.23}$$

Denote by $\mathcal{BC}$ the Banach space of bounded and continuous functions $\mathcal{BC} = \{z : \mathbb{R} \to T(T\mathcal{M})\}$ endowed with the norm

$$\|z\| = \sup_{\tau\in\mathbb{R}} |z(\tau)|.$$

Since (4.5) possesses an exponential dichotomy on $\mathbb{R}$, one may rewrite (4.21) as

$$z^s(\tau) = \Psi(\tau,\tau_0)z^s(\tau_0) + \int_{\tau_0}^{\tau} \Psi(\tau,s)\mathcal{P}(s)h(z(s),s)\mathrm{d}s,$$

$$z^u(\tau) = \Psi(\tau,\tau_0)z^u(\tau_0) + \int_{\tau_0}^{\tau} \Psi(\tau,s)(I-\mathcal{P}(s))h(z(s),s)\mathrm{d}s,$$

where $z^s(\tau) = \mathcal{P}(\tau)z(\tau)$, $z^u(\tau) = \big(I - \mathcal{P}(\tau)\big)z(\tau)$ and $\mathcal{P}(\tau) = X(\tau)P^sX^{-1}(\tau)$ is a projection-valued function associated with the exponential dichotomy of (4.5) on $\mathbb{R}$, which is guaranteed by Corollary 4. One may show that $\mathcal{P}(\tau)$ is invariant with respect to the evolution Ψ, i. e., $\mathcal{P}(\tau)\Psi(\tau,s) = \Psi(\tau,s)\mathcal{P}(s)$ [9]. Note that $|z(\tau)|$ is bounded as $\tau \to \pm\infty$. Together with (4.8), (4.9) this leads to

$$z^s(\tau) = \int_{-\infty}^{\tau} \Psi(\tau,s)\mathcal{P}(s)h\big(z(s),s\big)\mathrm{d}s,$$

$$z^u(\tau) = -\int_{\tau}^{\infty} \Psi(\tau,s)\big(I-\mathcal{P}(s)\big)h\big(z(s),s\big)\mathrm{d}s.$$

Then (4.21) takes the form

$$z(\tau) = \int_{-\infty}^{\tau} \Psi(\tau,s)\mathcal{P}(s)h\big(z(s),s\big)\mathrm{d}s - \int_{\tau}^{\infty} \Psi(\tau,s)\big(I-\mathcal{P}(s)\big)h\big(z(s),s\big)\mathrm{d}s.$$

In a vicinity of $x_\pm$ the potential V admits an estimate $\nabla V(x) = O(|x - x_\pm|)$. Hence, Lemma 2 together with (A_2) yields

$$|h_1(\tau)| < C_1\varepsilon^{\frac{(2+3\gamma)m_*-4-6\gamma}{4m_*}} \mathrm{e}^{-\sigma\delta_0^{\frac{m_*}{2}}\varepsilon^{-(1-\gamma/2)}}, \quad |\tau-\tau_*| > \delta_0\varepsilon^{\gamma/m_*},$$

for some positive constant C_1, $0 < \gamma < 2$ and sufficiently small ε. On the other hand, if τ is close to τ_*, we get for some constant $C_2 > 0$

$$|h_1(\tau)| < C_2\delta_0^{m_*-1}\varepsilon^{\gamma\frac{m_*-1}{m_*}}, \quad |\tau-\tau_*| < \delta_0\varepsilon^{\gamma/m_*}.$$

Then $h_1 \in \mathcal{BC}$ and there exists a positive constant C_3 independent of ε such that

$$\|h_1\| \leqslant C_3\varepsilon^{\gamma\frac{m_*-1}{m_*}}. \tag{4.24}$$

Besides, it follows from (4.23) that for sufficiently small $\|z\|$

$$|h_2(z,\tau)| < C_4|z(\tau)|^2 \tag{4.25}$$

with some constant $C_4 > 0$.

Define a map $\mathcal{F} : \mathcal{BC} \to \mathcal{BC}$ according to

$$\mathcal{F} : z \mapsto \int_{-\infty}^{\tau} \Psi(\tau,s)\mathcal{P}(s)h\big(z(s),s\big)\mathrm{d}s - \int_{\tau}^{\infty} \Psi(\tau,s)\big(I-\mathcal{P}(s)\big)h\big(z(s),s\big)\mathrm{d}s. \tag{4.26}$$

The estimate (4.25) implies

$$\|\mathcal{F}(0)\| \leqslant \sup_{\tau\in\mathbb{R}}\left(\int_{-\infty}^{\tau}|\Psi(\tau,s)\mathcal{P}(s)h_1(s)|\mathrm{d}s+\int_{\tau}^{\infty}|\Psi(\tau,s)\big(I-\mathcal{P}(s)\big)h_1(s)|\mathrm{d}s\right)\leqslant 2K\alpha^{-1}C_3\varepsilon^{\gamma\frac{m_*-1}{m_*}},$$

where K, α are the parameters of exponential dichotomy associated to (4.5).

Denote by $\mathcal{BC}_r$ the closed ball of radius r in $\mathcal{BC}$ centered at 0. Then (4.25), (4.26) imply for sufficiently small r

$$\|\mathcal{F}(z)-\mathcal{F}(0)\| \leqslant K\alpha^{-1}C_4\|z\|^2$$

and all $z \in \mathcal{BC}_r$. This shows that $\mathcal{F}$ is a contraction on $\mathcal{BC}_r$ whenever

$$\varepsilon^{\gamma\frac{m_*-1}{m_*}} < \frac{\alpha}{4KC_3}r, \quad r < \frac{\alpha}{4KC_4}. \tag{4.27}$$

Thus, we conclude that if for some $\kappa_0 > 0$ the conditions of Lemma 7 are valid, then for any $\rho > 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0) \cap \mathcal{E}_h^+(\rho) \cap \mathcal{E}_h^-(\rho)$ inequalities (4.27) hold and there exists a unique fixed point z_ε of $\mathcal{F}$. This finishes the proof of Proposition 2. □

Finally, we consider the case when two fixed equilibria $x_\pm \in X_c$ satisfy $x_\pm \notin X_\pm^l$ for all turning points $\tau_l, l = 1, \ldots, L$. This may occur only if $\mathcal{T}_\pm \neq \emptyset$ and both points $x_\pm \in X_{\max}$ (or $x_\pm \in X_{\min}$). Then there exists a sequence of turning points $\tau_{l_1} < \ldots < \tau_{l_M}$ such that $x_- \in X_-^{l_1}$, $x_+ \in X_+^{l_M}$ and $x_\pm \notin X_\pm^{l_k}, 2 \leqslant k \leqslant M-1$. Besides, one may take $x_0 \in X_c$ satisfying $x_0 \in X_-^{l_2}, x_0 \in X_+^{l_{M-1}}$ (it also follows that $x_0 \in X_\pm^{l_k}, 2 \leqslant k \leqslant M-1$). Assume that $Q_h^{tr}(x_-, x_0, \tau_{l_1})$ and $Q_h^{tr}(x_0, x_+, \tau_{l_M})$ are nonempty. Then for any $q_- \in Q_h^{tr}(x_-, x_0, \tau_{l_1})$ and $q_+ \in Q_h^{tr}(x_0, x_+, \tau_{l_M})$ define

$$\check{q}_0(\tau) = \begin{cases} q_-\left((\tau-\tau_{l_1})\varepsilon^{-2/m_{l_1}}\right), & \tau \leqslant \tau_{mid}-\delta, \\ \omega(\tau,\varepsilon), & |\tau-\tau_{mid}| < \delta, \\ q_+\left((\tau-\tau_{l_M})\varepsilon^{-2/m_{l_M}}\right), & \tau \geqslant \tau_{mid}+\delta, \end{cases} \tag{4.28}$$

where τ_{mid} is any middle point $\tau_{mid} \in (\tau_{l_1}, \tau_{l_M})$ such that $\tau_{mid} \neq \tau_l$ for all $l = 1, \ldots, L$ and δ is a small positive constant such that the interval $(\tau_{mid}-\delta, \tau_{mid}+\delta)$ does not contain the turning points. The function $\omega(\tau, \varepsilon)$ is a solution of the Lagrange equation (2.1) satisfying the following boundary conditions:

$$\omega(\tau,\varepsilon) = q_-\left((\tau-\tau_{l_1})\varepsilon^{-2/m_{l_1}}\right), \quad \tau = \tau_{mid}-\delta,$$
$$\omega(\tau,\varepsilon) = q_+\left((\tau-\tau_{l_M})\varepsilon^{-2/m_{l_M}}\right), \quad \tau = \tau_{mid}+\delta.$$

The existence of such a function $\omega(\tau, \varepsilon)$ for sufficiently small ε follows from the theory of singular perturbed differential equations [27] and is similar to the Shilnikov lemma [10, 26] (see also [5]). Besides, one may prove [27] that $|\omega(\tau, \varepsilon) - x_0| = O(\varepsilon)$ as $\varepsilon \to 0$.

Then we get the following

Proposition 3. *For any positive constants ρ, κ_0, $\gamma < 2$ and any $q_- \in Q_h^{tr}(x_-, x_0)$, $q_+ \in Q_h^{tr}(x_0, x_+)$ there exists a positive constant ε_1 such that for all $\varepsilon \in (0, \varepsilon_1) \cap \mathcal{E}_h^+(\rho) \cap \mathcal{E}_h^0(\rho) \cap \mathcal{E}_h^-(\rho) \cap \mathcal{E}_{tr}(x_-, x_0) \cap \mathcal{E}_{tr}(x_0, x_+)$ the system (1.3) possesses a doubly asymptotic trajectory connecting x_- and x_+ which stays in $\varepsilon^{\gamma\frac{m_*-1}{m_*}}$-neighborhood of the curve $\check{q}_0$, where $m_* = \min\{m_{l_1}, m_{l_M}\}$, $\mathcal{E}_h^0(\rho) = \mathcal{E}_h(x_0, \rho)$.*

Proof. The proof of this proposition is similar to the proof of Proposition 2. Indeed, the conditions of the proposition guarantee that the variational Eq. (4.4) along the curve $\check{q}_0$ defined by (4.28) has an exponential dichotomy on $\mathbb{R}$ with any exponent $\alpha < \min_k\{\lambda_k(x_\pm), \lambda_k(x_0)\}$. Then applying the contraction principle to (4.2) (where $\hat{q}_0$ is replaced by $\check{q}_0$), one obtains the existence of a doubly asymptotic trajectory connecting x_- and x_+ in a small neighborhood of $\check{q}_0$. □

Remark. Following [5, 25], we call the trajectories constructed in Proposition 2 (Proposition 3) as one-bump (two-bump) orbits. In a similar way one may construct multibump orbits. Namely, if one takes a sequence of turning points $\tau_k, k = 1, \ldots, M$ (here we consider turning points not in $\mathbb{T}^1$ as above, but in $\mathbb{R}$), a sequence of equilibria $x_k, k = 0, \ldots, M$, satisfying $x_{k-1} \in X_-^k, x_k \in X_+^k$, and a sequence of heteroclinics $q_k \in Q_h^{tr}(x_{k-1}, x_k), k = 1, \ldots, M$, one may define

$$\check{q}_0(\tau) = \begin{cases} q_1\left((\tau-\tau_1)\varepsilon^{-2/m_1}\right), & \tau \leqslant \tau_{mid,1} + \delta, \\ q_k\left((\tau-\tau_k)\varepsilon^{-2/m_k}\right), & \tau_{mid,k-1} + \delta \leqslant \tau \leqslant \tau_{mid,k} - \delta, \quad 2 \leqslant k \leqslant M-1 \\ \omega_k(\tau, \varepsilon), & |\tau - \tau_{mid,k}| < \delta, \quad 2 \leqslant k \leqslant M-1 \\ q_{k+1}\left((\tau-\tau_{k+1})\varepsilon^{-2/m_{k+1}}\right), & \tau_{mid,k} + \delta \leqslant \tau \leqslant \tau_{mid,k+1} - \delta, \quad 2 \leqslant k \leqslant M-1 \\ q_M\left((\tau-\tau_M)\varepsilon^{-2/m_M}\right), & \tau \geqslant \tau_{mid,M} - \delta, \end{cases}$$

with functions $\omega_k(\tau, \varepsilon)$ solving the Lagrange equation (2.1) and satisfying

$$\omega_k(\tau, \varepsilon) = q_{k-1}\left((\tau - \tau_{k-1})\varepsilon^{-2/m_{k-1}}\right), \quad \tau = \tau_{mid,k} - \delta,$$
$$\omega_k(\tau, \varepsilon) = q_k\left((\tau - \tau_k)\varepsilon^{-2/m_k}\right), \quad \tau = \tau_{mid,k} + \delta,$$

where $\tau_{mid,k}$ are some intermediate points and $\delta > 0$ such that $(\tau_{mid,k} - \delta, \tau_{mid,k} + \delta)$ does not contain the turning points. Then arguing as in Propositions 2 and 3, one may prove the existence of a doubly asymptotic trajectory connecting x_0 and x_M which stays in a small neighborhood of $\check{q}_0$.

It is essential that multiplicity of such connecting orbits follows from arbitrariness of all considered sequences (of turning points, equilibria and heteroclinics).

5. AN EXAMPLE

To illustrate the results obtained, we consider a classical example of conservative systems with two degrees of freedom — the double mathematical pendulum. It consists of two masses m_1, m_2 attached to sequentially connected arms of lengths l_1 and l_2, respectively. The upper end of the first arm is fixed and the system is subjected to the action of the constant gravity force with acceleration g. Following [16], denote by φ_1, φ_2 the angles of deviation of the arms from the vertical axis and introduce parameters

$$\delta = \frac{m_2}{m_1}, \quad \varepsilon = \left(\frac{l_2}{l_1}\right)^{1/2}, \quad \nu = \left(\frac{E}{2m_1 g l_1}\right)^{1/2},$$

where E is the energy of the system. If one takes $x_1 = \varphi_1$, $x_2 = \varphi_2 - \varphi_1$ as coordinates of the system, then in the limit $\delta \to 0$, $\varepsilon \to 0$ the equation of motion takes the form

$$x_1'' + \sin x_1 = 0, \tag{5.1}$$

$$\varepsilon^2 x_2'' + (3\cos x_1 + 4\nu^2 - 2)\sin x_2 = 0. \tag{5.2}$$

The general solution of (5.2) is

$$x_1(\tau) = \begin{cases} 2\mathrm{arctan}\left(\nu\frac{\mathrm{sn}(\tau-\tau_0,\nu)}{\mathrm{dn}(\tau-\tau_0,\nu)}\right), & \nu < 1; \\ 2\mathrm{arctan}(\sinh(\tau-\tau_0) \quad \text{or} \quad \pi, & \nu = 1; \\ 2\mathrm{arctan}\left(\frac{\mathrm{sn}(\nu(\tau-\tau_0),\nu^{-1})}{\mathrm{cn}(\nu(\tau-\tau_0),\nu^{-1})}\right), & \nu > 1, \end{cases}$$

where $\mathrm{sn}(\tau,k), \mathrm{cn}(\tau,k), \mathrm{dn}(\tau,k)$ are the Jacobi elliptic functions of module k and τ_0 is an arbitrary constant. If we put $\varphi = x_2 - \pi/2$, Eq. (5.3) describes a Lagrangian system defined on $T\mathcal{M}\times\mathbb{R}^1$, $\mathcal{M} = \mathrm{S}^1$ with a Lagrangian

$$\mathcal{L}(\varphi,\varphi',\varepsilon) = \frac{1}{2}|\varphi'|^2 - f_\nu(\tau)\sin\varphi, \tag{5.3}$$

$$f_\nu(\tau) = \begin{cases} 6\mathrm{dn}^2(\tau,\nu) + 4\nu^2 - 5, & \nu < 1; \\ 6\cosh^{-2}(\tau) - 1 \quad \text{or} \quad -1, & \nu = 1; \\ 6\mathrm{cn}^2(\nu\tau,\nu^{-1}) + 4\nu^2 - 5, & \nu > 1. \end{cases}$$

We will refer to this system as *the reduced system*. Note that for $\nu \neq 1$ the factor f_ν is a T-periodic function with $T = 4K(\nu)$, where $K(\nu)$ is the elliptic integral of the first kind:

$$K(\nu) = \begin{cases} \int\limits_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-\nu^2\sin^2\theta}}, & \nu < 1; \\ \int\limits_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-\nu^{-2}\sin^2\theta}}, & \nu > 1. \end{cases}$$

If $\nu < \frac{1}{\sqrt{2}}$ or $\nu > \frac{\sqrt{5}}{2}$, the factor $f_\nu(\tau) > 0$ for all τ. In that case it was proved [16] by Fenichel's geometric singular perturbation theory (see, e. g., [13, 30]) that for sufficiently small ε the reduced system possesses a hyperbolic periodic orbit whose invariant manifolds intersect transversally. On the contrary, when $\frac{1}{\sqrt{2}} < \nu < \frac{\sqrt{5}}{2}$ the factor f_ν has two zeroes of multiplicity 1 (or one zero of multiplicity 2 if $\nu \in \{\frac{1}{\sqrt{2}}, \frac{\sqrt{5}}{2}\}$) and the singular perturbation theory cannot be applied to the reduced system.

In what follows we assume that $\nu \in \left[\frac{1}{\sqrt{2}}, \frac{\sqrt{5}}{2}\right] \setminus \{1\}$. In this case the set X_c defined by (1.4) consists of two points $\{\pm\pi/2\}$ if $\nu \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{5}}{2}\right) \setminus 1$ and of one point $\{\pi/2\}$ if $\nu \in \left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{5}}{2}\right\}$. The variational Eq. (2.2) around $\varphi = \pm\pi/2$ is

$$\varepsilon^2 v'' \mp f_\nu(\tau) v = 0.$$

Let $\tau_\pm$ be the turning points such that

$$f_\nu(\tau_\pm) = 0, \quad \tau_+ > 0, \quad \tau_- < 0, \quad |\tau_\pm| < 2K(\nu).$$

Then (2.23) reads

$$S_e^+ = \int\limits_{\tau_+ < |s| < 2K(\nu)} |f_\nu(s)|^{1/2}\mathrm{d}s, \quad S_h^+ = \int\limits_{|s|<\tau_+} |f_\nu(s)|^{1/2}\mathrm{d}s, \quad S_e^- = S_h^+, \quad S_h^- = S_e^+, \quad \gamma^\pm = \ln 2.$$

Using properties of the elliptic functions, one gets

$$f_\nu'(\tau_\pm) = \pm\left(\frac{2}{3}(1+4\nu^2)(2\nu^2-1)(5-4\nu^2)\right)^{1/2}.$$

Noting that the conditions $(A_1), (A_2)$ hold, we obtain the reference system associated to $\tau_\pm$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\zeta^2} + f_\nu(\tau_\pm)\zeta\cos\varphi = 0, \quad \zeta = (\tau - \tau_\pm)\varepsilon^{-2/3}. \tag{5.4}$$

Applying the results of [17] to the reference system (5.4), we arrive at

Proposition 4. *For any $m \in \mathbb{Z}$ there exists a doubly asymptotic trajectory $\varphi_m^\pm(\zeta)$ of the system (5.4) associated to the turning point $\tau_\pm$ such that $\lim_{\zeta\to-\infty}\varphi_m^\pm(\zeta) = \mp\pi/2 - m\pi$ and $\lim_{\zeta\to+\infty}\varphi_m^\pm(\zeta) = \pm\pi/2 + m\pi$, i.e., the trajectories $\varphi_m^\pm(\zeta)$ connect the points $\mp\pi/2 - m\pi$ with $\pm\pi/2 + m\pi$ via m full rotations.*

Due to symmetry of (5.4) with respect to $(\zeta, \varphi) \to (-\zeta, -\varphi)$ one may show that the functions $\varphi_m^{\pm}(\zeta)$ are odd. If one considers the variational Eq. (4.11) along the trajectory $\varphi_m^{\pm}(\zeta)$, it also can be proved that $\varphi_m^{\pm}(\zeta)$ is transversal at least for $m = 0$.

To verify the conditions of Lemma 7, we note first that one may easily obtain an expression for the Poincaré map $\Phi(\tau)$ corresponding to $\varphi = \pm\pi/2$. Indeed, take, for example, $\varphi = -\pi/2$. Then (2.14), (2.16), (2.19), (2.20) yield up to a factor $\big(1 + O(\varepsilon)\big)$

$$\Phi(\tau_- - \delta) = \Xi^{-1}(\tau_- - \delta)R(\pi/4)Z(a)R(b)Z(c)R(-\pi/4)\Xi(\tau_- - \delta), \tag{5.5}$$

where

$$a = \varepsilon^{-1}\int\limits_{\tau_- - \delta}^{\tau_-} |f_\nu(s)|\mathrm{d}s + \frac{\gamma_-}{2}, \quad b = \varepsilon^{-1}\int\limits_{\tau_-}^{\tau_+} |f_\nu(s)|\mathrm{d}s, \quad c = \varepsilon^{-1}\int\limits_{\tau_+}^{4K(\nu)+\tau_- - \delta} |f_\nu(s)|\mathrm{d}s + \frac{\gamma_-}{2}.$$

Hence, the stable and unstable subspaces $E^{s,u}(-\pi/2; \tau_- - \delta)$ in coordinates z^- (see (4.19)) are spanned by the vectors $z^{s,u}$:

$$z^{s,u} = R(\pi/4)\left(\begin{array}{l} \sin b\mathrm{e}^{c-a} \\ \cos b\sinh(a+c) \pm (\cos^2 b\sinh^2(a+c) - \sin^2 b)^{1/2} \end{array}\right)(1 + O(\varepsilon)), \tag{5.6}$$

where the sign $'+'$ corresponds to z^u and $'-'$ to z^s.

Let ψ denote the angle between the stable and unstable subspaces. Then

$$\cos^2\psi = \frac{\sin^2 b\left(\mathrm{e}^{2(c-a)} + 1\right)^2}{\sin^2 b\left(\mathrm{e}^{2(c-a)} + 1\right)^2 + 4\mathrm{e}^{2(c-a)}(\cos^2 b\cosh^2(a+c) - 1)}\big(1 + O(\varepsilon)\big). \tag{5.7}$$

One may see that ψ oscillates as $\varepsilon \to 0$. However, since $\Lambda_- = 1$, the first condition of Lemma 7 holds as it follows from (5.5), (5.6). In the same manner one may check that the second condition of Lemma 7 is also fulfilled. Thus, applying Proposition 2 we get

Proposition 5. *For any $m \in \mathbb{Z}$ there exists a doubly asymptotic trajectory $\hat{\varphi}_m^{\pm}(\tau)$ of the system* (5.3) *such that $\lim_{\tau\to-\infty}\hat{\varphi}_m^{\pm}(\tau) = \pm\pi/2$ and $\lim_{\zeta\to+\infty}\varphi_m^{\pm}(\tau) = \mp\pi/2$, which stays in $O(\varepsilon^{2/3})$-neighborhood of the curve $\varphi_0^{\pm}\big((\tau - \tau_\pm - 4mK(\nu))\varepsilon^{-2/3}\big)$.*

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