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Classical Perturbation Theory and Resonances in Some Rigid Body Systems

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Abstract—We consider the system of a rigid body in a weak gravitational field on the zero level set of the area integral and study its Poincaré sets in integrable and nonintegrable cases. For the integrable cases of Kovalevskaya and Goryachev – Chaplygin we investigate the structure of the Poincaré sets analytically and for nonintegrable cases we study these sets by means of symbolic calculations. Based on these results, we also prove the existence of periodic solutions in the perturbed nonintegrable system. The Chaplygin integrable case of Kirchhoff's equations is also briefly considered, for which it is shown that its Poincaré sets are similar to the ones of the Kovalevskaya case.

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1. INTRODUCTION

Let us consider a nearly integrable Hamiltonian system

$$\dot{y} = -\frac{\partial \mathscr{H}}{\partial x}, \quad \dot{x} = \frac{\partial \mathscr{H}}{\partial y}, \quad \mathscr{H} = \mathscr{H}_0(y) + \mu \mathscr{H}_1(x, y) + o(\mu).$$

Everywhere below we assume that $\mathscr{H}$ is an analytical function and is 2π-periodic in x. Classical perturbation theory allows one to formally present this system as follows:

$$\dot{u} = -\frac{\partial \mathscr{K}}{\partial v}, \quad \dot{v} = \frac{\partial \mathscr{K}}{\partial u}, \quad \mathscr{K} = \mathscr{K}_0(u) + \mu \mathscr{K}_1(u) + \mu^2 \mathscr{K}_2(u) + \ldots + \mu^m \mathscr{K}_m(u, v) + o(\mu^m).$$

Here $\mathscr{K}$ is 2π-periodic in v. However, this formal procedure may lead to the appearance of so-called resonant harmonics in $\mathscr{K}$, which are unbounded as action variables tend to the Poincaré sets (resonant surfaces) [1].

The structure of the Poincaré sets plays an important role in the integrability of Hamiltonian systems. In particular, if these sets are complex enough, then the system cannot have a full set of analytical first integrals.

In addition, information about resonant harmonics in $\mathscr{K}$ can be used to prove the existence of periodic solutions. To be more precise, it was proved [2] that a *resonant* torus of an integrable system may produce a pair of periodic solutions (one elliptic and one hyperbolic) as we add a small perturbation. In some sense, this result complements the KAM theorem on the preservation of most of the *nonresonant* tori.

Unfortunately, the classical perturbation methods generally lead to cumbersome calculations, which complicates their implementation.

In our work, we consider the Hamiltonian system of a dynamically symmetric rigid body in a weak gravitational field on the zero level set of the area integral. For this system we obtain recurrent

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relations for the components of the generating function formal series and for finding $\mathscr{K}_i$. It is shown that in the integrable Kovalevskaya case, the numbers of the resonant harmonics lie on four lines and in the Goryachev – Chaplygin case they are located on two lines.

Based on the results of symbolic calculations and considering $O(\mu^3)$ terms of the perturbed Hamiltonian, we show that there always exist periodic solutions in nonintegrable cases for μ small enough. We also study the resonance sets of the Chaplygin integrable case of Kirchhoff's equations and prove that the resonances lie on four lines, as in the Kovalevskaya case.

2. AUXILIARY RESULTS AND DEFINITIONS

Let us briefly recall classical perturbation theory [1–3], which we are going to use below. Let us have a Hamiltonian system with the Hamiltonian $\mathscr{H}_0 + \mu\mathscr{H}_1$, where $\mathscr{H} = \mathscr{H}_0(y)$ and $\mathscr{H}_1 = \mathscr{H}_1(x, y)$ are analytical functions and the variable y is defined in a domain in $\mathbb{R}^n$, x is the angle variable defined on a torus $\mathbb{T}^n$, and μ is a small parameter. In all systems considered below, $\mathscr{H}_0$ has the form $\mathscr{H}_0(y) = (a_{11}y_1^2 + a_{22}y_2^2)/2$, which is also assumed in Corollary 1.

The main idea of classical perturbation theory is to find a canonical transformation $y, x \mod 2\pi \mapsto u, v \mod 2\pi$ which transforms the Hamiltonian $\mathscr{H}_0 + \mu\mathscr{H}_1$ to the form $\mathscr{K}_0(u) + \mu\mathscr{K}_1(u) + \mu^2\mathscr{K}_2(u) + \ldots$. The generating function of the canonical transformation has the form

$$S = S_0(u, x) + \mu S_1(u, x) + \mu^2 S_2(u, x) + \ldots$$

It is also assumed that $S_0 = \sum_{i=1}^n u_i x_i$, i. e., the transformation is close to the identical one. The correspondence between the variables y, x and u, v is given as follows:

$$y_i = \frac{\partial S}{\partial x_i}, \quad v_i = \frac{\partial S}{\partial u_i}, \quad i = 1, \ldots, n.$$

The function S satisfies the Hamilton – Jacobi equation

$$\mathscr{H}_0\left(\frac{\partial S}{\partial x}\right) + \mu\mathscr{H}_1\left(x, \frac{\partial S}{\partial x}\right) = \mathscr{K}_0 + \mu\mathscr{K}_1(u) + \mu^2\mathscr{K}_2(u) + \ldots$$

This equation can be solved formally, i. e., one can find functions S_i satisfying it. After a finite number of steps of the perturbation procedure, the Hamiltonian takes the form

$$\mathscr{K} = \mathscr{K}_0(u) + \mu\mathscr{K}_1(u) + \ldots + \mu^{m-1}\mathscr{K}_{m-1}(u) + \mu^k\mathscr{K}_m(u, v) + o(\mu^m). \tag{2.1}$$

It is known that for μ small enough, the system with such a Hamiltonian has a periodic solution in a vicinity of a resonant torus of the unperturbed system, under additional conditions satisfied by the system. To be more precise, the following result holds:

Theorem 1. *Let $u = u^0$ be an invariant resonant torus of the unperturbed system ($\mu = 0$); let $\omega_1, \ldots, \omega_n$ be the frequencies of the unperturbed τ-periodic solution. We assume that $\omega_n \neq 0$. Consider the following function k:*

$$k(\lambda_1, \ldots, \lambda_{n-1}) = \frac{1}{\tau}\int_0^\tau \mathscr{K}_m(\omega_1 t + \lambda_1, \ldots, \omega_{n-1}t + \lambda_{n-1}, \omega_n t, u_1^0, \ldots, u_n^0)\, dt.$$

Let λ^0 be a nondegenerate critical point of the function k. Then for small values of $\mu \neq 0$ there exists an isoenergetically nondegenerate τ-periodic solution of the perturbed Hamiltonian system and this solution analytically depends on μ.

Remark 1. Nonzero characteristic exponents of the periodic solution corresponding to λ^0 analytically depend on $\sqrt{\mu}$ and it is possible to investigate linear stability of the solution. In particular, for $n = 2$, the solution is unstable if

$$\frac{\partial^2 k}{\partial\lambda^2}\Big|_{\lambda=\lambda^0} \cdot \left(\omega_1^2\frac{\partial^2\mathscr{K}_0}{\partial u_2^2} - 2\omega_1\omega_2\frac{\partial^2\mathscr{K}_0}{\partial u_1\partial u_2} + \omega_2^2\frac{\partial^2\mathscr{K}_0}{\partial u_1^2}\right) > 0.$$

Taking into account the periodicity of $\mathscr{K}$ in u, for $n = 2$, the function $\mathscr{K}_m(u, v)$ in (2.1) has the form

$$\mathscr{K}_m(u, v) = \sum_{(\tau_1,\tau_2)\in\mathbb{Z}^2} k_m^\tau(u) e^{i(\tau_1 v_1 + \tau_2 v_2)}.$$

Here values $k_m^\tau(u)$ can be nonzero for some u and, from the above theorem, we obtain the following result.

Corollary 1. *Let us consider the system with the Hamiltonian* (2.1)*. Let τ_1, τ_2 be a pair of coprime numbers and let these numbers belong to the convex hull of all indices for which $k_\tau^m(u^0) \neq 0$, for $\langle u^0, \tau\rangle = a_{11}u_1^0\tau_1 + a_{22}u_2^0\tau_2 = 0$. Suppose that the Hamiltonian $\mathscr{K}$ is defined on the invariant torus $u = u^0$ and this torus is nondegenerate and isoenergetically nondegenerate. Then for small $\mu \neq 0$ in the vicinity of this torus there exists a pair of isoenergetically nondegenerate periodic solutions one of which is linearly stable and the other is unstable.*

We now give a definition of a resonant harmonic of $\mathscr{K}_k(u, v)$. First, let the perturbation in the Hamiltonian $\mathscr{H}_0 + \mu\mathscr{H}_1$ be a trigonometric polynomial on the angle variables

$$\mathscr{H}_1 = \sum_{(\tau_1,\tau_2)\in\mathbb{Z}^2} h_\tau e^{i(\tau_1 v_1 + \tau_2 v_2)}. \tag{2.2}$$

Here $h_\tau = \bar{h}_{-\tau} \in \mathbb{C}$ are constants. It can be shown [4] that in this case the terms S_i of the generating function are calculated recursively as follows:

$$S_1^\tau = ih_\tau/\langle u, \tau\rangle, \quad S_m^\tau = \frac{1}{2i\langle u, \tau\rangle} \sum_{u+v=m} \sum_{\sigma+\delta=\tau} \langle \sigma, \delta\rangle S_u^\sigma S_v^\delta, \quad m > 1.$$

If S_m^τ is known, the function k_τ^m from (2.2) can be calculated from the relation

$$S_m^\tau = ik_\tau^m/\langle u, \tau\rangle, \quad \tau \neq 0. \tag{2.3}$$

From (2.3) we find that $k_m^\tau = k_m^\tau(u_1, u_2)$ is a rational function of u_1 and u_2. Moreover, its denominator is a product of the values of the form $\langle u, \tau\rangle$ for various τ. Therefore, we find that, for a given $\tau \neq 0$ and for all $u \neq 0$, $\langle \tau, u\rangle = 0$, one of the following cases occurs:

1. k_m^τ is zero,
2. k_m^τ is nonzero (for almost all u),
3. k_m^τ is not defined.

Definition 1. *For a given $\tau \neq 0$, we say that k_m^τ is a resonant harmonic in $\mathscr{K}_m$ if either case* 2 *or case* 3 *holds.*

Remark 2. This definition is consistent for a broader class of perturbations. In particular, it will be consistent for the systems considered below.

One can see (2.3) that, for a resonant harmonic k_m^τ, the function S_m^τ is not defined on the line $\langle \tau, u\rangle = 0$ and the canonical transformation S is not defined on the same line. Therefore, if we know the numbers of the resonant harmonics of the system, we can obtain the Poincaré sets of the system up to a set of measure zero.

As an illustration of the above definition, we present the numbers of the resonant harmonics of the system (Fig. 1) with a polynomial perturbation (2.2), where

$$h_{0,1} = h_{1,1} = h_{0,2} = -h_{0,-1} = -h_{-1,-1} = -h_{0,-2} = i.$$

Note that if the numbers $(\tau_1, \tau_2) = \tau$ are coprime and $k_i^\tau \equiv 0$ for $i < m$, and $k_m^\tau \not\equiv 0$, then k_m^τ is defined on the corresponding line. Moreover, if τ belongs to the convex hull of all resonant numbers (for all steps of the perturbation procedure up to m), then the Hamiltonian $\mathscr{K}$ is defined for $u \neq 0$, $\langle u, \tau\rangle = 0$ and Corollary 1 can be applied.

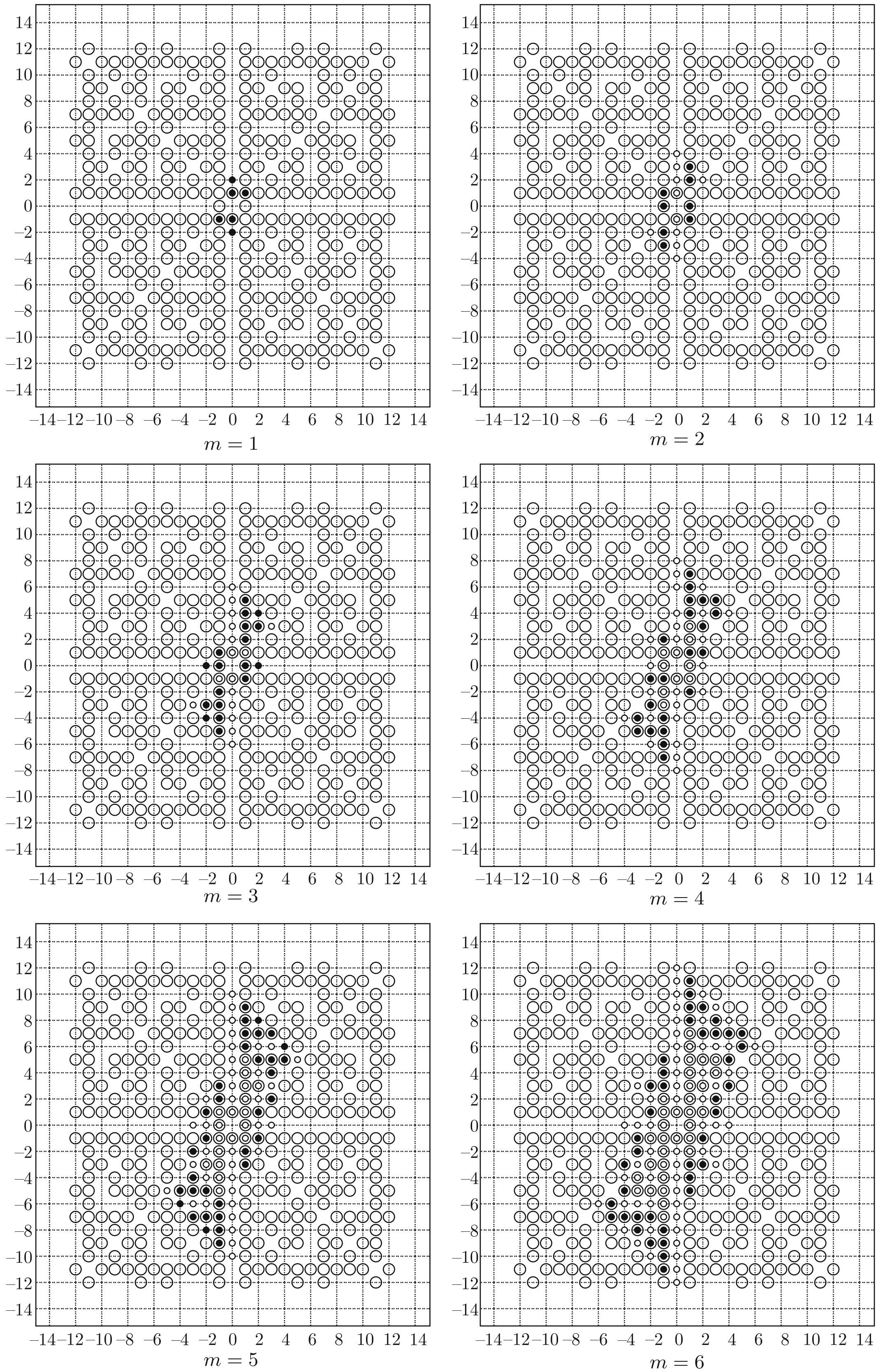

Fig. 1. Numbers of nonzero harmonics in $\mathscr{K}_m$ for the case of trigonometric polynomial perturbation ($\tau \neq 0$). Numbers τ of harmonics well defined on $\langle \tau, u \rangle = 0$, $u \neq 0$ are shown in color (it may be possible to apply Corollary 1 to these harmonics). Pairs of coprime number are also highlighted.

3. RIGID BODY IN A WEAK GRAVITATIONAL FIELD

3.1. Equations of Motion and Formal Solution of the Hamilton – Jacobi Equation

In the special canonical variables, the Hamiltonian of a dynamically symmetric rigid body with a fixed point in a weak gravitational field takes the form [5]

$$\begin{aligned}\mathscr{H} = \frac{1}{2A}G^2 + \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L^2 + \mu\Big[\frac{x}{r}\Big(\frac{H}{G}\sqrt{1-\frac{L^2}{G^2}}\sin l \\ + \frac{L}{G}\sqrt{1-\frac{H^2}{G^2}}\sin l\cos g + \sqrt{1-\frac{H^2}{G^2}}\cos l \sin g\Big) \\ + \frac{z}{r}\Big(\frac{LH}{G^2} - \sqrt{1-\frac{L^2}{G^2}}\sqrt{1-\frac{H^2}{G^2}}\cos g\Big)\Big].\end{aligned}$$

Here $(x, 0, z)$ are the coordinates of the center of mass in the principal axes of inertia, $r = \sqrt{x^2+z^2}$, and μ is a small parameter. Let $z = 0$ and $H = 0$, then the form of the Hamiltonian simplifies to

$$\mathscr{H} = \frac{1}{2}\Big(\frac{1}{C} - \frac{1}{A}\Big)L^2 + \frac{1}{2A}G^2 + \mu\Big[\frac{L}{G}\sin l\cos g + \cos l\sin g\Big].$$

Let us introduce the following notation: $1/C - 1/A = a_{11}$, $1/A = a_{22}$, $L = y_1$, $G = y_2$, $l = x_1$, $g = x_2$. The Hamiltonian $\mathscr{H}$ takes the form

$$\begin{aligned}\mathscr{H} = \frac{a_{11}}{2}y_1^2 + \frac{a_{22}}{2}y_2^2 + \frac{y_1}{y_2}\mu\frac{i}{4}\big(-e^{i(x_1+x_2)} - e^{i(x_1-x_2)} + e^{i(-x_1+x_2)} + e^{i(-x_1-x_2)}\big) \\ + \mu\frac{i}{4}\big(-e^{i(x_1+x_2)} + e^{i(x_1-x_2)} - e^{i(-x_1+x_2)} + e^{i(-x_1-x_2)}\big).\end{aligned} \tag{3.1}$$

In accordance with the classical perturbation procedure, we find a canonical transformation $y, x \mod 2\pi \to u, v \mod 2\pi$ which transforms $\mathscr{H}_0 + \mu\mathscr{H}_1$ to the form (2.1). The Hamilton – Jacobi equation can be represented as

$$\begin{aligned}\Big(u_2 + \mu\frac{\partial S_1}{\partial x_2} + \ldots\Big)\cdot\mathscr{H}_0\Big(u_1 + \mu\frac{\partial S_1}{\partial x_1} + \ldots, u_2 + \mu\frac{\partial S_1}{\partial x_2} + \ldots\Big) \\ + \mu\Big(u_1 + \mu\frac{\partial S_1}{\partial x_1} + \ldots\Big)\cdot\mathscr{H}_{11}(x_1,x_2) + \mu\Big(u_2 + \mu\frac{\partial S_1}{\partial x_2} + \ldots\Big)\cdot\mathscr{H}_{12}(x_1,x_2) \\ = \Big(u_2 + \mu\frac{\partial S_1}{\partial x_2} + \ldots\Big)\cdot\Big(\mathscr{K}_0 + \mu\mathscr{K}_1(u) + \mu^2\mathscr{K}_2(u) + \ldots\Big).\end{aligned} \tag{3.2}$$

Here, $y_1/y_2\cdot\mathscr{H}_{11} + \mathscr{H}_{12} = \mathscr{H}_1$. If we consider $m-1$ steps of the perturbation procedure, i. e., $S_i = 0$ for $i > m-1$, then $\mathscr{K}_m(u, x)$ may depend both on u and x. One can show that

$$\mathscr{K}_m(u,x) = \mathscr{K}_m(u) - \sum_\tau i\langle\tau, u\rangle S_m^\tau e^{i(\tau,x)}, \quad m > 0.$$

Here $\mathscr{K}_m(u)$ is the mth term in the Hamiltonian $\mathscr{K}$ when m steps of the perturbation procedure are done.

From (3.2), for $m = 0$, we obtain $\mathscr{H}_0 = \mathscr{K}_0$. Similarly, for $m = 1$ and $m = 2$, we have

$$\begin{aligned}\mathscr{K}_1 = \frac{u_1h_0^{11} + u_2h_0^{12}}{u_2} = 0, \quad S_1^\tau = -\frac{u_1h_\tau^{11} + u_2h_\tau^{12}}{iu_2\langle\tau,u\rangle}, \\ -u_2\mathscr{K}_2 = \sum_\xi\left(S_1^\xi S_1^{-\xi}(1/2\cdot u_2\langle-\xi,\xi\rangle + \xi_2\langle-\xi,u\rangle) - i(\xi_1S_1^\xi h_{-\xi}^{11} + \xi_2S_1^\xi h_{-\xi}^{12})\right), \\ u_2i\langle\tau,u\rangle S_2^\tau = \sum_{\xi+\eta=\tau}\left(S_1^\xi S_1^\eta(1/2\cdot u_2\langle\xi,\eta\rangle + \xi_2\langle\eta,u\rangle) - i(\xi_1S_1^\xi h_\eta^{11} + \xi_2S_1^\xi h_\eta^{12})\right).\end{aligned}$$

For an arbitrary $m > 2$, we obtain

$$
\begin{aligned}
-u_2\mathscr{K}_m &= \sum_{\xi+\eta=0}\sum_{\substack{n+k=m\\ n,k>0}}\left(\frac{1}{2}u_2\langle\xi,\eta\rangle+\xi_2\langle\eta,u\rangle\right)S_n^{\xi}S_k^{\eta} - i\sum_{\xi+\eta=0}S_{m-1}^{\xi}(\xi_1 h_\eta^{11}+\xi_2 h_\eta^{12})\\
&+\frac{i}{2}\sum_{\xi+\eta+\zeta=0}\sum_{\substack{n+k=m\\ n>0,k>1}}\sum_{\substack{k_1+k_2=k\\ k_1,k_2>0}}\xi_2\langle\eta,\zeta\rangle S_n^{\xi}S_{k_1}^{\eta}S_{k_2}^{\zeta},\\
u_2 i\langle\tau,u\rangle S_m^{\tau} &= \sum_{\xi+\eta=\tau}\sum_{\substack{n+k=m\\ n,k>0}}\left(\frac{1}{2}u_2\langle\xi,\eta\rangle+\xi_2\langle\eta,u\rangle\right)S_n^{\xi}S_k^{\eta} - i\sum_{\xi+\eta=\tau}S_{m-1}^{\xi}(\xi_1 h_\eta^{11}+\xi_2 h_\eta^{12})\\
&+\frac{i}{2}\sum_{\xi+\eta+\zeta=\tau}\sum_{\substack{n+k=m\\ n>0,k>1}}\sum_{\substack{k_1+k_2=k\\ k_1,k_2>0}}\xi_2\langle\eta,\zeta\rangle S_n^{\xi}S_{k_1}^{\eta}S_{k_2}^{\zeta}+\sum_{\substack{n+k=m\\ n,k>0}} i\tau_2 S_n^{\tau}\mathscr{K}_k.
\end{aligned}
\tag{3.3}
$$

These equations can be solved for any m and, in particular, it is seen that the definition of the resonant harmonics can be correctly applied to the system with the Hamiltonian (3.1).

3.2. Integrable Cases

The expressions obtained for S_i can be cumbersome. By means of symbolic calculations, for $A = 2$, $C = 1$ (the Kovalevskaya case), we obtain the following results for the first three steps of the perturbation procedure:

$$
\begin{aligned}
&S_1^{1,1} = S_1^{-1,-1} = S_1^{-1,1} = S_1^{1,-1} = \frac{1}{2u_2},\\
&S_2^{2,2} = \bar{S}_2^{-2,-2} = \frac{-i\cdot u_1}{8(u_1+u_2)u_2^3}, \quad S_2^{2,0} = \bar{S}_2^{-2,0} = \frac{i}{4u_1u_2^2}, \quad S_2^{2,-2} = \bar{S}_2^{-2,2} = \frac{i\cdot u_1}{8(u_1-u_2)u_2^3},\\
&S_2^{0,2} = \bar{S}_2^{0,-2} = \frac{-i}{4u_2^3},\\
&S_3^{3,3} = S_3^{-3,-3} = \frac{-2u_1+u_2}{24u_2^5(u_1+u_2)}, \quad S_3^{3,1} = S_3^{-3,-1} = S_3^{1,1} = S_3^{-1,-1} = \frac{1}{8u_2^4(u_1+u_2)},\\
&S_3^{-3,1} = S_3^{3,-1} = S_3^{1,-1} = S_3^{-1,1} = \frac{-1}{8u_2^4(u_1-u_2)}, \quad S_3^{3,-3} = S_3^{-3,3} = -\frac{2u_1+u_2}{24u_2^5(u_1-u_2)},\\
&S_3^{1,3} = S_3^{-1,-3} = S_3^{1,-3} = S_3^{-1,3} = \frac{-2u_1+u_2}{8u_2^5(u_1-u_2)}.
\end{aligned}
$$

Taking into account (3.2), we note that k_m^{τ} can be zero on the line $\langle u,\tau\rangle = 0$, while $S_m^{\tau} \not\equiv 0$. Similarly, the following results are obtained for the Goryachev – Chaplygin case:

$$
\begin{aligned}
&S_1^{1,1} = S_1^{-1,-1} = \frac{u_1+u_2}{u_2\,(3u_1+u_2)}, \quad S_1^{1,-1} = S_1^{-1,1} = \frac{u_1-u_2}{u_2\,(3u_1-u_2)},\\
&S_2^{2,2} = \bar{S}_2^{2,2} = \frac{i(2u_2\,(-u_1-u_2)\,(3u_1+u_2)+2u_2\,(u_1+u_2)^2+(u_1+u_2)^2\,(3u_1+u_2))}{2u_2^3\,(3u_1+u_2)^3},\\
&S_2^{2,-2} = \bar{S}_2^{-2,2} = \frac{i\,(u_1-u_2)\,(2u_2\,(u_1-u_2)-2u_2\,(3u_1-u_2)-(u_1-u_2)\,(3u_1-u_2))}{2u_2^3\,(3u_1-u_2)^3}.
\end{aligned}
\tag{3.4}
$$

Based on the results of symbolic calculations, we can conclude that, in the integrable systems considered, all nonzero resonant harmonics in $\mathscr{K}_m(u,x)$ lie on the four lines for $m \leqslant 7$. The numbers of the resonant harmonics are presented above (the explicit expressions for k_m^{τ} are cumbersome and omitted). Moreover, the following proposition holds.

Proposition 1. *For the Kovalevskaya case, the numbers τ of nonzero resonant harmonics $k_m^{\tau}(u) \not\equiv 0$ of $\mathscr{K}_m(u,v)$ can lie only on the four lines: $\tau_1 = 0$, $\tau_2 = 0$, $\tau_1 = \pm\tau_2$.*

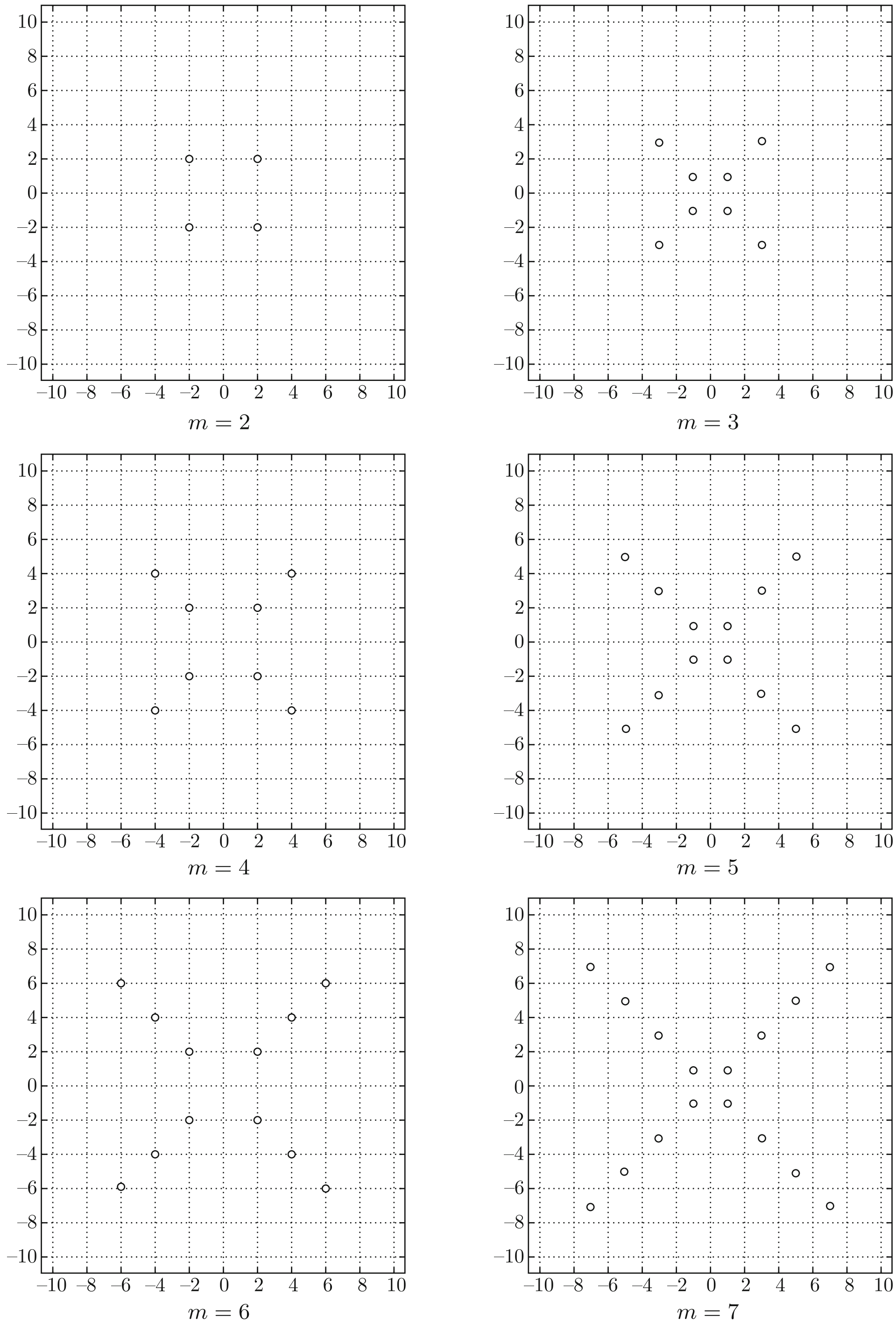

Fig. 2. Numbers of nonzero harmonics in $\mathscr{K}_m$ for the Goryachev – Chaplygin case $A = 4$, $C = 1$ ($\tau \neq 0$). Numbers τ of harmonics well defined on $\langle \tau, u \rangle = 0$, $u \neq 0$ are shown in color.

Proof. First, consider the case of an identical transformation, i.e., $S = S_0$. The Hamiltonian and the additional first integral have the form

$$\mathscr{K}(u, v) = \mathscr{K}_0(u) + \mu \mathscr{K}_1(u, v) + o(\mu),$$
$$\mathscr{F}(u, v) = \mathscr{F}_0(u) + \mu \mathscr{F}_1(u, v) + o(\mu).$$

Here, $\mathscr{K}_0(u) = (u_1^2 + u_2^2)/2$ and $\mathscr{F}_0(u) = (u_2^2 - u_1^2)^2/2$. From $\{\mathscr{F}, \mathscr{K}\} = 0$, we have $\{\mathscr{K}_0, \mathscr{F}_1\} = \{\mathscr{F}_0, \mathscr{K}_1\}$.

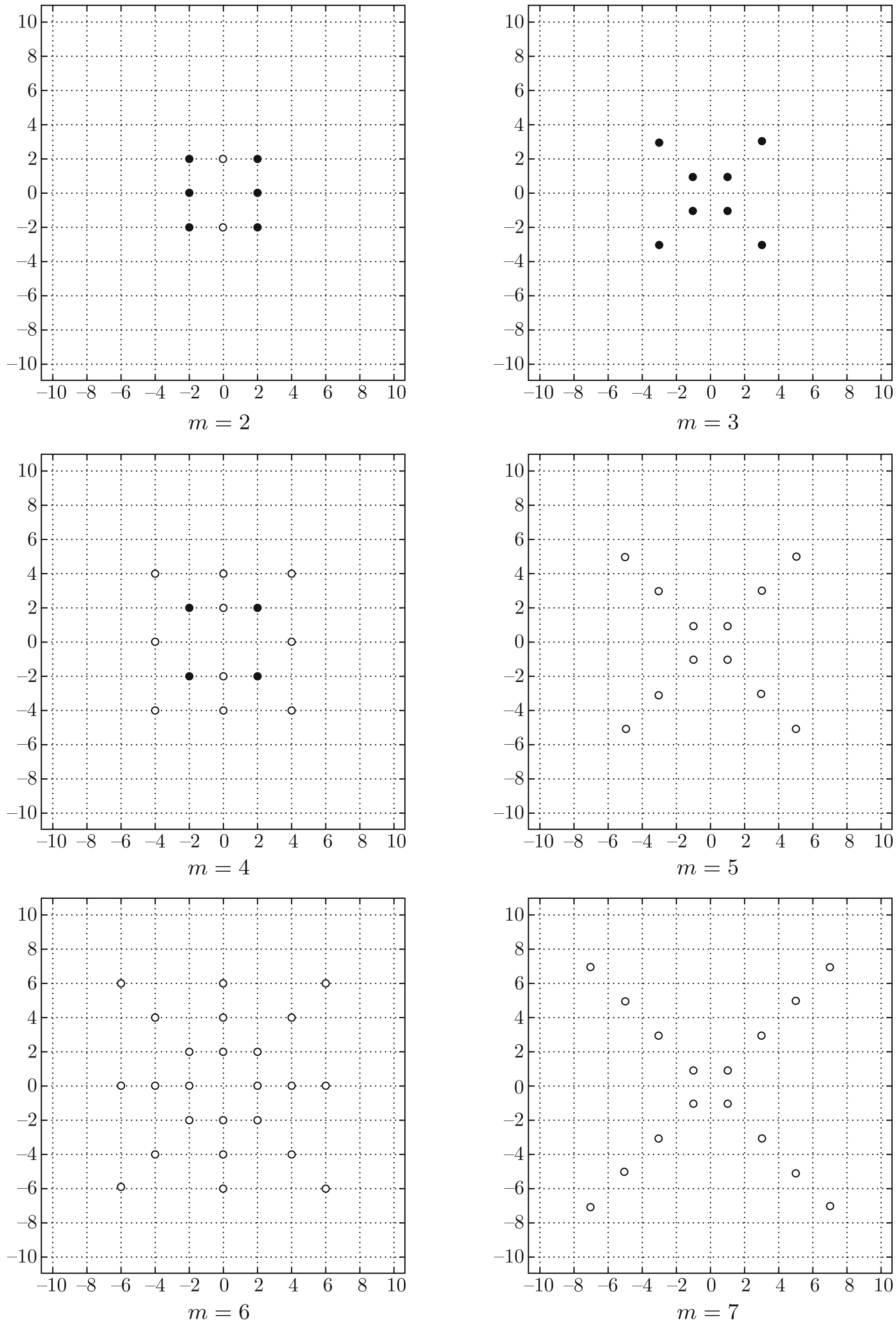

Fig. 3. Numbers of nonzero harmonics in $\mathscr{K}_m$ for the Kovalevskaya case $A = 2$, $C = 1$ ($\tau \neq 0$). Numbers τ of harmonics well defined on $\langle \tau, u \rangle = 0$, $u \neq 0$ are shown in color.

For $m = 1$, one easily finds that

$$f_m^\tau(u) = \frac{(u_2^2 - u_1^2)(\tau_1 u_1 - \tau_2 u_2)}{\tau_1 u_1 + \tau_2 u_2} k_m^\tau(u) = \lambda(u, \tau) k_m^\tau(u). \tag{3.5}$$

The function $\lambda(u, \tau)$ is defined for all u if either $\tau_1 = 0$ or $\tau_2 = 0$ or $\tau_1 = \pm\tau_2$. For other $(\tau_1, \tau_2) = \tau \neq 0$, the function $\lambda(u, \tau)$ is not defined on the line $\langle u, \tau \rangle = 0$. Suppose that τ is not on the four lines considered and for some nonzero u, $\langle u, \tau\rangle = 0$ we have $k_1^\tau(u) \neq 0$. Therefore, f_1^τ is not defined in u, which contradicts the existence of the analytical first integral.

Moreover, Eq. (3.5) holds for all $m > 1$. This follows from the fact that, for the kth step of the perturbation procedure, $\mathscr{F}_m = \mathscr{F}_m(u)$, $m < k$. Suppose that, up to the $(k-1)$th step, the resonant numbers lie on the four lines and, for the kth step, the number τ of a resonant harmonic k_k^τ does not belong to the lines considered. In this case, the function $f_k^\tau(u)$ is not defined on the line $\langle \tau, u\rangle = 0$. At the same time, the generating function is defined everywhere except on the four lines, i. e., the additional integral has to be defined for $u \neq 0$, $\langle \tau, u\rangle = 0$. The contradiction proves the result. □

A similar result holds for the Goryachev – Chaplygin case.

Lemma 1. *For the Goryachev – Chaplygin case, the numbers τ of nonzero resonant harmonics $k_m^\tau(u) \not\equiv 0$ of $\mathscr{K}_m(u, v)$ can lie only on the three lines: $\tau_1 = 0$, $\tau_1 = \pm\tau_2$.*

Based on this lemma, we prove the following

Proposition 2. *For the Goryachev – Chaplygin case, the numbers τ of nonzero resonant harmonics $k_m^\tau(u) \not\equiv 0$ of $\mathscr{K}_m(u, v)$ can lie only on the two lines $\tau_1 = \pm\tau_2$.*

Proof. For the first two steps, we have shown (3.4) that $S_1^\tau \not\equiv 0$ or $S_2^\tau \not\equiv 0$ only for $\tau_1 = \pm\tau_2$. Now we consider the case $m > 2$. For given τ and m, the function S_m^τ can be presented as the sum of four groups of terms (3.3). We will show that, for $\tau_1 = 0$, the sum of terms in each group equals zero. Without loss of generality, we assume here that $A = 4$ and $C = 1$.

Let us consider the first group of terms in (3.3). Taking into account the obvious equalities $\xi + \eta = \tau = \eta + \xi$ and $n + k = m = k + n$, we find that the first group of terms can be presented as a sum of the values $\Sigma_1^i \cdot C_1^i$, where C_1^i are some numbers and Σ_1^i takes the form

$$\begin{aligned}
\Sigma_1^i &= u_2\langle \xi, \eta\rangle + \xi_2\langle \eta, u\rangle + \eta_2\langle \xi, u\rangle \\
&= u_2\left(\frac{3}{4}\xi_1\eta_1 + \frac{1}{4}\xi_2\eta_2\right) + \xi_2\left(\frac{3}{4}\eta_1 u_1 + \frac{1}{4}\eta_2 u_2\right) + \eta_2\left(\frac{3}{4}\xi_1 u_1 + \frac{1}{4}\xi_2 u_2\right) \\
&= \frac{3}{4}(u_2\xi_2\eta_2 + u_2\xi_1\eta_1 + u_1\xi_2\eta_1 + u_1\xi_1\eta_2) = 0.
\end{aligned}$$

In the last equality we use the fact that $\xi_1 = \xi_2 = -\eta_1 = \eta_2$.

Now we consider the second group. Note that h_η^{11} and h_η^{12} are nonzero only for four values of η. Therefore, for $\tau_1 = 0$ and $\tau_2 \neq 0$, the sum $\xi + \eta$ is equal to either $(0, 2)$ or $(0, -2)$, i. e., there are four possibilities

$$\begin{aligned}
&\xi = (1, 1), \quad \eta = (-1, 1), && \xi = (-1, 1), \quad \eta = (1, 1), \\
&\xi = (-1, -1), \quad \eta = (1, -1), && \xi = (1, -1), \quad \eta = (-1, -1).
\end{aligned}$$

It is not hard to show that in all these cases the term $\xi_1 h_\eta^{11} + \xi_2 h_\eta^{12}$ equals zero.

Similarly to the first group, one can prove that the third group can be presented as a sum of the values $\Sigma_3^i \cdot C_3^i$, where C_3^i are some numbers and Σ_3^i takes the form

$$\begin{aligned}
\Sigma_3^i &= \xi_2\langle \eta, \zeta\rangle + \eta_2\langle \zeta, \xi\rangle + \zeta_2\langle \xi, \eta\rangle \\
&= \xi_2\left(\frac{3}{4}\eta_1\zeta_1 + \frac{1}{4}\eta_2\zeta_2\right) + \eta_2\left(\frac{3}{4}\zeta_1\xi_1 + \frac{1}{4}\zeta_2\xi_2\right) + \zeta_2\left(\frac{3}{4}\xi_1\eta_1 + \frac{1}{4}\xi_2\eta_2\right) \\
&= \frac{3}{4}(\xi_2\eta_2\zeta_2 + \xi_2\eta_1\zeta_1 + \xi_1\eta_2\zeta_1 + \xi_1\eta_1\zeta_2) = 0.
\end{aligned}$$

The last equality follows from the fact that, for any ξ, η, ζ which belong to the lines $\tau_1 = \pm\tau_2$, we have $\xi_1 = \pm\xi_2$, $\eta_1 = \pm\eta_2$, $\zeta_1 = \pm\zeta_2$. Therefore, all four terms in the last sum have the same absolute value. Since $\xi_1 + \eta_1 + \zeta_1 = 0$ and $\xi_2 + \eta_2 + \zeta_2 \neq 0$, there are two positive and two negative terms in the sum considered.

Since, by the assumption, for $n < m$, all nonzero S_n^τ belong to the lines $\tau_1 = \pm\tau_2$, the last group equals zero, which proves the result. □

In addition, we consider another system with the toric configuration space for which it is also possible to prove that the resonant numbers lie on several lines. Consider Kirchhoff's equations of motion of a rigid body moving in a perfect incompressible fluid, possessing a single-valued potential of velocities and at rest at infinity. The Hamiltonian has the form

$$\mathscr{H} = \frac{1}{2}(\mathbf{A}M, M) + (\mathbf{B}M, \gamma) + \frac{1}{2}(\mathbf{C}\gamma, \gamma).$$

Here M and γ are three-dimensional vectors of "impulsive moment" and "impulsive force", respectively.

We assume that the rigid body has three planes of symmetry. In this case, we can consider $\mathbf{B}$ to be zero matrix, $\mathbf{A}$ and $\mathbf{C}$ are arbitrary symmetric matrices. In addition, we assume that $\mathbf{A} = \mathrm{diag}(1, 1, a)$, $\mathbf{C} = \mathrm{diag}(\mu, -\mu, 0)$ and the system is considered on the level sets of the first integrals $(M, \gamma) = 0$, $\gamma^2 = 1$. When $a = 2$, we obtain the Chaplygin integrable case. In the special canonical variables, we have (here $H = 0$)

$$\mathscr{H} = \frac{1}{2}\left(G^2 + L^2 a - L^2\right) + \frac{\mu}{2}\left(-\frac{L^2}{G^2}\cos^2 g \cos 2l + \frac{L}{G}\sin 2g \sin 2l + \sin^2 g \cos 2l\right).$$

In the Chaplygin case, the additional first integral has the form

$$\mathscr{F} = (M_1^2 - M_2^2 + \mu\gamma_3^2)^2 + 4M_1^2 M_2^2.$$

Therefore, in the special canonical variables, we have $\mathscr{F}_0 = (G^2 - L^2)^2$ and $\mathscr{H}_0 = (G^2 + L^2)/2$, i.e., up to a constant multiplier, the functions $\mathscr{F}_0$ and $\mathscr{H}_0$ coincide with the ones of the Kovalevskaya case. Similarly to the Kovalevskaya and Goryachev – Chaplygin cases, the following result can be proved.

Proposition 3. *For the integrable Chaplygin case of Kirchhoff's equations, the numbers τ of nonzero resonant harmonics $k_m^\tau(u) \not\equiv 0$ of $\mathscr{K}_m(u, v)$ can lie only on the four lines: $\tau_1 = 0$, $\tau_2 = 0$, $\tau_1 = \pm\tau_2$.*

3.3. Nonintegrable Cases and Periodic Solutions

Unlike the Kovalevskaya and Goryachev – Chaplygin systems, in the nonintegrable cases the numbers of resonant harmonics lie on more than four lines and the number of lines increases as we proceed to the higher steps of the perturbation procedure. Taking into account Corollary 1, we can expect the birth of many pairs of periodic solutions (linearly stable and unstable) in the system. To be more precise, if we consider the Hamiltonian $\mathscr{K}$ up to $o(\mu^3)$ terms, the following holds.

Proposition 4. *Let $C = 1$ and $\tau_1 = 2$, $\tau_2 = 0$, then for $A > 1$, $A \neq 4$ and for all $u \neq 0$, $\langle u, \tau\rangle = 0$, for small $\mu \neq 0$, in a vicinity of the resonant torus u, there are two periodic solutions of the system with the Hamiltonian* (3.1).

Proof. Let $u_1 = 0$ and $u_2 = -2\lambda a_{11}$, then we obtain

$$k_2^{2,0} = \frac{A^3(A-4)}{64\lambda^2(A-1)^2}.$$

Clearly, for $A \neq 4$ and $\lambda \neq 0$, we have $k_2^{2,0} \neq 0$ (which can also be seen in Figs. 2–4) and the result follows from Corollary 1. □

Proposition 5. *Let $C = 1$ and $\tau_1 = 3$, $\tau_2 = 1$, then for $A > 1$, $A \neq 4/3$, $A \neq 2$, $A \neq 4$ and for all $u \neq 0$, $\langle u, \tau\rangle = 0$, for small $\mu \neq 0$, in a vicinity of the resonant torus u, there are two periodic solutions of the system with the Hamiltonian* (3.1).

Proof. Let $u_1 = -\lambda a_{22}$ and $u_2 = 3\lambda a_{11}$, then we obtain

$$k_3^{3,1} = -i\frac{A^5(4-3A)^2(A-4)(A-2)(3A-2)}{12288\lambda^3(A-1)^6}.$$

For the values of A considered here and for $\lambda \neq 0$, we have $k_3^{3,1} \neq 0$ and Corollary 1 can be applied. □

Note that the harmonic under consideration is zero only in the integrable cases and for $A = 4/3$. Similar results can be proved for various m and τ in accordance with Fig. 4. For instance, the following is also true.

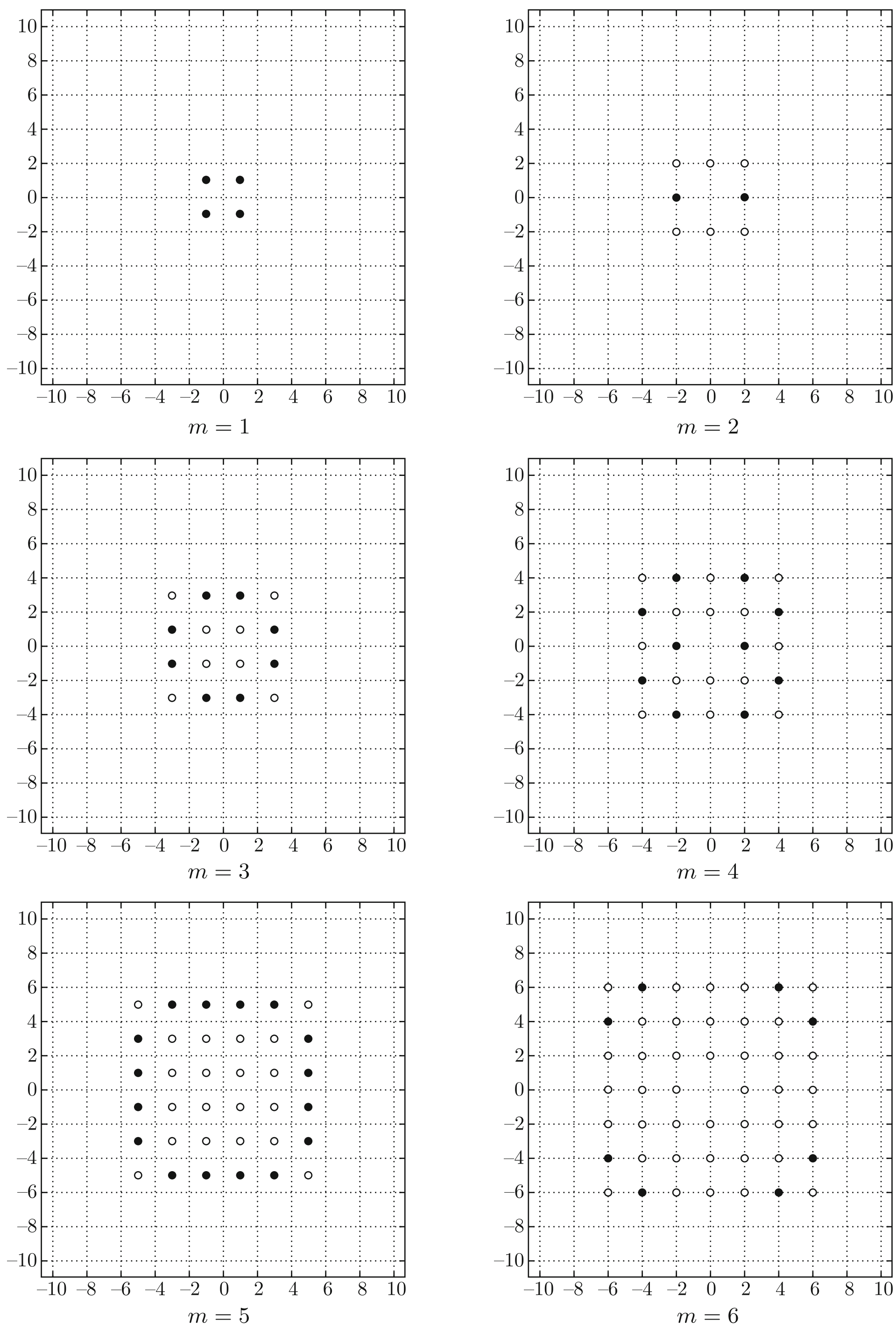

Fig. 4. Numbers of nonzero harmonics in $\mathscr{K}_m$ for the nonintegrable case $A = 8$, $C = 1$ ($\tau \neq 0$). Numbers τ of harmonics well defined on $\langle \tau, u\rangle = 0$, $u \neq 0$ are shown in color.

Proposition 6. *Let $C = 1$ and $\tau_1 = 1$, $\tau_2 = 3$, then for $A > 1$, $A \neq 2$, $A \neq 4$ and for all $u \neq 0$, $\langle u, \tau\rangle = 0$, for small $\mu \neq 0$, in a vicinity of the resonant torus u, there are two periodic solutions of the system with the Hamiltonian* (3.1).

Proof. For $u_1 = -3\lambda a_{22}$ and $u_2 = \lambda a_{11}$ we have

$$k_3^{1,3} = i\frac{A^5(A-4)^3(A-2)(A+2)}{4096\lambda^3(A-1)^6}.$$

Therefore, $k_3^{1,3} \neq 0$ for the values of A under consideration. □

4. CONCLUSION

It is well known [6] that a natural mechanical system with the Hamiltonian $H = T(p, q) + V(q)$ on a two-dimensional compact analytical configuration space has no additional first integral independent of the Hamiltonian if the genus of the configuration space is greater than one. At the same time, there are a few integrable systems with a toric configuration space, including systems considered above. Moreover, there are several results on the integrability of a Hamiltonian system with the Hamiltonian $H = T(p) + V(q)$ on a torus. In [4], a criterion for the existence of a full set of formally analytical first integrals was proved for $V(q)$ being a trigonometric polynomial. In [7–10], the connection between the arrangement of the spectrum of the potential and the existence of polynomial first integrals is studied. In particular, it is shown that if there is a polynomial integral of degree three in momenta, then all nonzero harmonics are on a single line.

The Goryachev – Chaplygin integrable system illustrates the difference between the case of a perturbation in the form of a trigonometric polynomial (or series) and the case where the potential depends on the momenta. As was proved above, in this system, all resonant harmonics lie on two lines, not one, even though the additional first integral is a polynomial of degree three. Moreover, unlike the Kovalevskaya case, these lines are not orthogonal in the kinetic metric.

Based on a few steps of classical perturbation theory, we can conclude that the nonintegrable cases considered above are similar to the case of a trigonometric polynomial perturbation, i. e., the number of lines on which the resonances are located is increasing when we consider the higher approximations of the perturbation calculations. In particular, this leads to the birth of many periodic solutions from the resonant tori.

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