ISSN 1560-3547, Regular and Chaotic Dynamics, 2016, Vol. 21, No. 5, pp. 556–580. © Pleiades Publishing, Ltd., 2016.

The Spatial Problem of 2 Bodies on a Sphere. Reduction and Stochasticity

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Received August 17, 2016; accepted September 13, 2016

Abstract—In this paper, we consider in detail the 2-body problem in spaces of constant positive curvature S^2 and S^3. We perform a reduction (analogous to that in rigid body dynamics) after which the problem reduces to analysis of a two-degree-of-freedom system. In the general case, in canonical variables the Hamiltonian does not correspond to any natural mechanical system. In addition, in the general case, the absence of an analytic additional integral follows from the constructed Poincaré section. We also give a review of the historical development of celestial mechanics in spaces of constant curvature and formulate open problems.

MSC2010 numbers: 70F15, 01A85

DOI: 10.1134/S1560354716050075

Keywords: celestial mechanics, space of constant curvature, reduction, rigid body dynamics, Poincaré section.

1. HISTORICAL REVIEW OF THE DEVELOPMENT OF CELESTIAL MECHANICS IN SPACES OF CONSTANT CURVATURE

In the literature concerned with celestial mechanics in spaces of constant curvature (hereinafter referred to as SCCs), there are several review papers [2, 50, 106] which describe sufficiently thoroughly the history of the development of individual branches of this area of mechanics, but do not give a general picture. On the other hand, recently there have been an increasing number of publications which very "creatively"[1)] misrepresent both the current situation in this area and the history of obtaining various results. In view of this, in this paper we give a fairly detailed historical review, in which we have endeavored not only to give a faithful account of the state of the art in this area of dynamics, but also to describe a historically trustworthy chronological order of obtaining the main results and to point out their authorship. We are aware of the fact that our review bears some subjectivity, but we are ready to maintain a debate on this interesting area of mechanics, where there are still many unsolved problems.

To justify the study of dynamics in spaces of constant curvature (SCCs), we present two classical, but still up-to-date quotations, one of which belongs to E. Schrödinger [103, p. 14], who considered the quantization of a hydrogen atom on a sphere.

"As far as I know, this is a new problem, which I found difficult to tackle in any other way. It may appear foolish to pay attention to the extremely feeble curvature of the Universe in dealing with the hydrogen atom, because even the influence of those much stronger fields of gravitation in which all our observations are actually made is (if the frame is properly chosen) entirely negligible. But this problem, by obliterating the sharp cut between "elliptic and hyperbolic orbits" (the classical orbits here are all closed) and by resolving

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[1)] The concept of a "creative way of writing" was proposed in [49]. Interestingly, some authors [120] who have read such a popular history seriously believe that the whole celestial mechanics in SCCs has been constructed by Bolyai, Lobachevsky and ... Diacu!

the continuous spectrum into an intensely crowded line spectrum, has extremely interesting features, well worth investigating by a method which proves hardly more complicated here than in the flat case..."

The second quotation belongs to H. Weyl, who in his book *Space, Time, Matter* [118, p. 76] considered multidimensional equations of motion of a rigid body[2)].

"... On the other hand, the fact that we have freed ourselves from the limitation to a definite dimensional number and that we have formulated physical laws in such a way that the dimensional number appears accidental in them, gives us an assurance that we have succeeded fully in grasping them mathematically..."

Indeed, the laws of celestial mechanics in curved spaces are much more complicated and diverse than those in the Euclidean case and their understanding is of great importance for dynamical systems theory. Moreover, the above-mentioned differences between flat space and SCC can be used to explain the discrepancies between astronomical observations and theoretical predictions (along with other possible explanations based on general relativity theory and on taking account of atmospheric refraction, nonsphericity of planets etc.).

The recent surge of interest in exploring SCCs was largely initiated by the papers of P. Higgs [59] and the Belarussian authors Yu. A. Kurochkin and V. S. Otchik [76].

Remark. The paper by Higgs [59] holds the third place in the number of citations among all his works. The first two of his works are devoted to a theoretical prediction of the boson, which he discovered experimentally in 2012 (for this discovery he was awarded the Nobel prize in 2013).

These papers were written mainly for the study of the quantum problem and continue the investigations started by E. Schrödinger [103]. Later, N. A. Chernikov, a researcher from the Dubna Nuclear Center, published the paper [38], in which he formulated the Kepler laws for L^2 and even developed a relativistic problem on L^2.

Analysis of the dynamics of a material point in spaces of constant curvature from the point of view of classical mechanics was undertaken by V. V. Kozlov in [70, 73], where the Kepler problems and the problem of two centers were examined. In the paper [69], published in 1995, V. V. Kozlov posed the problem of systematically investigating the n-body problem in curved space and of generalizations of the Sundman theorem. As far as we know, this problem has not been solved yet. By the way, the Levi-Civita regularization in SCCs is discussed in [53]. A more detailed analysis of the two- and three-body problem was carried out by A. V. Borisov, I. S. Mamaev, A. A. Kilin, V. A. Chernoivan [26, 29, 39] using [73, 76]. The monograph [21] is concerned with various integrable problems generalizing the problem of two centers, with stationary configurations of the two-body problem, nonintegrability of the general two-body problem on S^2, poses and investigates the bounded two- and three-body problems, and presents a stability analysis of libration points. The authors of [29] investigate the interesting classical (nonrelativistic) problem of the perihelion of Mercury shifting due to the curvature of space. The paper [27] is concerned with the study of a transformation relating the planar and curved problems of two centers.

Available reviews on dynamics in SCCs. The historical aspects of mechanics in SCCs were first pointed out by P. Dombrowski and J. Zitterbarth in the paper [50]. Using this paper, A. V. Borisov and I. S. Mamaev published Russian translations of the most important classical works of W. Killing, H. Liebmann, P. Serret and others as a separate book [22], and added the references and comments given in [50]. Previously, they had published the monograph [21], where one of the chapters is entirely devoted to mechanics in SCCs, but where, in referring to the main works, they mention only [35, 59, 73, 103] (in which there are no references to the classical works either).

Based on the above-mentioned two books [21, 22], A. V. Shchepetilov [106] published a monograph in the English language in which all historical aspects of mechanics in SCCs were gathered once again. This review is the most complete, but it still lacks some important details that are interesting from the point of view of classical mechanics.

The main historical aspects of advances in the celestial mechanics of SCCs are presented in Tables 1 and 2. In what follows, some comments and explanations on individual items of these tables are given.

[2)]It is well known that the motion of a four-dimensional rigid body about a fixed point is isomorphic to the free motion of a rigid body on S^3.

Table 1. Celestial mechanics in spaces of constant curvature.

Problems	Author, year, results
1. Newton's laws	P. Serret [104] 1860 (Newtonian potential for $\mathbf{S}^3$)
	E. Schering [100] 1870 (Newtonian potential for $\mathbf{L}^3$)
2. Kepler's laws	P. Serret [104] 1860 (first Kepler's law)
	W. Killing [68] 1885
	C. Neumann [94] 1886
	H. Liebmann [79] 1902 (Kepler's laws for $\mathbf{L}^3$, $\mathbf{S}^3$)
	N. A. Chernikov [38] 1992
	V. V. Kozlov [70] 1994
3. Kepler's equation	V. V. Kozlov [70] 1994 (Kepler's equation for $\mathbf{S}^2$)
4. The Laplace – Runge – Lenz vector in the Kepler problem	P. W. Higgs [59] 1979
	N. Katayama, [64] 1990
5. Bertrand's theorem	G. Darboux [40] 1884
	H. Liebmann [80] 1903, [81] 1905
	J. J. Slawianowski [108] 1980 (see a detailed comment in [70])
	M. Ikeda, N. Katayama [60] 1982
	V. V. Kozlov, A. O. Kharin [73] 1992
6. The problem of two centers and generalizations	W. Killing [68] 1885
	W. S. Otchik [96] 1991
	V. V. Kozlov, A. O. Kharin [73] 1992 (addition of Hooke's center)
	A. V. Borisov, I. S. Mamaev [21], (see also [88]) 1999 (homogeneous field), [24] 2005 (addition of imaginary centers)
7. The bounded two-body problem	V. A. Chernoivan, I. S. Mamaev [39] 1999 (chaotic dynamics and analytic nonintegrability, see also [26])
	S. L. Ziglin [121] 2001 (meromorphic nonintegrability according to Ziglin – Morales – Ramis)
	A. J. Maciejewski, M. Przybylska [86] 2003 (meromorphic nonintegrability according to Ziglin – Morales – Ramis)
8. The two-body problem	A. V. Shchepetilov [109] 2000
	A. V. Borisov, A. A. Kilin, I. S. Mamaev [29] 2004 (chaotic dynamics and analytical nonintegrability)
	A. V. Shchepetilov [105] 2006 (meromorphic nonintegrability according to Ziglin – Morales – Ramis)
10. Rigid body dynamics	W. K. Clifford [36] 1876, [37] 1882
	R. S. Ball [9] 1881
	R. S. Heath [58] 1884
	W. Killing [68] 1885
	D. Francesco [43] 1902

Table 2. Special issues of dynamics in SCCs.

Problems	Author, year, results
1. The theory of potential and figures of equilibrium	E. Schering [101] 1873
	V. V. Kozlov [71] 2000 (Ivory's theorem)
	I. A. Bizyaev, A. V. Borisov, I. S. Mamaev [13] 2015 (potential of a homogeneous spheroid)
2. Hamilton's hodograph	A. V. Borisov, I. S. Mamaev [25] 2005
	J. F. Carinena, M. F. Ranada, M. Santander [33] 2007
3. Trajectory isomorphism with a plane	P. Appell [6] 1891 (Kepler's problem)
	P. W. Higgs [59] 1979 (Kepler's problem)
	A. Albouy [1] 2003, [2] 2013 (two centers)
	A. V. Borisov, I. S. Mamaev [27] 2007
	I. A. Bizyaev, A. V. Borisov, I. S. Mamaev [14] 2014
4. Superintegrable combinations of Hooke's centers with a high-degree integral	E. Onofri, M. Pauri [95] 1978
	V. V. Kozlov, Yu. N. Fedorov [72] 1994
	A. V. Borisov, A. A. Kilin, I. S. Mamaev [19] 2009
	F. Tremblay, A. V., Turbiner, P. Winternitz [111] 2009
	I. A. Bizyaev, A. V. Borisov, I. S. Mamaev [14] 2014
5. Choreographies in the two- and three-body problem	A. V. Borisov, A. A. Kilin, I. S. Mamaev [29] 2004
	Montanelli H., Gushterov N. I. [90, 91] 2016

Comment (concerning the papers by F. Diacu). We briefly discuss the papers by F. Diacu gaining popularity, which we omit to mention in the main text of our paper. In his historical reviews (see, e.g., [44, 45]), he uses compilations of the well-known papers, however, he does not pay proper attention to them. By adding his reasoning and some references that are not relevant to the topics under discussion (for example, references to Einstein, Gauss and others), F. Diacu has created "his own" history, which makes one to believe that it is he who has revived interest in the dynamics in SCCs. Of interest in this connection is, for example, his passage concerning a common alma mater which he shared with H. Liebmann (see [45]). We note that, acting in the role of a historian of science, Diacu fails to avoid inaccuracies in referencing. For example, in his popular paper [45] he makes no reference to the work of Higgs [59], and the reference to the work of Killing [67] is erroneous and has nothing to do with the dynamics in SCCs. Moreover, the book [46] mentions a paper that has never existed. In particular, he writes about the nonrecognition of the papers by S. P. Novikov in the Soviet Union and his "only hope to get published in a Western journal with a good reputation". Further, the author asserts that J. Moser helped the Soviet scientist to publish a paper in the US journal *Annals of Mathematics*. However, there is no such paper by S. P. Novikov in this journal; he had his quite voluminous papers published without trouble in *Izvestiya Akademii Nauk SSSR* (mathematical series) and was awarded the Fields Prize for it. The scientific publications of Diacu are devoted to partial solutions of the n-body problem in SCCs; however, it does not seem possible to single out and classify his results. The publications are poorly structured, and no genealogical relationship can be traced between his investigations and classical mechanics in Euclidean spaces. Also, "distracting" terminology (for example, rotor-pulsator) is often used. But the physical essence is not elucidated, and the results themselves require verification by other, clearer, methods. As for the book [44], it actually presents a combination of elementary trigonometric substitutions and notation invented by the author for scalar and vector products. Of course, the above comments do not cancel the necessity of carrying out an additional analysis of his works in order to draw concrete conclusions on the results.

The laws of gravitation and the theory of potential. Traditionally, to prove the Newtonian law in Euclidean space, one uses either the theory of potential, where this law arises as a spherically symmetric solution of the Poisson equation, or the Bertrand theorem, according to which all trajectories of a particle in the field of the attracting center turn out to be closed only in the case of Newtonian and Hookean[3)] potentials (interestingly, the original version of the proof of the theorem was known already to Lagrange, and some technical details are still being discussed [62]).

It is of interest that, when postulating the law of gravitation in S^3, P. Serret [104] actually used the fact that the sphero-conical trajectories are closed, while E. Schering [100], using a generalization of the Poisson equation, pointed out the possibility of obtaining an analogous potential. In the book [104], also in the analysis of the behavior of sphero-conical curves, P. Serret formulated an analog of the first Kepler law. The same law was independently obtained by Darboux in [40] (the reference to Serret was added by him in the expanded edition of his work [41] in an appendix to the mechanics course of Despeyrous). For a more complete discussion of Kepler's laws in SCCs, see [2].

A development of the theory of Newtonian potential in SCCs is contained in the work of E. Schering [100, 101] (1870 and 1873) and in the much later work of V. V. Kozlov [71] (2000). In [68] (1885) W. Killing set forth in a brief and not very comprehensible form the results of [101], which will hardly be intelligible to the modern reader and should be interpreted from a modern perspective. The paper [71] contains assertions generalizing the Newton theorem for the potential of a homogeneous ball; however, the formulated analog of the Ivory theorem is not proved (see also the discussions in [113]).

Interestingly, the attempts at generalizing to SCCs the Maclaurin ellipsoids, which are figures of equilibrium of a rotating fluid, were not successful and required the addition of internal flows [13]. We note that there are still many outstanding questions in the theory of potential and in the theory of rotating liquid ellipsoids (and figures of equilibrium in general) in SCCs.

Remark. It is interesting that a somewhat different approach is possible to the search for possible laws of gravitation in SCCs. For example, in the dynamics of point vortices on a sphere, to exclude the antipodal singularity in solving the Poisson equation on a two-dimensional sphere, one introduces background vorticity (see, e.g., [17]). Without such a background vorticity the model of point vortices (containing antipodes) was postulated and developed in [18]. In a similar way, one can proceed in curved celestial mechanics by introducing background "antimatter". An example of such an approach with uniform distribution of antimatter can be found in [15]. However, we note that the authors of [15] consider a two-dimensional analog of the Newtonian law of gravitation (which depends on the logarithm, has no physical meaning and is of purely mathematical interest). In addition, the paper [15] contains a number of inaccuracies and obvious errors. For example, Theorem 9.2 is wrong: the functions presented in it are not integrals of motion.

Nevertheless, a more reasonable addition of "antimatter" (not necessarily uniformly distributed) can lead to obtaining new interesting versions of the gravity law. For example, the analog of the Newtonian law of gravitation with uniform distribution of "antimatter" in S^3 has the form

$$U(\theta) = -k(\pi - \theta)\cot\theta,$$

where θ is the length of the arc between the bodies. This law does not satisfy the conditions of the Bertrand theorem, but is undoubtedly of interest to researchers.

The only reasonable choice between different models can be made only using astronomical measurements, which unfortunately cannot be performed so far. We note that an analogous problem exists in relativistic theories (general relativity theory etc.), where only indirect measurements have been performed, still giving rise to various debates. In this sense, the works of Lipschitz [84] and Mordukhai-Boltovsky [93], who defined the law of gravitation in a different way, may be not only of historical interest.

Remark. In some studies, the choice of a generalization of the gravitational interaction to SCCs is not motivated by anything at all. For example, the authors of [82] postulate a somewhat strange law of gravitation on S^2 for which the potential of interaction of bodies is inversely proportional

[3)] Which means the potential of an isotropic oscillator.

to the length of the arc joining them. It is interesting that, after numerically constructing several projections of the system trajectories, the authors assert that there are chaotic ones among them. Yet in order to assert this, one has obviously to construct a Poincaré section of the reduced system, which they fail to do.

The discussion of gravitation laws on an imaginary Lobachevsky space (given by a one-sheet hyperboloid) goes back to C. Grosche [57]. This was preceded by the works [11, 56, 61], which developed the method of quantization of the Kepler problem on a sphere and a pseudo-sphere (given by a two-sheet hyperboloid) by using the Feynman integral along paths. In all above-mentioned works, an analog of the Newtonian potential arises from the Poisson equation.

The Keplers laws on an imaginary Lobachevsky space (considered together with L^3 and S^3) are dealt with in [77, 78, 97]. Similar analyses can be found in [34] (for different SCCs), where the imaginary Lobachevsky space is called a *de Sitter space*.

One of the criticisms made against the above studies is their actual uselessness to mechanics due to the fact that the metric represented by the kinetic energy is indefinite (pseudo-Riemannian). Even in Euclidean space the dynamical systems with a pseudo-Riemannian metric are usually not dealt with in classical mechanics. In addition, the systems considered in these studies do not exhibit any new effects and, as a rule, reduce to well-known systems with insignificant modifications. These results are appropriate for a book of problems in theoretical physics rather than for scientific journals with high citation rates.

Comment. The contribution of Lobachevsky and Bolyai to the discovery of the law of gravitation in the Lobachevsky space has been grossly overestimated (as for Bolyai, the credit is simply undeserved). However, they are cited almost in all papers on SCCs.

When Bolyai is mentioned, a frequently used reference is [112], which is inaccurate and can mislead the reader, since, in fact, the page indicated in [106] contains only an interpretation of Bolyai's notes on the work of Lobachevsky [85]. What is really meant is the edition of F. Engel and P. Stäckel on the history of non-Euclidean geometry, the second volume of which is devoted to the life and the scientific heritage of W. Bolyai and J. Bolyai [112]. According to P. Stäckel, who together with K. Kürschak inspected the numerable hand-written drafts of Bolyai and translated them into German, the commentary on the publication of Lobachevsky was very personal and was not meant for publication [112, p. 138]. As a matter of fact, P. Stäckel only attempted to trace the development of thought of János Bolyai using extracts from his many "deciphered" drafts left in the period 1848–1851.

But even apart from that, the text on the frequently mentioned page 156 [112] is rather vague. Its meaning is by no means unambiguous, however, it is obvious that this text is concerned with geodesics, (astronomical) measurements, the validity of Euclidean geometry and with the standard (Euclidean) Newton law, and presents no generalization of this law to curved spaces. Thus, Bolyai — almost ten years after the work of Lobachevsky — only tried to critically comprehend the results of the Russian scientist, and these notes can be regarded only as attempts at verifying the Lobachevsky geometry. In any case, Bolyai's contribution to the discovery of the law of gravitation for the Lobachevsky space is merely a myth.

Moreover, the discovery of an analog of the Newtonian law for L^3 should not be attributed to Lobachevsky either. It can only be regarded as a hypothesis, which he set up by using a formula found by him for the area of the sphere in L^3 and by using analogy with Euclidean space, in which the area of the sphere increases and, accordingly, the gravity force decreases in proportion to r^2. In fact, Lobachevsky himself wrote in [85, p. 159] that "this is a pure conjecture which should be confirmed by other, more convincing, arguments".

Generalizations of the Bertrand theorem. In the case of the sphere S^2, Darboux [40, 41] obtained a generalization of the Bertrand theorem and found an analog of the Hookean and Newtonian centers[4)].

Remark. Generally speaking, Bertrand formulated two theorems: the first concerns potentials for which all bounded trajectories of the particle are closed, and the second concerns the central (not necessarily potential) forces for which the particle trajectories are conical sections (conics). Bertrand proved only the first theorem, the other was proved by Darboux. Here and in Table 1 we mean the first theorem of Bertrand.

[4)] By the way, Darboux also formulated the general Bertrand theorem on the plane $\mathbb{R}^2$ for the case when the conditions of potentiality of forces are omitted.

Darboux also solved a more general problem of closed trajectories of a point moving on the surface of revolution under the action of potential forces. Darboux's studies on closed trajectories of a point in a potential field on the surface of revolution were continued in [99, 119]. However, the closedness conditions formulated in them have a form that is too general, and do not lead to finding a concrete potential or form of the surface of revolution.

The Laplace – Runge – Lenz vector. We mention the works of P. Higgs [59] (1979) and N. Katayama [64] (1990), in which the Laplace – Runge – Lenz vector is presented for the Kepler problem on S^3 and L^3, respectively. In addition, there are many, mainly physical, works which are concerned with a generalization of the Laplace – Runge – Lenz vector to SCCs (see, e.g., [10, 34]).

In [65], an analog of the Laplace – Runge – Lenz vector on S^3 and L^3 for the MICZ-system (McIntosh – Cisneros – Zwanziger) is presented which describes the motion of a particle in the asymptotic field of a self-modular monopole (the physical meaning of this system is described, for example, in [51]). This result was independently rediscovered in [16, 27] and [55], respectively, for S^3 and L^3. In [55], a more general case on L^3 in which a generalization of the Laplace – Runge – Lenz vector is possible is presented.

The problem of two centers. In the planar case the (spatial) problem of two Newtonian centers was considered by Euler, who mentioned it in his correspondence with Lagrange (see [3] for details). The integrability of the problem of two centers in SCCs was pointed out by W. Killing [68] in 1885. However, we note that already Liouville [83] found (without relation to celestial mechanics) a more general potential on S^2, which admits separation of variables in sphero-conical coordinates. More recently, the integrability of the problem of two centers on S^2 was independently rediscovered in [73] and that on S^3, L^3 in [21]. In addition, in [73] V. V. Kozlov and A. O. Kharin show that on the sphere S^2 this problem remains integrable if elastic attraction (repulsion) is added to the point that is the middle of the segment between the centers.

It is well known that in the planar case the problem of two fixed centers has an integrable limiting case when one of the centers tends to infinity, resulting in the problem of particle motion in a constant homogeneous field and in the field of a Newtonian center. This problem was first considered by Lagrange, and its analysis is presented in the book [31] in relation to the classical model of motion of an electron in an atom placed in a homogeneous electric field.

On the sphere, such a limiting case is impossible due to its compactness, and an analog of the Lagrange problem for the Lobachevsky space was presented in the thesis of I. S. Mamaev [88] and in the book [21]. In addition, in [87] (see also [21]) a separation of variables was obtained in the spatial problem of two centers on S^3 and L^3. It turned out that the elimination of a cyclic variable results in an additive term equivalent to the Hookean center (which was pointed out by V. V. Kozlov and A. O. Kharin [73]) with an intensity proportional to the constant of a cyclic integral. These results were republished in the English language in [115][5)].

Another well-known generalization of the planar problem of two centers, which has found application in the dynamics of satellites [74], was presented by G. Darboux [42]. In this case, two more imaginary centers are added to two Newtonian centers in a perpendicular direction. An analog of this problem for the space of constant curvature was found in [24].

A qualitative and topological analysis of the problem of 2 centers on S^2 is presented in [115, 117], in particular, a global regularization of this system is contained in [117]. These and many other results are presented in the book [116], though, as a rule, without references to original sources.

Central projection. The remarkable ideas of Serret, Darboux and Halphen were developed by Appell [7], who, by means of the central projection (which was independently used by Higgs and called by him the *gnomonic projection*) reduced the problem of the central field on a sphere to the planar case. This result allows one in a natural way to extend the Bertrand theorem from the planar case to SCCs. Afterwards Appell generalized the idea of the central projection and introduced a homographic transformation [7, 8].

[5)] We note that the main results in [115] (including formulation of the problem and separation of variables), were obtained by the author not on her own, but were communicated to her during her period of probation at the Udmurt State University under the guidance of A. V. Borisov.

The general idea of Appell was developed by A. Albouy and led to the creation of projective dynamics, which is applicable to a wide class of dynamical systems possessing quadratic first integrals (see [1, 2]) and to establishing a trajectory isomorphism between the problems of two centers on a plane and a sphere. A more explicit approach containing various generalizations of the problem of two centers was presented in [27].

Superintegrable systems with a high-degree integral. We first note that, due to the existence of a trajectory isomorphism given by the central projection (and its generalizations), integrable and superintegrable systems of flat spaces $\mathbb{R}^n$ and spaces of constant curvature turn out to be equivalent. Therefore, they should be considered jointly.

In [95], the most general conditions for superintegrability of a natural system on $\mathbb{R}^2$ are presented using methods due to Bertrand [12] and examples of specific superintegrable potentials are given. Afterwards these Onofri – Pauri potentials were rediscovered in [19, 20, 111]. Interestingly, in the physical literature one can find references only to the paper [111] (published in the same year as [19, 20]), and these systems are called TTW systems.

In [20], a general procedure of reduction to the system in S^{n-1} is proposed in $\mathbb{R}^n$ for a special kind of potential (the Jacobi potential). This reduction is different from the central projection and can also be used to construct integrable and superintegrable systems in SCCs. We note that the relation between the Onofri – Pauri potential and the potential of some set of oscillators on the sphere is also shown in [20] (see also [32]).

In [63], examples of superintegrable systems in $\mathbb{R}^3$ are presented which are carried over to S^3 using the central projection, see [14].

An extensive list of references concerned with superintegrable systems is given in the review [89]. We note that it becomes increasingly more difficult in this area to find relations of newly published results to the well-known ones (since, as a rule, neither the authors nor the editors of journals trouble themselves with it). As an example, we refer to the papers [98, 110], which anounce the discovery of new superintegrable systems, but mention nowhere the fact that the metric generated by kinetic energy has a zero curvature and that hence there is a natural relation with the Onofri – Pauri systems.

The n-body problem and stationary configurations. In the n-body problem, due to compactness of the sphere, static (equilibrium) configurations can arise. In this case, a great number of existence results can be obtained by the simplest generalization (adaptation) of the known results, which have been obtained either in the Thomson problem of equilibrium configurations of charges on the sphere [4] or in the problem of equilibrium configurations of vortices on the sphere [5, 30, 75] (although the addition of the “antipode” in the case of Newtonian potential imparts some specificity). Nevertheless, the stability of these configurations requires a separate study [48].

Stationary configurations (relative equilibria) in the two-body problem on S^2 were explored in [29], their linear stability was analyzed in [66], and the existence and stability of these configurations in L^2 was investigated in [54]. The libration points in the bounded three-body problem on S^2 and L^2 are explored in [66], and the results of this paper imply, in particular, that Moulton’s theorem on the number of collinear stationary configurations does not hold in curved space. For more complex stationary configurations there are a great many unsolved problems in the flat space too, see, e.g., [102], and those in SCCs have almost not been investigated.

Relative choreographies in the two-body problem on S^2 are explored in [29]. Generalizations of the well-known flat choreographies in the three-body problem to the case S^2 and L^2 have been obtained in [90, 91].

The generalized Sitnikov problem in curved space, where the effect of stability interchanges is observed, is studied in [52].

In [47], for the case of n equal masses, a partial solution is presented for which, during motion, the bodies form a regular whose lateral lengths change with time.

2. THE TWO-BODY PROBLEM IN SPACES OF CONSTANT CURVATURE

In this paper we consider the problem of the motion of two material points on the two-dimensional S^2 and three-dimensional S^3 spheres whose interaction potential depends only on the distance between them. As is well known, in Euclidean space the two-body problem reduces to the problem of the motion of a point in the central field, which is due to Galilean invariance and the existence of integrals leading to a uniform and rectilinear motion of the center of mass. In curved space such a relationship is absent, and the two-body problem is one of the central problems in the celestial mechanics of SCCs. Its role in this area is analogous to that of the three-body problem in the celestial mechanics of Euclidean space and the Euler – Poisson equations in rigid body dynamics.

The system under consideration is invariant under the action of the groups $SO(3)$ for S^2 and $SO(4)$ for S^3, respectively; due to this fact it can, as is well known, be reduced in both cases to a Hamiltonian system with two degrees of freedom. On the other hand, an explicit application of the reduction procedure involves considerable difficulties, since the symmetry group is noncommutative and hence it is impossible to use the classic Routh approach. In Euclidean celestial mechanics, where difficulties arise only for $n > 2$ bodies, the corresponding reduction mechanisms were developed by Lagrange, Jacobi and others (relevant references can be found in [28], and a systematic exposition of these approaches can be found in the Russian-language review [114]).

Since analysis of the general two-body problem is difficult, many dynamical effects caused by the curvature of space can be explored by using the bounded two-body problem. For the case S^2 it was originally considered in [21, 39] and examined in more detail in [26]. The results of computer experiments that are presented in these studies point to the fact that in the general case this system has no additional analytic integral. The results of meromorphic nonintegrability (according to Ziglin – Morales – Ramis [92]) — both for an analog of the Newtonian center and for that of the Hookean center — were obtained in [86, 121]. An extension of these results to the unbounded two-body problem on S^2, L^2 was performed by A. V. Shchepetilov [105].

We note that the bounded three-body problem in SCCs turns out to be much more difficult as compared to the planar case. An analog of the libration points in the bounded three-body problem on S^2 and partially on L^2 was investigated by A. A. Kilin [66], where, in particular, it was shown that the equilibrium points of these systems undergo bifurcations that are not encountered in Euclidean space.

The problem of explicit reduction of the two-body problem in SCCs was first considered by A. V. Shchepetilov [107, 109]. In these papers, to obtain a reduced system, he uses the method of restricting the symplectic form, as proposed in the Marsden – Weinstein procedure. The Hamiltonians of reduced systems presented in these papers have an unduly complex form. Moreover, for the sake of greater generality, the author considers the quantum mechanical two-body problem in SCCs, which makes calculations even more cumbersome. We also note that, in addition to the above difficulties, the reduction method based on the restriction of the symplectic form has an irreparable drawback, namely, it does not allow one to solve the problem of reconstructing the dynamics of the complete system.

Remark. The papers [107, 109] and all subsequent publications of the author [105, 106, etc.] which are concerned with the problem of reduction contain an extremely scanty description of the calculations performed by the author. The author excludes key steps from the operations performed, and, as a result, it is almost impossible to verify the results obtained. It turns out that it is easier to perform reduction without recourse to these papers.

The above-mentioned difficulties were completely eliminated in [29], where the two-body problem on S^2 was reduced to a natural Hamiltonian system with two degrees of freedom. This was done by using the classical approach developed by Bour in the planar three-body problem (see [114]), who used the well-known methods of rigid body dynamics. This, in particular, allowed a numerical analysis of the system, which demonstrated the absence, in the general case, of an additional analytic first integral. For a development of these classical ideas, see [105], where the author presented a simple form of the Hamiltonians of a reduced system on S^2 and L^2 which generalizes in a natural way the reduced Hamiltonian of [29]; this allowed the author of [105] to extend the results on meromorphic nonintegrability from the bounded two-body problem to the general one.

In this paper, we perform reduction of the two-body problem on S^2 and S^3 using methods of Poisson geometry (going back to S. Lie) and some techniques of rigid body dynamics (in particular, the dynamics of multidimensional tops). One of the classical key steps in the proposed procedure is the representation of the reduced system in noncanonical Hamiltonian form with a degenerate Poisson bracket. Although redundant variables arise in this case, the equations of motion, the Hamiltonian and the first integrals have a much simpler form than those in canonical variables.

The numerical analysis of the Poincaré section of the reduced system, as performed in Section 5, allows a conclusion about its real-analytic nonintegrability and the presence of chaotic trajectories. Note that the case S^2 examined in [29] holds for zero value of one of the integrals.

To conclude this section, we outline possible avenues for further development of the results obtained in this paper.

1. One of the important problems is the investigation of relative equilibria of the system, in particular, the analysis of stability and bifurcations. In this case there can exist configurations in which the distance between the points remains constant, but their motion in absolute space can be fairly complex.

2. Another outstanding problem is that of meromorphic nonintegrability (according to Ziglin–Morales–Ramis) of the two-body problem on S^3. In this case, unlike the two-body problem on S^2, the (reduced) Hamiltonian has an additional parameter related to a Casimir function. If one does not introduce (symplectic) canonical coordinates, which render the form of the Hamiltonian much more complicated, the issue of an additional meromorphic integral makes sense. As reference solutions one can probably choose here collisional solutions, as is done in [105].

3. It is of natural interest to generalize the reduction described above to the case of the Lobachevsky space. Since the reduction has been performed in algebraic form, such an extension should not present any difficulties in this case. Moreover, by changing the structure of the algebra, one can obtain a reduced Hamiltonian that depends on the curvature of space as a parameter. It is of interest to find out how the stationary configurations will bifurcate depending on the curvature of space and to study the chaotic behavior in phase space.

3. THE n-BODY PROBLEM ON S^2

3.1. Parameterization by the Group $SO(3)$

Consider the problem of n bodies on a two-dimensional sphere $S^2 \subset \mathbb{R}^3$. Let $OXYZ$ be a fixed coordinate system and let $\boldsymbol{R}_\alpha = (X_\alpha, Y_\alpha, Z_\alpha)$ be the Cartesian coordinates of the point mass μ_α, $\alpha = 1, \ldots, N$.

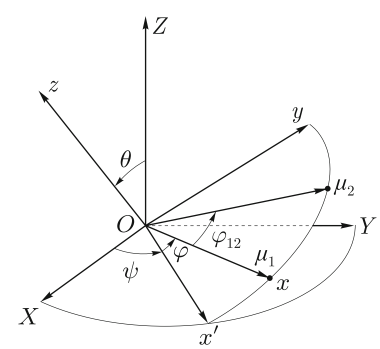

Fig. 1. Euler angles.

Let us choose a pair of particles μ_1, μ_2 and suppose that a moving orthogonal coordinate system $Oxyz$ is attached to them in such a way that the axis Ox passes through the point μ_1 and the plane Oxy contains both masses μ_1, μ_2 (see Fig. 1). In this case, the radius vectors of the point masses $\boldsymbol{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ in the moving axes $Oxyz$ characterize the *relative* position of the particles (i. e., the configuration of the bodies irrelative to its position on the sphere S^2). Let $\boldsymbol{q} = (q_1, \ldots, q_n)$ denote the corresponding generalized (local) coordinates which completely parameterize the relative position (configuration) of the particles in the case of material points $n = 2N - 3$. We will describe the orientation of the moving axes relative to the fixed axes by the Euler angles θ, φ, ψ so that the position of the particles in the fixed axes is described by

$$\boldsymbol{R}_\alpha(\theta,\varphi,\psi,\boldsymbol{q}) = \mathbf{Q}(\theta,\varphi,\psi)\cdot\boldsymbol{r}_\alpha(\boldsymbol{q}),$$

$$\mathbf{Q}=\mathbf{Q}_\psi\mathbf{Q}_\theta\mathbf{Q}_\varphi=\begin{pmatrix}\cos\varphi\cos\psi-\cos\theta\sin\psi\sin\varphi & -\sin\varphi\cos\psi-\cos\theta\sin\psi\cos\varphi & \sin\theta\sin\psi\\ \cos\varphi\sin\psi+\cos\theta\cos\psi\sin\varphi & -\sin\varphi\sin\psi+\cos\theta\cos\psi\cos\varphi & -\sin\theta\cos\psi\\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta\end{pmatrix},$$

$$\mathbf{Q}_\theta=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix},\ \mathbf{Q}_\varphi=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix},\ \mathbf{Q}_\psi=\begin{pmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{3.1}$$

where $\mathbf{Q}$ is the matrix of the direction cosines.

Assuming that the forces of interaction are potential, we construct the Lagrangian of the system

$$L = T - U,$$

where T and U are the kinetic and potential energy, respectively. We use a relation that is well known in rigid body dynamics:

$$\mathbf{Q}^{-1}\dot{\mathbf{Q}}=\begin{pmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{pmatrix}=$$

$$=\begin{pmatrix}0 & -\dot\psi\cos\theta-\dot\varphi & \dot\psi\sin\theta\cos\varphi-\dot\theta\sin\varphi\\ \dot\psi\cos\theta+\dot\varphi & 0 & -\dot\psi\sin\theta\sin\varphi-\dot\theta\cos\varphi\\ -\dot\psi\sin\theta\cos\varphi+\dot\theta\sin\varphi & \dot\psi\sin\theta\sin\varphi+\dot\theta\cos\varphi & 0\end{pmatrix},$$

where $\boldsymbol{\omega}=(\omega_x,\omega_y,\omega_z)$ are the projections of the angular velocity of the moving frame onto the moving axes $Oxyz$, and we represent the kinetic energy of the system as

$$T=\sum_\alpha \mu_\alpha\big(\dot{\boldsymbol{R}}_\alpha,\dot{\boldsymbol{R}}_\alpha\big)=\frac{1}{2}\big(\boldsymbol{\omega},\mathbf{I}(\boldsymbol{q})\boldsymbol{\omega}\big)+\big(\boldsymbol{\omega},\boldsymbol{\xi}(\boldsymbol{q},\dot{\boldsymbol{q}})\big)+\frac{1}{2}\sum_{i,j}G_{ij}(\boldsymbol{q})\dot q_i\dot q_j,$$

$$\mathbf{I}(\boldsymbol{q})=\sum_\alpha\mu_\alpha\left(\boldsymbol{r}_\alpha^2\mathbf{E}-\boldsymbol{r}_\alpha\otimes\boldsymbol{r}_\alpha\right),\quad \boldsymbol{\xi}=\sum_\alpha\mu_\alpha\boldsymbol{r}_\alpha\times\dot{\boldsymbol{r}}_\alpha=\sum_{\alpha,i}\mu_\alpha\boldsymbol{r}_\alpha\times\frac{\partial\boldsymbol{r}_\alpha}{\partial q_i}\dot q_i,$$

$$G_{ij}=\sum_\alpha\mu_\alpha\left(\frac{\partial\boldsymbol{r}_\alpha}{\partial q_i},\frac{\partial\boldsymbol{r}_\alpha}{\partial q_j}\right),$$

where $\boldsymbol{q}=(q_1,\ldots,q_n)$, the symbols $(\cdot,\cdot)$, $\times$ and $\otimes$ correspond, respectively, to the scalar, vector and tensor products in $\mathbb{R}^3$, and $\mathbf{E}$ is the identity matrix. Since the particles interact only with each other, the potential energy of the system does not depend on the Euler angles:

$$U = U(\boldsymbol{q}).$$

3.2. Integrals of Motion and Reduction

As is well known, due to the invariance of the Lagrangian under rotations (i.e., under the change of fixed axes) the projections of the angular momentum of the system onto the fixed axes are preserved:

$$\boldsymbol{M}=\sum_\alpha\mu_\alpha\boldsymbol{R}_\alpha\times\dot{\boldsymbol{R}}_\alpha=\mathbf{Q}\Big(\mathbf{I}(\boldsymbol{q})\boldsymbol{\omega}+\boldsymbol{\xi}\Big)=\text{const}.$$

Using the Noether theorem, we obtain the corresponding expressions for the components of this vector in the Euler angles:

$$M_X = \cos\psi\frac{\partial L}{\partial\dot\theta} + \frac{\sin\psi}{\sin\theta}\left(\frac{\partial L}{\partial\dot\varphi} - \cos\theta\frac{\partial L}{\partial\dot\psi}\right),\quad M_Y = \sin\psi\frac{\partial L}{\partial\dot\theta} - \frac{\cos\psi}{\sin\theta}\left(\frac{\partial L}{\partial\dot\varphi} - \cos\theta\frac{\partial L}{\partial\dot\psi}\right),$$
$$M_Z = \frac{\partial L}{\partial\dot\psi}. \tag{3.2}$$

We define the generalized momenta of the system in a standard way:

$$P_\theta = \frac{\partial L}{\partial\dot\theta},\quad P_\varphi = \frac{\partial L}{\partial\dot\varphi},\quad P_\psi = \frac{\partial L}{\partial\dot\psi},$$
$$p_i = \frac{\partial L}{\partial\dot q_i},\quad i = 1,\ldots,n.$$

Then the equations of motion can be represented in the canonical Hamiltonian form

$$\dot\theta = \frac{\partial H}{\partial P_\theta},\quad \dot P_\theta = -\frac{\partial H}{\partial\theta},\quad \dot\varphi = \frac{\partial H}{\partial P_\varphi},\quad \dot P_\varphi = -\frac{\partial H}{\partial\varphi},\quad \dot\psi = \frac{\partial H}{\partial P_\psi},\quad \dot P_\psi = -\frac{\partial H}{\partial\psi},$$
$$\dot q_i = \frac{\partial H}{\partial p_i},\quad \dot p_i = -\frac{\partial H}{\partial q_i}.$$

The Hamiltonian function is expressed in a natural way in terms of the projections of the angular momentum vector onto the moving axes $\boldsymbol{m} = (m_x, m_y, m_z)$:

$$H = \frac{1}{2}\Big(\boldsymbol{m}, \mathbf{A}(\boldsymbol{q})\boldsymbol{m}\Big) + \Big(\boldsymbol{m}, \boldsymbol{k}(\boldsymbol{q},\boldsymbol{p})\Big) + \frac{1}{2}\sum_{i,j} C_{ij}(\boldsymbol{q})p_i p_j + U(\boldsymbol{q}),$$
$$m_x = \frac{\partial L}{\partial\omega_x} = \frac{\sin\varphi}{\sin\theta}\,(P_\psi - P_\varphi\cos\theta) + P_\theta\cos\varphi,$$
$$m_y = \frac{\partial L}{\partial\omega_y} = \frac{\cos\varphi}{\sin\theta}\,(P_\psi - P_\varphi\cos\theta) - P_\theta\sin\varphi, \tag{3.3}$$
$$m_z = \frac{\partial L}{\partial\omega_z} = P_\varphi,\quad \boldsymbol{k} = \mathbf{B}(\boldsymbol{q})\boldsymbol{p},$$

where $\mathbf{A}(\boldsymbol{q})$, $\mathbf{B}(\boldsymbol{q})$, $\mathbf{C}(\boldsymbol{q})$ are the 3×3, $3\times n$ and $n\times n$ matrices which are the blocks of the $(3+n)\times(3+n)$ matrix, which arises when the quadratic form corresponding to the kinetic energy is inverted:

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{C} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I} & \left\|\frac{\partial\xi_i}{\partial\dot q_j}\right\| \\ \hline \left\|\frac{\partial\xi_i}{\partial\dot q_j}\right\|^T & \mathbf{G} \end{array}\right)^{-1}.$$

In order to obtain a reduced system, we pass from the canonical momenta P_θ, P_φ, P_ψ to the variables m_x, m_y, m_z. It turns out that the set of variables $\boldsymbol{m}$, $\boldsymbol{q}$, $\boldsymbol{p}$ is closed relative to the Poisson bracket:

$$\{m_i, m_j\} = -\varepsilon_{ijk}m_k,\quad \{q_i, p_j\} = \delta_{ij}.$$

This Poisson bracket is degenerate and possesses the Casimir function

$$C_1 = m_x^2 + m_y^2 + m_z^2. \tag{3.4}$$

Since the Hamiltonian (3.3) is expressed only in terms of these variables, we obtain the closed system of equations

$$\dot{\boldsymbol{m}} = \boldsymbol{m}\times\frac{\partial H}{\partial\boldsymbol{m}},\quad \dot q_i = \frac{\partial H}{\partial p_i},\quad \dot p_i = -\frac{\partial H}{\partial q_i}, \tag{3.5}$$

which *defines the reduced system* for this problem (since the variables $\boldsymbol{m}$, $\boldsymbol{q}$, $\boldsymbol{p}$ are invariant under the left group action $SO(3)$, i. e., under the change of the fixed axes).

In order to obtain a reduced system in canonical variables, it is necessary to define on the level set of the integral (3.4) $C_1 = M_0^2$ the cylindrical coordinates (Andoyer variables [23])

$$m_x = \sqrt{M_0^2 - p_0^2}\sin q_0, \quad m_y = \sqrt{M_0^2 - p_0^2}\cos q_0, \quad m_z = p_0,$$
$$q_0 \in [0, 2\pi), \quad p_0 \in [-M_0, M_0], \tag{3.6}$$

which commute canonically:

$$\{q_0, p_0\} = 1.$$

3.3. Reconstruction

Let us assume that we are given a solution to the system (3.5):

$$\boldsymbol{m} = \boldsymbol{m}^{(0)}(t), \quad \boldsymbol{q} = \boldsymbol{q}^{(0)}(t), \quad \boldsymbol{p} = \boldsymbol{p}^{(0)}(t).$$

We need to determine the time dependence for the Euler angles.

1. As a first step, we define the fixed axes $OXYZ$ in such a way that $\boldsymbol{M}\|OZ$, so the following relations hold:

$$M_X = 0, \quad M_Y = 0, \quad M_Z = P_\psi = M_0.$$

Using them, we find from (3.2)

$$P_\theta = 0, \quad P_\varphi - P_\psi\cos\theta = m_z^{(0)}(t) - M_0\cos\theta = 0.$$

Finally, taking (3.3) into account, we find

$$P_\theta = 0, \quad P_\varphi = m_z^{(0)}(t), \quad P_\psi = M_o = \text{const},$$
$$\cos\theta = \frac{m_z^{(0)}(t)}{M_0}, \quad \operatorname{tg}\varphi = \frac{m_x^{(0)}(t)}{m_y^{(0)}(t)}.$$

2. Using these relations, we find the quadrature for the angle ψ:

$$\dot\psi = \frac{\sin\varphi\omega_x + \cos\varphi\omega_y}{\sin\theta} = \frac{M_0\left(m_x^{(0)}(t)\omega_x^{(0)}(t) + m_y^{(0)}(t)\omega_y^{(0)}(t)\right)}{M_0^2 - m_z^{(0)}(t)},$$
$$\omega_x = \frac{\partial H}{\partial m_x}, \quad \omega_y = \frac{\partial H}{\partial m_y}.$$

3.4. Example — the 2-body Problem on S^2

In this case, the system has only one mutual variable, which is the angle between the radius vectors of the points (see Fig. 1)

$$\varphi_{12} = q_1 \in (0, \pi).$$

Let us denote the radius of the sphere by a. Then the radius vectors of particles in the moving coordinate system $Oxyz$ are

$$\boldsymbol{r}_1 = (a, 0, 0), \quad \boldsymbol{r}_2 = (a\cos q_1, a\sin q_1, 0).$$

Performing the above operations, we obtain the Hamiltonian of the system (3.3) in the form

$$H = \frac{1}{2a^2}\left(\big(\boldsymbol{m}, \mathbf{A}(q_1)\boldsymbol{m}\big) - \frac{2}{\mu_1}m_3p_1 + \frac{\mu_1+\mu_2}{\mu_1\mu_2}p_1^2\right) + U(q_1),$$
$$\mathbf{A}(q_1) = \begin{pmatrix} \frac{\mu_1+\mu_2\cos^2 q_1}{\mu_1\mu_2\sin^2 q_1} & \frac{\cos q_1}{\mu_1\sin q_1} & 0 \\ \frac{\cos q_1}{\mu_1\sin q_1} & \frac{1}{\mu_1} & 0 \\ 0 & 0 & \frac{1}{\mu_1} \end{pmatrix}.$$

In the canonical variables (3.6) the Hamiltonian becomes

$$H = \frac{1}{2a^2\mu_1}\left(p_0^2 + \left(1+\frac{\mu_1}{\mu_2}\right)p_1^2 + \frac{\mu_1+\mu_2\sin^2(q_0+q_1)}{\mu_2\sin^2 q_1}\left(M_0^2 - p_0^2\right) - 2p_0p_1\right) + U(q_1).$$

Remark. In [29] a different procedure, based on the Lagrangian representation, is proposed for reducing the 2-body problem on S^2. We recast it here in "Hamiltonian terms". First of all, we eliminate the cyclic variable ψ, on which the Hamiltonian $\left(\frac{\partial H}{\partial \psi} = 0\right)$ does not depend. To do this, it suffices to set

$$P_\psi = M_0 = \text{const}$$

in the Hamiltonian (3.3).

Then, using the freedom of choice of the fixed axes $OXYZ$, we define them so that the axis OZ is directed along the total angular momentum $\boldsymbol{M}$. Consequently, the system admits two invariant relations

$$M_X = 0, \quad M_Y = 0.$$

From (3.2) we conclude that in the canonical variables these relations can be represented as

$$F_1 = P_\theta = 0, \quad F_2 = P_\varphi - \cos\theta P_\psi = P_\varphi - M_0\cos\theta = 0.$$

The submanifold $\mathcal{M}_0 = \{F_1 = 0, F_2 = 0\}$, which corresponds to these relations, is not Poisson, but

$$\{F_1, F_2\} = -M_0\sin\theta \neq 0.$$

Hence, in this case, the Dirac reduction [21] is possible and the Dirac bracket of the variables φ, P_φ, $\boldsymbol{q}$, $\boldsymbol{p}$ turns out to be canonical:

$$\{\varphi, P_\varphi\}_{\mathcal{D}} = \{\varphi, P_\varphi\} + \frac{\{\varphi, F_1\}\{P_\varphi, F_2\} - \{\varphi, F_2\}\{P_\varphi, F_1\}}{\{F_1, F_2\}} = 1,$$

$$\{q_i, p_j\}_{\mathcal{D}} = \{q_i, p_j\} + \frac{\{q_i, F_1\}\{p_j, F_2\} - \{q_i, F_2\}\{p_j, F_1\}}{\{F_1, F_2\}} = \delta_{ij}.$$

When restricted to $\mathcal{M}_0$, the Hamiltonian (3.3) takes the form

$$H = \frac{1}{2R_0^2}\left(\frac{(P_\varphi - p_1)^2}{\mu_1} + \frac{p_1^2}{\mu_2} + \left(M_0^2 - P_\varphi^2\right)\frac{\mu_1\sin^2\varphi + \mu_2\sin^2(\varphi+q_1)}{\mu_1\mu_2\sin^2 q_1}\right) + U(q_1),$$

which, after the canonical transformation, coincides with the Hamiltonian of [29].

4. THE 2-BODY PROBLEM IN S^3

Above we have considered the situation where the configuration space of the system, $\mathcal{N}$, can almost everywhere be parameterized by the product

$$G \times \mathcal{N}_{\boldsymbol{q}}. \tag{4.1}$$

In this case, the potential energy depends only on the variables on $\mathcal{N}_{\boldsymbol{q}}$ and the (noncommutative) Lie group G is such that the kinetic energy of the system is invariant under left translations on it (more precisely, under the lift of this action to the phase space). In this case, a system reduced by the group action G can be represented naturally (almost everywhere) in Hamiltonian form on the Poisson manifold

$$\mathcal{M}_{red} = \mathfrak{g}^* \times T^*\mathcal{N}_{\boldsymbol{q}}, \tag{4.2}$$

where $\mathfrak{g}^*$ is the coalgebra of the Lie algebra of the group G and $T^*\mathcal{N}_{\boldsymbol{q}}$ is the tangent bundle of $\mathcal{N}_{\boldsymbol{q}}$.

For example, in the 2-body problem on S^2, $\mathcal{N}_q$ is an interval that parameterizes the distance between particles and $G \approx SO(3)$. In the problem of 3 and more bodies in S^3, this method admits a natural generalization (i.e., three noncoplanar radius vectors in $\mathbb{R}^4$ uniquely define some orthogonal frame), so we shall not consider this case here.

In the 2-body problem in S^3 the representation (4.1) is impossible, since the symmetry group is six-parametric $G \approx SO(4)$ and $\mathcal{N}_{\boldsymbol{q}}$ is also one-dimensional, so that $\dim G + \dim \mathcal{N}_{\boldsymbol{q}} = 7$, whereas the configuration space is only six-dimensional. This occurs due to the fact that the pair of straight lines in $\mathbb{R}^4$ that emanate from the origin of coordinates does not allow one to uniquely define the orientation of the entire moving frame in $\mathbb{R}^4$, since the rotation in the plane of the orthogonal plane of the particles remains undefined.

In view of this, we use the following algorithm to perform reduction.

1. Choose an extended configuration space $\widetilde{\mathcal{N}}$ that contains redundant variables and has the form
$$\widetilde{\mathcal{N}} = G \times \mathcal{N}_{\boldsymbol{q}},$$
where G is the symmetry group of the system. In this case, the Lagrangian function turns out to be degenerate in velocities. As a result, in Hamiltonian representation the phase space of the system has an invariant submanifold (on which the Legendre transformation can be reversed for all generalized velocities except for the redundant ones).

2. Perform reduction by the group action G and obtain a Hamiltonian system on the invariant submanifold of the Poisson manifold (4.2).

3. Perform reduction of the resulting system by symmetries corresponding to the redundant variables.

4.1. A Particle in the Central Field

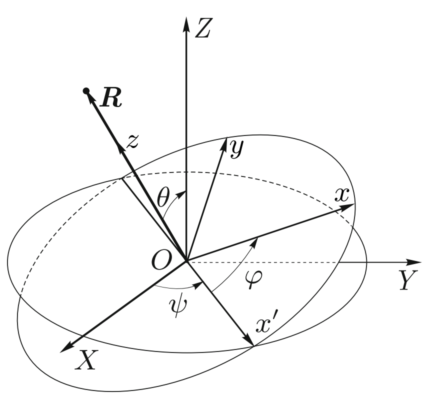

Fig. 2

To illustrate the proposed algorithm, we first consider a simpler example, a particle in $\mathbb{R}^3$ under the action of the central field. Let us choose a fixed orthogonal coordinate system $OXYZ$ with origin O at the center of the force field.

To apply methods of rigid body dynamics, we need to choose a moving orthogonal coordinate system $Oxyz$ attached to the particle, such that the axis Oz is directed along the radius vector of the particle $\boldsymbol{R}$ (see Fig. 2). It can be seen that the orientation of the axes in the plane Oxy is undefined. Hence, the configuration space thus defined
$$\widetilde{\mathcal{N}} = SO(3) \times \mathcal{N}_{\boldsymbol{q}},$$
where $\mathcal{N}_{\boldsymbol{q}} = \{q_1 = |R| \in (0, \infty)\} \approx \mathbb{R}^1$, contains a redundant variable, the angle of proper rotation φ. Indeed, writing the coordinates of the particle in the fixed coordinate system as
$$\boldsymbol{R} = \mathbf{Q}\boldsymbol{r},$$
where $\mathbf{Q}$ is the orthogonal matrix (3.1), $\boldsymbol{r} = (0, 0, q_1)$, and differentiating with respect to time, we obtain a Lagrangian function that is independent of φ and $\dot{\varphi}$:
$$L = \frac{1}{2}\mu(\dot{\boldsymbol{R}}, \dot{\boldsymbol{R}}) - U(|\boldsymbol{R}|) = \frac{1}{2}\mu q_1^2\big(\dot{\theta} + \sin^2\theta\dot{\psi}^2\big) + \frac{1}{2}\mu\dot{q}_1^2 - U(q_1),$$
here, μ is the mass of the particle and U is its potential energy. Thus, the resulting system turns out to be invariant not only under the change of the fixed axes, but also under rotations about the moving axis Oz.

After a Legendre transformation which is defined on the invariant submanifold
$$P_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = 0,$$

passing to the left-invariant functions in the coalgebra $so^*(3)$, we obtain a Hamiltonian system on $\widetilde{\mathcal{M}}_{red} = so^*(3) \oplus T^*\mathcal{N}_{\boldsymbol{q}}$:

$$\dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1},$$

$$H = \frac{1}{2\mu q_1^2}(m_x^2 + m_y^2) + \frac{1}{2\mu}p_1^2 + U(q_1),$$

where $\boldsymbol{m} = (m_x, m_y, m_z)$, on the fixed level set

$$m_z = P_\varphi = 0.$$

Since the angle φ is redundant, this system is invariant under the rotation group action generated by the Hamiltonian

$$H_\varphi = m_z.$$

The invariants under this action are

$$q_1,\ p_1,\ M_0^2 = m_1^2 + m_2^2,$$

and the function M_0^2 is an integral of motion (since on the level set $m_3 = 0$ it coincides with the Casimir function $\boldsymbol{m}^2$). Finally, we obtain a reduced system with one degree of freedom

$$H = \frac{1}{2\mu}p_1^2 + U_r(q_1), \quad U_r = U(q_1) + \frac{M_0^2}{2\mu q_1^2}.$$

4.2. The 2-body Problem in S^3

As above, we assume the sphere S^3 to be embedded into the four-dimensional Euclidian space $\mathbb{R}^4$. We choose two orthogonal coordinate systems in $\mathbb{R}^4$ with origin at the center of the sphere: a fixed coordinate system, $OX_1X_2X_3X_4$, and a moving coordinate system, $Ox_1x_2x_3x_4$, where the axis Ox_1 is directed along the radius vector of the first body and the second body lies in the plane Ox_1x_2. As in the previous case, the rotation angle of the axes Ox_3 and Ox_4 in the plane Ox_3x_4 is not defined and is a redundant variable.

Let $\mathbf{Q}$ be an orthogonal 4×4 matrix defining a transformation from the moving axes to the fixed ones so that

$$\begin{gathered} \boldsymbol{R}_\alpha = \mathbf{Q}\boldsymbol{r}_\alpha, \quad \alpha = 1, 2, \\ \boldsymbol{r}_1 = (a, 0, 0, 0), \quad \boldsymbol{r}_2 = (a\cos q_1, a\sin q_1, 0, 0), \end{gathered} \tag{4.3}$$

where a is the radius of the sphere S^3.

The kinetic energy of the system can be written as

$$T = \frac{1}{2}\sum_\alpha \mu_\alpha \dot{\boldsymbol{R}}_\alpha^2 = -\frac{1}{2}\operatorname{Tr}\hat{\boldsymbol{\omega}}\mathbf{J}\hat{\boldsymbol{\omega}} - \mu_2 a^2\omega_{12}\dot{q}_1 + \frac{1}{2}\mu_2 a^2\dot{q}_1^2,$$

$$\hat{\boldsymbol{\omega}} = \mathbf{Q}^T\dot{\mathbf{Q}}, \quad \mathbf{J} = \sum_\alpha \mu_\alpha \boldsymbol{r}_\alpha \otimes \boldsymbol{r}_\alpha = \begin{pmatrix} a^2(\mu_1 + \mu_2\cos^2 q_1) & \mu_2 a^2 \sin q_1 \cos q_1 & 0 & 0 \\ \mu_2 a^2 \sin q_1 \cos q_1 & \mu_2 q^2 \sin^2 q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\hat{\boldsymbol{\omega}}$ is the skew-symmetric 4×4 matrix of the angular velocity of the moving frame and μ_1, μ_2 are the masses of the bodies. Finally, we obtain a Lagrangian function in the form

$$\begin{aligned} L = \frac{a^2}{2}\Big[&(\mu_1 + \mu_2)\omega_{12}^2 + (\mu_1 + \mu_2\cos^2 q_1)(\omega_{13}^2 + \omega_{14}^2) + \mu_2\sin^2 q_1(\omega_{23}^2 + \omega_{24}^2) \\ &+ 2\mu_2 \sin q_1 \cos q_1(\omega_{13}\omega_{23} + \omega_{14}\omega_{24}) - 2\mu_2\omega_{12}\dot{q}_1 + \mu_2\dot{q}_1^2\Big] - U(q_1). \end{aligned}$$

The redundancy of the configuration space leads to the degeneration

$$\frac{\partial L}{\partial \omega_{34}} = 0.$$

After the Legendre transformation

$$m_{ij} = \frac{\partial L}{\partial \omega_{ij}},$$

which is defined on the level set $m_{34} = 0$, we obtain a Hamiltonian system on the manifold $so^*(4) \times T^*\mathcal{N}_{\boldsymbol{q}} = \{(\hat{\boldsymbol{m}}, q_1, p_1)\}$, where $\hat{\boldsymbol{m}} \in so^*(4)$ is the skew-symmetric matrix of angular momentum. The Hamiltonian is given by

$$\begin{aligned} H &= \sum_{i<j} m_{ij}\omega_{ij} + \dot{q}_1 p_1 - L\big|_{\omega_{ij}\to m_{ij}} \\ &= \frac{1}{2a^2}\left[\frac{m_{12}^2}{\mu_1} + \frac{m_{13}^2 + m_{14}^2}{\mu_1} + \frac{\mu_1 + \mu_2\cos^2 q_1}{\mu_1\mu_2\sin^2 q_1}(m_{23}^2 + m_{24}^2)\right. \\ &\quad \left. - 2\frac{\cos q_1}{\mu_1 \sin q_1}(m_{13}m_{23} + m_{14}m_{24}) + \frac{2}{\mu_1}m_{12}p_1 + \frac{\mu_1+\mu_2}{\mu_1\mu_2}p_1^2\right] + U(q_1). \end{aligned} \tag{4.4}$$

The Poisson brackets on $so^*(4)$ are given for the elements of the momentum matrix by

$$\{m_{ij}, m_{kl}\} = \delta_{ik}m_{jl} + \delta_{jl}m_{ik} - \delta_{il}m_{jk} - \delta_{jk}m_{il}, \tag{4.5}$$

while q_1 and p_1 remain canonical:

$$\{q_1, p_1\} = 1.$$

The Casimir functions of this Poisson structure are

$$C_1 = \sum_{i<j} m_{ij}^2, \quad C_2 = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}. \tag{4.6}$$

As is well known, in this case the integrals of motion corresponding to the group action $SO(4)$ are the projections of angular momentum onto the fixed axes, which are given by the elements of the skew-symmetric matrix

$$\hat{\boldsymbol{M}} = \mathbf{Q}\hat{\boldsymbol{m}}\mathbf{Q}^T = \|M_{ab}\| = \left\|\sum_{i,j} Q_{ai}Q_{bj}m_{ij}\right\|, \tag{4.7}$$

where $\mathbf{Q}$ is the orthogonal matrix defined above (4.3) and the indices i, j, k relate here (and in the sequel) to the projections onto the moving axes $Ox_1x_2x_3x_4$, whereas the indices a, b relate to the projections onto the fixed axes $OX_1X_2X_3X_4$.

The symmetry group corresponding to the redundant variable is generated by the Hamiltonian

$$H_\varphi = m_{34}. \tag{4.8}$$

The invariants of this action which form a set closed relative to the Poisson bracket (4.5) have the form

$$Z_0 = m_{12}, \quad Z_1 = m_{13}^2 + m_{14}^2, \quad Z_2 = m_{23}^2 + m_{24}^2, \quad Z_3 = 2(m_{13}m_{23} + m_{14}m_{24}),$$

and, when restricted to $m_{34} = 0$, their commutation relations q_1 and p_1 are represented as

$$\begin{gathered} \{Z_0, Z_1\} = Z_3, \quad \{Z_0, Z_2\} = -Z_3, \quad \{Z_0, Z_3\} = 2(Z_2 - Z_1), \\ \{Z_1, Z_2\} = 2Z_0Z_3, \quad \{Z_1, Z_3\} = 4Z_0Z_1, \quad \{Z_2, Z_3\} = -4Z_0Z_2, \\ \{q_1, p_1\} = 1. \end{gathered}$$

This bracket is degenerate, and its Casimir functions are obtained using the functions (4.6) as follows:

$$\tilde{C}_1 = C_1\big|_{m_{34}=0} = Z_0^2 + Z_1 + Z_2, \quad \tilde{C}_2 = C_2^2\big|_{m_{34}=0} = 4Z_1Z_2 - Z_3^2$$

The Hamiltonian (4.4) is also expressed in terms of the variables $\boldsymbol{Z}$, q_1 and p_1:

$$H = \frac{1}{2a^2\mu_1}\left(Z_0^2 + Z_1 + A(q_1)Z_2 - B(q_1)Z_3 + 2Z_0p_1 + \left(1 + \frac{\mu_1}{\mu_2}\right)p_1^2\right) + U(q_1),$$

$$A(q_1) = \frac{\mu_1 + \mu_2\cos^2 q_1}{\mu_2\sin^2 q_1}, \quad B(q_1) = \frac{\cos q_1}{\sin q_1}.$$

Thus, we obtain a *reduced system of the* 2*-body problem on* S^3 *in algebraic form* (*for a part of variables*).

To represent the system in canonical form, we supplement the variables q_1, p_1 with a pair of canonically conjugate variables q_0, p_0 which parameterize the fixed level set of the Casimir functions

$$\tilde{C}_1 = M_0^2, \quad \tilde{C}_2 = \Delta^2.$$

We use a parameterization which is a natural generalization of (3.6):

$$Z_0 = p_0, \quad Z_3 = \rho \sin 2q_0,$$
$$Z_1 = \frac{1}{2}\big(M_0^2 - p_0^2 + \rho\cos 2q_0\big), \quad Z_2 = \frac{1}{2}\big(M_0^2 - p_0^2 - \rho\cos 2q_0\big),$$
$$\rho^2 = \big(M_0^2 - p_0^2\big)^2 - \Delta^2, \quad \{q_0, p_0\} = 1.$$

We see that for $\Delta = 0$ these relations lead to a system coinciding with the reduced system on S^2. In the general case, when $\Delta \neq 0$, in canonical variables the Hamiltonian of the reduced system does not correspond to any natural mechanical system.

To reconstruct the solutions of the complete system, we supplement the brackets (4.5) with the commutation relations of the momenta $\hat{\boldsymbol{m}}$ with elements of the orthogonal matrix $\mathbf{Q} \in SO(4)$ defined above (4.2):

$$\{m_{ij}, Q_{ak}\} = \delta_{ik}Q_{aj} - \delta_{jk}Q_{ai}. \tag{4.9}$$

According to (4.9), the columns of the matrix $\mathbf{Q}$ are projections of the unit vectors of the moving frame onto the fixed axes. We denote them as

$$\boldsymbol{E}_i = (Q_{1i}, Q_{2i}, Q_{3i}, Q_{4i}).$$

In this case, the radius vectors of the particles are given by

$$\boldsymbol{R}_1 = a\boldsymbol{E}_1, \quad \boldsymbol{R}_2 = a\cos q_1 \boldsymbol{E}_1 + a \sin q_1 \boldsymbol{E}_2.$$

Consequently, to reconstruct the motion of the bodies for the given values of the first integrals (4.7) and the known solutions of the reduced system $\boldsymbol{Z}(t)$, $q_1(t)$, $p_1(t)$, we need to have equations governing the evolution of the unit vectors $\boldsymbol{E}_1(t)$, $\boldsymbol{E}_2(t)$.

First of all, we note that, as a consequence of this definition of $\mathbf{Q}$, the components of the vectors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ do not depend on the redundant variable, i.e., they are invariant under the action given by the Hamiltonian (4.8):

$$\{m_{34}, E_{1a}\} = \{m_{34}, E_{2a}\} = 0, \quad a = 1, \ldots, 4.$$

Moreover, their commutation relations with the variables of the reduced system $Z_0, \ldots, Z_3$ (on the level set $m_{34} = 0$) can be represented as

$$\{Z_0, E_{1a}\} = E_{2a}, \quad \{Z_0, E_{2a}\} = -E_{1a},$$
$$\{Z_1, E_{1a}\} = \{Z_3, E_{2a}\} = -2\Big(\sum_b M_{ab}E_{1b} + Z_0E_{2a}\Big),$$
$$\{Z_2, E_{2a}\} = \{Z_3, E_{1a}\} = -2\Big(\sum_b M_{ab}E_{2b} - Z_0E_{1a}\Big),$$
$$\{Z_1, E_{2a}\} = \{Z_2, E_{1a}\} = 0, \quad a, b = 1, \ldots, 4,$$

where M_{ab} are the elements of the matrix of the projections of momentum onto the fixed axes (i.e., the integrals of motion).

Remark. The derivation of these relations is based on the equations

$$\sum_b Q_{bi}M_{ba} = \sum_j Q_{ai}m_{ij},$$

which are derived immediately from the definition (4.7).

Commuting the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ with the Hamiltonian (4.2), we obtain equations governing their evolution:

$$a^2\mu_1\dot{\boldsymbol{E}}_1 = \hat{\boldsymbol{M}}\big(\boldsymbol{E}_1 - B(q_1)\boldsymbol{E}_2\big) + B(q_1)Z_0\boldsymbol{E}_1 - p_1\boldsymbol{E}_2,$$
$$a^2\mu_1\dot{\boldsymbol{E}}_2 = \hat{\boldsymbol{M}}\Big(A(q_1)\boldsymbol{E}_2 - B(q_1)\boldsymbol{E}_1\Big) + \Big(Z_0\big(1 - A(q_1)\big) + p_1\Big)\boldsymbol{E}_1 - B(q_1)Z_0\boldsymbol{E}_2.$$

5. NUMERICAL ANALYSIS OF THE 2-BODY PROBLEM

Consider the problem of the motion of two particles on S^3 which interact with each other with the (Newtonian) potential

$$U = \gamma\cot q_1,$$

where γ is the constant of gravitation.

As variables of the reduced system we choose$(\boldsymbol{Y}, q_1, \tilde{p}_1)$:

$$Z_0 = 2Y_0, \quad Z_1 = \frac{Y_1 + Y_2}{2}, \quad Z_2 = \frac{Y_1 - Y_2}{2}, \quad Z_3 = Y_3, \quad p_1 = \frac{\tilde{p}_1}{\sin q_1}.$$

In this case, the equations of motion can be represented as

$$\begin{aligned}
\sin^2 q_1\dot{Y}_0 &= \frac{Y_2\sin 2q_1 - Y_3\cos 2q_1}{4a^2\mu_1} - \frac{Y_3}{4a^2\mu_2},\\
\sin^2 q_1\dot{Y}_1 &= 2Y_0\frac{Y_3\cos 2q_1 - Y_2\sin 2q_1}{a^2\mu_1} + \frac{2Y_0Y_3}{a^2\mu_2},\\
\sin^2 q_1\dot{Y}_2 &= 2Y_0\frac{Y_3\cos 2q_1 - Y_1\sin 2q_1}{a^2\mu_1} + \frac{2Y_0Y_3}{a^2\mu_2} - \frac{2Y_2\tilde{p}_1}{a^2\mu_1},\\
\sin^2 q_1\dot{Y}_3 &= \frac{2Y_2\tilde{p}_1}{a^2\mu_1}\sin q_1 + \frac{2Y_0(Y_1 - Y_2)}{a^2\mu_1\mu_2}(\mu_1 + \mu_2\cos 2q_1),\\
\sin^2 q_1\dot{\tilde{p}}_1 &= \frac{(\mu_1 + \mu_2)(2\tilde{p}_1^2 + Y_1 - Y_2)}{2a^2\mu_1\mu_2}\cos q_1 - \frac{Y_3}{2a^2\mu_1}\sin q_1 + \frac{\tilde{p}_1Y_0}{a^2\mu_1}\sin 2q_1 + \gamma\sin q_1,\\
\sin q_1\dot{q}_1 &= \frac{(\mu_1 + \mu_2)\tilde{p}_1}{a^2\mu_1\mu_2}\sin q_1 + \frac{2Y_0}{a^2\mu_1}Y_0.
\end{aligned} \tag{5.1}$$

The Casimir functions have the form

$$\tilde{C}_1 = 4Y_0^2 + Y_1, \quad \tilde{C}_2 = Y_1^2 - Y_2^2 - Y_3^2.$$

The values $q_1 = 0$ and $q_1 = \pi$ correspond to collision of particles, in this case Eqs. (5.1) have a singularity. To regularize the system (5.1), we make the change of variables

$$dt = \sin^2 q_1 d\tau.$$

To illustrate the behavior of the trajectories of the regularized system, we take a Poincaré map and restrict the system to the four-dimensional manifold of the level set of the first integrals

$$\mathcal{M}^4 = \{(\boldsymbol{Y}, q_1, \tilde{p}_1) \mid \tilde{C}_1 = M_0^2, H = h\}$$

whence we obtain a four-dimensional flow with the integral $\tilde{C}_2|_{\mathcal{M}^4}$. We parameterize it by the variables $(Y_0, Y_2, q_1, \tilde{p}_1)$.

Then we fix $\tilde{C}_2 = \Delta^2$ and thus obtain a family of three-dimensional flows. As the secant for this flow we choose

$$Y_0 = 0.$$

Numerically integrating and finding intersections of the trajectories with the above section, we finally obtain a family of point two-dimensional maps, which we parameterize by $(q_1, \tilde{p}_1)$.

A Poincaré section for different fixed parameters is presented in Figs. 3 and 4.

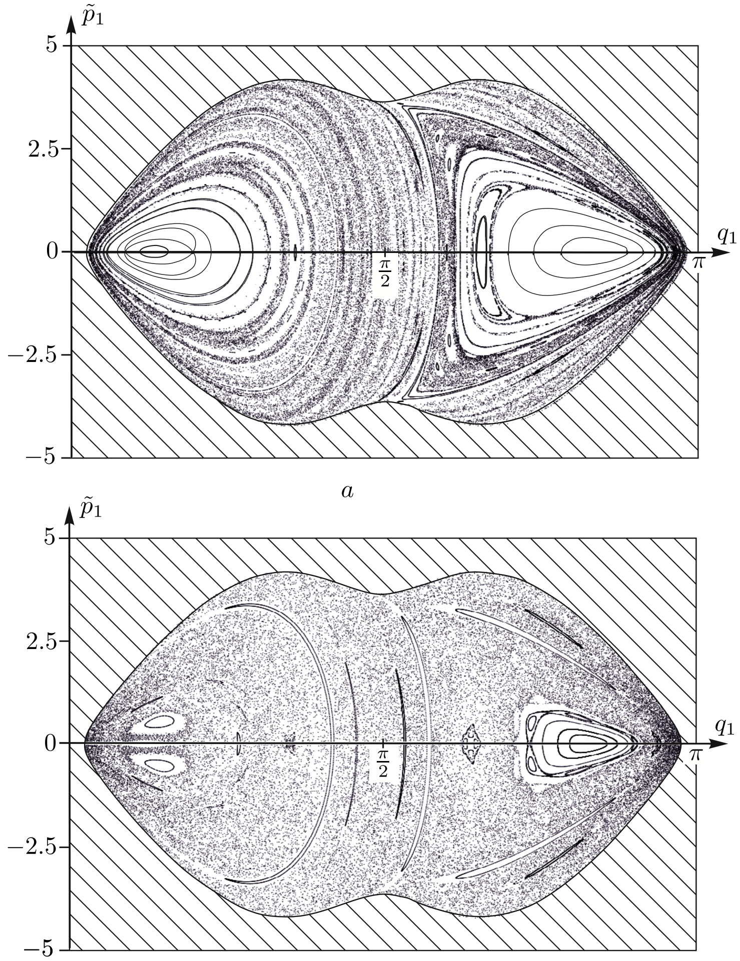

Fig. 3. Poincaré section (secant $Y_0 = 1$) for the fixed parameters $\mu_1 = 1$, $\mu_2 = 2$, $a = 1$, $h = 60$, $M_0^2 = 200$, $\Delta^2 = 150$: a) $\gamma = 0.004$, b) $\gamma = 0.05$ (the Poincaré section in the shaded area is undefined for these parameters).

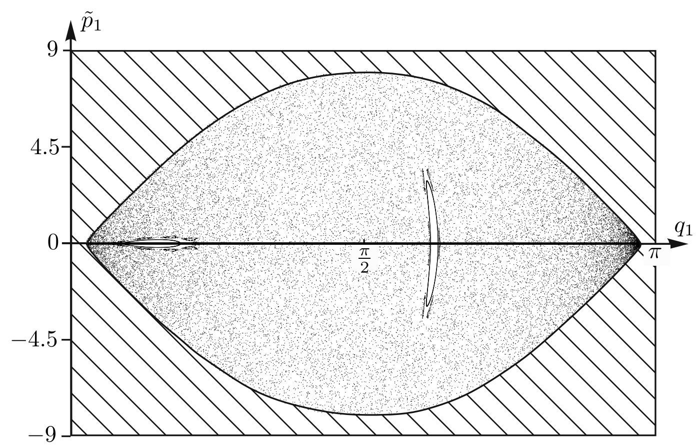

Fig. 4. Poincaré section (secant $Y_0 = 0$) for the fixed parameters $\mu_1 = 1$, $\mu_2 = 2$, $a = 1$, $\gamma = 5$, $h = 80$, $M_0^2 = 200$, $\Delta^2 = 0$ (the Poincaré section in the shaded area is undefined for these parameters).

ACKNOWLEDGMENTS

The authors express gratitude to Valery V. Kozlov, Andrey A. Oshemkov and Alexander A. Kilin for useful comments in the process of preparing this paper. Special thanks are due to Alain Albouy for constant and inspiring discussions on issues concerning celestial mechanics in SCCs (especially as far as the history is concerned), which have contributed to the understanding of the problems in this area and to the development of new problems.

This work was supported by the Russian Scientific Foundation (project No. 14–50–00005).

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