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Historical and Critical Review of the Development of Nonholonomic Mechanics: the Classical Period

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Abstract—In this historical review we describe in detail the main stages of the development of nonholonomic mechanics starting from the work of Earnshaw and Ferrers to the monograph of Yu. I. Neimark and N. A. Fufaev. In the appendix to this review we discuss the d'Alembert – Lagrange principle in nonholonomic mechanics and permutation relations.

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INTRODUCTION

In this review we shall try to describe the most important aspects of development of the theory of nonholonomic systems. We note that the introduction of new geometric and topological methods, as well as methods of qualitative analysis and high-performance computer calculations, has given a great impetus to nonholonomic mechanics lately. Some basic avenues of research in nonholonomic mechanics have already been outlined in the previous review [15].

In this paper we do not claim to present a comprehensive survey of the literature, as is done, for example, in the recent book [121]. Instead, we undertake a critical review of the most important and interesting studies in this area. Such an approach is of particular importance in view of an increase in the number of new journals and an avalanche-like increase in the number of publications on the dynamics of nonholonomic systems. In our opinion, an uncritical citation of the literature does not allow one to distinguish major results from minor and even erroneous results[1)]. We shall pay particular attention to works that are of interest from the viewpoint of our understanding of the concept of constructive approach to analysis of dynamical systems, as presented in [5].

This review was motivated, among others, by the sketch [74], which is concerned less with analyzing the historical development of nonholonomic mechanics than with advocating an approach that is mainly developed in *Journal of Geometric Mechanics*. This branch of geometric mechanics (the general principles of which are laid down in [57]) is a more abstract continuation of the ideas of J. Marsden's school, which are set forth in [11]. A synthesis of both approaches is presented in the book [84].

In our view, at the initial stage of its development geometric mechanics made it possible to describe some theorems of dynamics in a geometrically clearer form (as is well known, in his *Mécanique Analytique* [72] Lagrange completely ignored drawings and figures). But later the focus was not on the solution of complicated and interesting problems (although there are some exceptions here), but on the development of a peculiar geometric language, mainly using the old problems that had already been solved in classic works[2)].

The approach of geometric mechanics is illustrated in the paper by Duistermaat [42], in which the well-known work of Chaplygin [35] about the rolling motion of a ball is presented in terms of geometric mechanics. Although the volume of the paper [42] is several times larger than that of [35] (101 pages and 31 pages, respectively), the problems that had not been solved by Chaplygin remained unresolved in [42]. Another example is the global geometric approach to the reduction of dynamical systems, which is criticized in [17]. It turns out that constructively such a global view mostly does not lead to any new results, and the proposed generalizations of the classical results usually do not work in practice. In our view, such a formal approach considerably increases the gap between mechanics and practical applications.

We note that both the abstract approach and the constructive approach have their adherents. The open problems of the 20th century mathematics as formulated by D. Hilbert and H.Poincaré are an illustrative example. Afterwards, the former point of view was backed up by bourbakists and partially by A. Einstein and P. Dirac, who linked the value of a physical theory with its beauty. In its extreme form it leads to a widening of the gap between mathematics and applications and real problems and transforms it into a branch of linguistics or scholasticism. Unfortunately, the number of adherents of this point of view has considerably increased lately, since the activities of the scientific community are evaluated not by the quality of scientific results, but by the number of published and cited works. Indeed, "linguistic" texts are much faster to write and publish than really profound studies (and even than those which are not erroneous). In spite of many criticisms, the formal approach in mathematics continues to develop. Many publications have appeared recently in which the well-known classical results, recast in new terms, are passed off as new ones.

The latter point of view (held by H. Poincaré) is that the development of science, including mathematics, is stimulated primarily by the solution of concrete problems and not by the creation of abstract theories. We make here several quotations which give an exhaustive estimate of this approach in mathematics.

[1)] The existing citation databases do not even distinguish erroneous studies from really valuable ones.

[2)] Despite the fact that J.Marsden was for many years involved in our efforts to promote the journal *Regular and Chaotic Dynamics*, our views on scientific issues were essentially different, and our approach in [15] was characterized as "somewhat anti-reductionist" in [46].

Jacobi [59](see also [67]) about Euler's work:

"Euler's work has the great merit that it presents, wherever possible, all cases in which problems can be solved completely using given methods and means... Therefore, his examples always show a complete content of his method according to the state of science of that time and, as a rule, when it is possible to add a new example to Euler's examples, it is an enrichment of science, for it rarely happened that a case solvable by his methods escaped his attention..."

P. Halmos [52]:

"The heart of mathematics consists of concrete examples and concrete problems. Big general theories are usually afterthoughts based on small but profound insights; the insights themselves come from concrete special cases..."

E. Zehnder [85] about J. Moser:

"These notes owe much to Jürgen Moser's deep insight into dynamical systems and his broad view of mathematics. They also reflect his specific approach to mathematics by singling out inspiring typical phenomena rather than designing abstract theories..."

We note that, unlike Hamiltonian mechanics, the dynamics of nonholonomic systems possesses much more diverse properties, which manifests itself in unusual motions (such as those of a rattleback etc.), many of which can be observed experimentally. These unusual and, at first glance, enigmatic properties stimulate the further process of study of nonholonomic systems.

Nowadays, nonholonomic problems are solved by using many branches of mathematics: the theory of groups and Lie algebras, topology, qualitative methods, the theory of conditionally periodic functions, the theory of general Poisson structures and methods of effective computer analysis. In addition, nontrivial intuition and a good ability to perform analytic calculations are required. These qualities were inherent in S. A. Chaplygin, who after the work of Hertz brought nonholonomic mechanics to an entirely new level. Most of the modern methods for analyzing nonholonomic systems (methods of integration, Hamiltonization etc.) go back to his work. The abstract level of the mathematical language often involves many additional obstacles in solving more complicated problems of nonholonomic mechanics. This is why the authors of "general theories" rarely achieve real progress in solving new problems, and their works either use the same elementary systems, for which everything is clear as it is, or introduce new abstract systems which are absolutely useless. For example, the introduction of Weinstein's groupoids and algebroids into the range of problems of nonholonomic mechanics, as is done in [74], will hardly fill engineers and physicists with optimism.

Remark. By the way, the recent remarkable book [73] on Poisson structures gives almost no examples, and the exposition is focused on the proof of general results. Moreover, the authors of [73] do not point out the deeper properties of Poisson manifolds which are associated with a chaotic embedding of symplectic leaves. Various examples of such sort have been found recently in nonholonomic mechanics [9].

This work is only the first part of the review presented by us. The second part will address the recent results (from 1985 until now) and formulate a number of unsolved problems. In order not to encumber the main historical material with formulae, they have been presented in the Appendix, which discusses various forms of equations of nonholonomic systems and their fundamental differences from Hamiltonian equations (in particular, the impossibility of obtaining them from the variational principle).

1. THE FIRST PERIOD: UNTIL THE APPEARANCE OF THE WORK OF HERTZ IN 1894 AND THE GENERAL CONCEPT OF NONHOLONOMIC SYSTEMS

In that period, various versions of the problem of a rigid body rolling without slipping (going back to L. Euler and S. Poisson) were considered. As an example, we mention the treatise of S. Earnshaw (1844) [43], concerned with the motion of a homogeneous ball on a rotating plane (which still attracts the attention of researchers [19]).

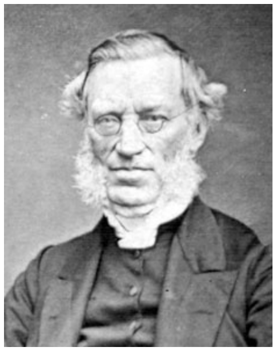

Fig. 1. Samuel Earnshaw (1805–1888)

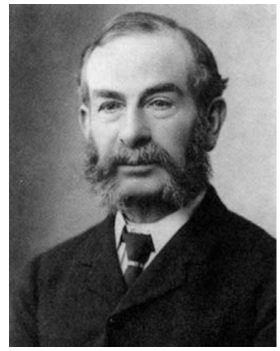

Fig. 2. Edward John Routh (1831–1907)

The equations of motion of a rigid body (disk, hoop or coin) rolling on a plane were obtained by G. Slesser (1861) [104] from the general principles of dynamics by an explicit elimination of constraint reaction forces [3)].

N. Ferrers (1872) [48] explicitly showed that the equations of motion of a system with constraints given in differential form cannot be represented in the form of Lagrange equations. However, subject to certain conditions, such a representation turns out to be possible for some variables. As an example, Ferrers used the problem of a rolling hoop in which the angle between the normal to the plane of the disk and the vertical (nutation angle) evolves according to the standard Lagrange equations (of genus 2)[4)]. We note that the works [48, 104] still contained no hint of the general formalism of the dynamics of nonholonomic systems.

E. Routh (1884) [99] considered the problem of a homogeneous ball rolling on a rotating axisymmetric surface. It was Routh who pointed out the possibility of using the Lagrange equations with undetermined multipliers (of genus 1) to describe the motion of nonholonomic systems.

In a more general form which is particularly suitable to describe rigid body dynamics, the equations of motion were obtained by A. Vierkandt (1892) [112]. In this work the author is already well aware of the specificity of nonholonomic constraints and of the fact that these constraints arise in the case of rolling. All of his results are in good agreement with the modern analysis of the problem (see, e.g., [20]). In [112], special attention was paid to the motion of a homogeneous (round) disk, for which Vierkandt noted that the motion of the reduced system is, as a rule, periodic. He gave a general description of the motion and dynamics of the point of contact, in particular, he pointed out steady motions in which the point of contact moves in a circle.

2. THE SECOND PERIOD: 1894–1912 (INTENSIVE DEVELOPMENT, THE WORK OF CHAPLYGIN)

The general understanding of the inapplicability of the Lagrange equations and variational principles in nonholonomic mechanics is due to H. Hertz (1894), who discussed these issues in his fundamental work "Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt" [56]. We note that this work is mainly devoted to the concept of hidden cyclic parameters (coordinates, masses), which Hertz contrasted to the usual concept of interaction as a result of the action of forces. To better understand his views, we present some fragments from this work.

"The application of Hamilton's principle to a material system does not exclude the existence of fixed connections between the chosen coordinates. But at any rate it requires that these connections be mathematically expressible by finite equations between the coordinates: it does not permit the occurrence of connections which can only be represented by differential equations. But nature itself does not appear to entirely exclude connections of this kind. They arise, for example, when bodies of three dimensions roll

3) Nowadays most mechanical engineers proceed in this way.

4) Afterwards this was reflected in problems of the Hamiltonization of a reduced system, as described in [14]. It should also be noted that the solution of the overwhelming majority of specific problems does not require a theoretical development of nonholonomic mechanics, and all equations can be obtained using the general principles of dynamics.

on one another without slipping. By such a connection, examples of which frequently occur, the position of the two bodies with respect to each other is only limited by the condition that they must always have one point of their surfaces common; but the freedom of motion of the bodies is further diminished by a degree. From the connection, then, there can be deduced more equations between the changes of the coordinates than between the coordinates themselves; hence there must amongst these equations be at least one non-integrable differential equation. Now Hamilton's principle cannot be applied to such a case; or, to speak more correctly, the application, which is mathematically possible, leads to results which are physically false. Let us restrict our consideration to the case of a sphere rolling without slipping upon a horizontal plane under the influence of its inertia alone. It is not difficult to see, without calculation, what motions the sphere can actually execute. We can also see what motions would correspond to Hamilton's principle; these would have to take place in such a way that with constant *vis viva* the sphere would attain given positions in the shortest possible time. We can thus convince ourselves, without calculation, that the two kinds of motions exhibit very different characteristics. If we choose any initial and final positions of the sphere, it is clear that there is always one definite motion from the one to the other for which the time of motion, *i.e.* the Hamilton's integral, is a minimum. But, as a matter of fact, a natural motion from every position to every other is not possible without the co-operation of forces, even if the choice of the initial velocity is perfectly free. And even if we choose the initial and final positions so that a natural free motion between the two is possible, this will nevertheless not be the one which corresponds to a minimum of time. For certain initial and final positions the difference can be very striking. In this case a sphere moving in accordance with the principle would decidedly have the appearance of a living thing, steering its course consciously towards a given goal, while a sphere following the law of nature would give the impression of an inanimate mass spinning steadily towards it...

This defence is not quite convincing. For rolling without slipping does not contradict either the principle of energy or any other generally accepted law of physics..."

These somewhat obscure and philosophical arguments of Hertz obviously point to the inapplicability of Hamilton's principle for nonholonomic systems. Moreover, they point to the possibility of realizing nonholonomic constraints by rolling rigid bodies without slipping on each other and to the validity of the law of conservation of energy for such systems (in contrast to systems with friction). Even the figurative observations of Hertz in which he compared the sphere with a living thing can be interpreted as arguments in favor of the absence of determinacy when systems with nonholonomic constraints[5)] are described using Hamilton's principle. The observations of Hertz were developed by H. Poincaré (1897) [92] in his popular work "Les idées de Hertz sur la Mécanique". In particular, he gave an elementary proof of the nonholonomicity of constraints in the problem of rolling of a sphere on a plane and noted that Hamilton's priciple (principle of least action) cannot be applied to the description of nonholonomic systems.

Remark. We note that Hertz and Poincaré drew a correct conclusion that Hamilton's principle cannot be applied to general nonholonomic systems. However, their example of a homogeneous sphere rolling on a plane is not quite correct (since it admits a representation in Hamiltonian form), which was pointed out later by Capon in [30].

We briefly recall what the difference is between nonholonomic and holonomic constraints. Let $\boldsymbol{q} = (q_1, \ldots, q_n)$ be the generalized coordinates on the configuration space of the system. The constraints dealt with in nonholonomic mechanics are, as a rule, linear and homogeneous in generalized velocities and do not explicitly depend on time:

$$f_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{j=1}^{n} a_{ij}(\boldsymbol{q}) \frac{dq_j}{dt} = 0, \quad i = 1, \ldots, m < n. \tag{2.1}$$

Nonholonomic constraints, as opposed to holonomic constraints, cannot be represented in the finite integral form

$$F_i(\boldsymbol{q}, t) = 0, \quad i = 1, \ldots, \tilde{m} < n. \tag{2.2}$$

Analysis of the reducibility of the constraints (2.1) to the form (2.2) is closely related to the integrability of the system of Pfaffian equations. Indeed, after multiplication by dt the constraint equations (2.1) can be represented in the form of Pfaffian equations.

[5)] Much later, this was pointed out within the framework of vakonomic mechanics [5] (which cannot be applied to systems with rolling and which suggests realizing nonintegrable constraints in a hydrodynamical way).

Consequently, the criterion for verification of the nonholonomicity of constraints can be obtained using the Frobenius theorem [49] (1877) (see, e.g., [16]).

Remark. We refer the reader to an interesting review paper by T. Hawkins [55], in which the Frobenius theorem is considered from various points of view and is linked to the studies by Darboux, Cartan and Clebsch. From the modern point of view these problems were characterized by Arnold as contact geometry (see, e.g., [4]). We note that although the review paper [55] is complete, it does not mention, for example, the fundamental books by S. Lie [75–77], in which the Frobenius theorem is formulated and its transparent proof is given. However, it is interesting that Lie [77] did not mention Frobenius at all when discussing the contribution of various mathematicians to the creation of group theories and, in particular, the Pfaff problem, although many mathematicians of that time were mentioned by him, even though critically.

Of special mention is the work of J. Hadamard (1895) [51] (written under the supervision of Appel) who studied in detail the reducibility of constraints (2.1) to the form (2.2). We note that he formulated a model of rolling in which a nonholonomic constraint is realized between the rigid body and the surface, which is given by the condition that the velocity of the point of contact and the projection of the angular velocity of the rigid body onto the normal to the fixed surface be zero. Afterwards this model was also developed by H. Beghin (1929) [6]. Lately, it has been widely used in robotics. In [51], considerable attention is paid to various issues of the dynamics of nonholonomic systems. A further refinement to the above model was made in [18].

On the other hand, the non-Hamiltonian property in the general case of equations of nonholonomic mechanics follows from qualitative considerations (unfortunately, this was understood only much later [68]). For example, the stability analysis of various particular solutions (as a rule, steady or permanent rotations) for specific nonholonomic systems has shown that they may exhibit *asymptotic stability*, which is impossible for Hamiltonian systems.

The most interesting studies are those concerned with the stability of rotations about the vertical axis, the so-called *rattleback*, which displays a striking dependence of stability on the direction of rotation (reversal). In 1895 Walker was the first to establish this dependence, which is not typical of Hamiltonian systems [116]. As is well known, a characteristic feature of the rattleback is that the dynamical and geometrical axes at the point of contact do not coincide [6)].

We mention the works of E. Carvallo (1899) [33], J. Boussinesq (1899) [25, 26], R. Routh (1899) [98], and C. Bourlet (1899) [24] on bicycle theory, which provided some description of bicycle motion and an explanation of its unique stability[7)].

Of special note is an error that was made by Lindelöf in [78] (1895) in the problem of a rolling body of revolution and that was related to the application of the Lagrange equations in the presence of nonintegrable constraints and to the substitution of nonholonomic constraints in the kinetic energy. Later the erroneous result obtained by Lindelöf due to this error was included in the first edition of the treatise by P. Appel (1896) [2]. We also note that an error (related to the application of Hamilton's principle) was made by C. Neumann (1886) [88], which was corrected by him later (1899) [87].

The errors made served as an impetus for finding them and contributed to the development of nonholonomic mechanics, which became a separate branch of dynamics. For example, S. A. Chaplygin (1897) [34], after analyzing Lindelöf's error in detail, obtained a correct form for the equations of motion and examined in detail the motion of a body of revolution (in particular, the motion of a disk) on a plane. As a result, he showed the integrability by quadratures of an arbitrary body of revolution and pointed out the possibility of adding a balanced and uniformly rotating rotor along the axis of revolution (without loss of integrability). In addition, Chaplygin obtained the first general form of nonholonomic mechanics equations in which undetermined multipliers are eliminated and terms of nonholonomicity are singled out explicitly.

[6)] We note that a number of new nonholonomic systems with reversal have been found in recent years (see, e.g., [8, 12]).

[7)] Nowadays the stability of a bicycle was analyzed by A. Ruina [62, 82] et al., who corrected some misprints and errors in the above-mentioned works. As a result, they obtained opposite conclusions and thus "shook" the classical bicycle theory, which linked stability with the gyroscopic effect. An explanation was found in nontrivial effects of nonholonomic systems, for example, asymptotic stability in rattleback dynamics.

An error similar to that of Lindelöf was also made by E. Crescini (1889) [39] when he integrated the equations for a heavy body of revolution rolling on a horizontal plane[8] using the Hamilton – Jacobi method.

Slightly later, the erroneous results (including those of Lindelöf) obtained in the description of a body of revolution rolling on a plane were analyzed in detail by D. Korteweg (1899) [64], who illustrated the impossibility of eliminating nonholonomic constraints in a free Lagrangian using a model example (among erroneous works he also mentioned the work of P. Molenbrock (1890) [83] and G. Schouten (1899) [102]). Korteweg attached particular importance to the work of A. Vierkandt [112], in which a correct method for obtaining the equations of motion is used. Korteweg himself [63] reduced the problem of a circular heavy disk on the plane to a hypergeometric equation and noted[9] that in the case of a homogeneous disk the problem had been solved by Vierkandt[10].

P. Appel corrected the above-mentioned error contained in his treatise in the later editions. In doing so, he obtained a compact (and universal) form of equations, for both holonomic and nonholonomic mechanics, which contain the energy of accelerations (prior to Appel, this form of equations was investigated by Gibbs [50], but without explicit consideration of nonholonomic constraints). As regards the Appel equations, one should agree with the opinion of G. Hamel (1904) [53]:

> "The second attempt of Appel, when he replaces the living force T with a new acceleration function S, is methodically not very good either, despite its aesthetic merits. First one has to perform a transformation of the second derivatives of coordinates, because of this the living force T completely loses its central role, so that a deep abyss separates systems with nonholonomic and holonomic conditional equations, and this does absolutely not correspond to the difference between both problems. In fact, holonomic conditional equations are only a special case of nonholonomic ones. . . "

We note that when writing equations of motion for specific problems (involving, for example, the rolling motion), one usually uses equations with undetermined multipliers or in quasi-velocities, which in the general form were obtained by Hamel (1904) [53] (who continued the studies carried out by L. Boltzmann). Hamel himself suggested that the resulting form of equations be called the Euler – Lagrange equations, and this name is often used in various textbooks.

As an interesting historical fact we note that when writing the extensive work [53], Hamel studied the Lie theory quite well and mastered the technique of obtaining the equations of motion in a form that was very modern at that time. In particular, in the above-mentioned work Hamel presented the general form of dynamical equations for nonholonomic systems, which in the case of integrable constraints turns into Poincaré equations on the Lie group, which can be applied only to holonomic systems and were obtained by Poincaré (1901) [91] (only three years before publication of Hamel's work, which contains no reference to Poincaré).

Various forms of equations, which differ in the method of eliminating undetermined multipliers, were obtained by V. Volterra (1897) [113] and P. V. Voronets (1901) [114] (in contrast to Hamel, they did not use the formalism of Lie theory and were not well familiar with its tools). Moreover, it turned out that the Volterra equations contain some errors (analogous to those of Lindelöf) which arose due to the substitution of constraint equations into the kinetic energy and therefore apply only to holonomic systems (for more on such a substitution in nonholonomic mechanics, see [13]). In the literature one can also find equations in the form due to G. Maggi (1901) [80], I. Tzénoff (1920) [58], Tatarinov and others, which are also used periodically depending on the problems under consideration and the preferences of researchers.

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[8] By the way, the work of Crescini was communicated by V. Volterra, who also made errors in deriving nonholonomic mechanics equations.

[9] From the modern point of view, the qualitative analysis made by Vierkandt is much more important than the reduction to a hypergeometric equation.

[10] Here we are confronted with a really interesting historical fact that the studies of the rolling motion of a disk appeared one after another within a period of 11 years in Germany (Vierkandt), Holland (Korteweg), Russia (Chaplygin), France (Appel), and England (Gallop). The studies by Vierkandt were published in a popular journal, the studies by Lindelöf and Chaplygin were published in little-known journals, but, in spite of the above-mentioned virtues, the work of Vierkandt remained almost unnoticed.

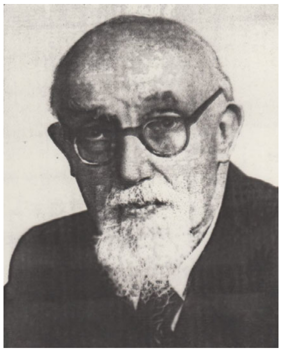

Fig. 3. Georg Karl Wilhelm Hamel (1877–1954)

Fig. 4. Sergey Alexeyevich Chaplygin (1869–1942)

Remark. Later, in his textbook on theoretical mechanics G. Hamel (1940) [54] compared various forms of the equations of motion due to Voronets, Tzénoff and Volterra and showed that all of them can be obtained from his general approach.

The development of nonholonomic mechanics was stimulated already in that period mainly by a successful solution of appropriate problems elucidating (qualitative) differences of the motion of nonholonomic systems from the motion of holonomic systems. The most important studies in this vein were, of course, those of S. A. Chaplygin (1903 and 1912) [35, 36], which deal with specific nonholonomic constraints often encountered in applications, namely, the *Chaplygin constraints*:

$$\dot{x}_i = \sum_{j=1}^{k} a_{ij}(\boldsymbol{q})\dot{\boldsymbol{q}}_j, \quad i = 1, \ldots, m$$

where $\boldsymbol{q} = (q_1, \ldots, q_k)$ and $\boldsymbol{x} = (x_1, \ldots, x_m)$ are the generalized coordinates on configuration space (of dimension $n = k + m$).

In the case where the problem reduces to a system with two degrees of freedom (i.e., $k = 2$) and the Lagrangian of the free system does not explicitly depend on the coordinates $\boldsymbol{x}$, Chaplygin devised the reducing multiplier method, which after rescaling time makes it possible to represent the equations of motion in Hamiltonian form. We note that Chaplygin considered the motion of a Chaplygin sleigh[11)] as an example illustrating the reducing multiplier method.

Remark. It turned out that the reducing multiplier method allows one to represent the equations of motion in Hamiltonian form after rescaling time [16] in another well-known Chaplygin problem of a dynamically asymmetric ball rolling on a horizontal plane.

Remark. As a historical remark we note that, although the Chaplygin sleigh is tradionally linked to the work of Chaplygin (1912) [36] and Carathéodory (1933) [31], it was considered somewhat earlier by A. Brill (1909) [29] as an example of the mechanism of a nonholonomic planimeter.

As an interesting fact, we note that in writing the equations of motion for specific nonholonomic problems Chaplygin used the principles of dynamics rather than various general forms of nonholonomic mechanics equations.

A distinctive feature of the work of Chaplygin is that he analyzed the equations by applying various geometric methods that existed at that time and by elucidating the geometric nature of motion (his work on the motion of a dynamically asymmetric ball is an illustrative example).

We also point out the Suslov problem (1900) [108] of the motion of a rigid body about a fixed point subject to a nonholonomic constraint, which was considered in his well-known treatise on theoretical mechanics, which was republished several times and still enjoys popularity among Russian specialists in mechanics. P. V. Voronets, a Ukrainian mathematician and scientist in the

[11)] In his reasoning, applied to the sleigh, he essentially used the quasi-coordinate introduced by him, which gave rise to a debate concerning the correctness of the method [45]

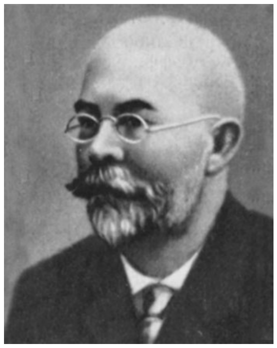

Fig. 5. Gavriil Konstantinovich Suslov (1857–1935)

Fig. 6. Pyotr Vasilyevich Voronets (1871–1923)

area of mechanics found (when he was a student of Suslov) not only a new form of nonholonomic mechanics equations, but also considered a number of examples, in particular, the problem of a body with a flat base rolling on a sphere (1911) [119] and the problem of a body of an arbitrary form rolling on the surface of a sphere.

In addition, Voronets must undoubtedly be credited with having applied the considerations of nonholonomic mechanics (including a nonholonomic basis) to Hamiltonian systems. This made it possible to considerably advance in the issues of reduction and in finding partial solutions in the n-body problem on a plane with various potentials [120]. We note that some historical issues concerning reduction, both in nonholonomic mechanics and in various dynamical systems, are addressed in the review paper [17].

Much of the research at that time was aimed at studying problems concerning the representation of general nonholonomic systems in Hamiltonian form, but this research did not lead to any significant advances. Dautheville (1909) [40] and Quanjel (1906) [95] used the Legendre transformation for a free system. As a result of this transformation, additional (nonholonomic) terms[12)] are obtained in the equations of motion after passage to the momenta.

3. THE THIRD PERIOD: 1913–1966 (PRINCIPLES OF NONHOLONOMIC GEOMETRY)

Unfortunately, in that period, when quantum mechanics and general relativity theory were created, the classical mechanics was pushed to the sidelines. The main focus was on the development of *nonholonomic geometry*. The most significant works in this direction were those of J. A. Schouten (1928) [103], G. Vránceanu (1936) [115], E. Cartan (1940) [32], and V. Vagner (1941) [109]. We note that to date the methods of nonholonomic geometry have found no application in nonholonomic mechanics [13)].

We note that the development of geometric methods has contributed to the creation of the Rashevsky – Chow theorem, which was independently formulated in the works of P. K. Rashevsky (1938) [96] and W. L. Chow (1939) [38] and was of great importance for control theory.

Apparently, among the theoretical works on nonholonomic geometry only the work of V. Vagner is of interest for the study of dynamics. His work presents a correct realization of constraints in the Suslov problem, as opposed to the erroneous realization presented by Suslov himself [107].

[12)] Regarding the work of Pöschl [94], submitted to *Comp. Rend. Acad. Sci.* by Appel, we note that it is concerned not with nonholonomic systems, but with the nonholonomic basis in Hamiltonian systems (i.e., these equations are the Poincaré equations [91] transformed to Hamiltonian form; as is well known, this transformation was performed by N. G. Chetaev). In contrast to Poincaré and Chetaev, Pöschl [94] was not well familiar with Lie's group formalism and, like Volterra [114], obtained similar equations, which in the general case are also inapplicable to nonholonomic constraints, by using, as Hamel put it, a forbidden operation of substituting constraints into the Lagrangian of the free system.

[13)] Nevertheless, there exist modern works (see, e.g., Koiller [61]) whose authors believe in the possibility of its further application.

A. D. Billimovich (1914) [7] considered nonstationary constraints. We note that, generally speaking, by increasing the dimension of the phase space one can reduce nonstationary constraints to stationary constraints, but they can be considered separately. The nonholonomic Billimovich pendulum is a valuable example, which is still encountered in the literature.

Formally the issue of the existence of an invariant measure was investigated by C. Blackall (1941) [10]. In particular, he gave the simplest model example of a nonholonomic system in which there is no smooth invariant measure.

We note that in that period the correctness of the variational principle and Hamilton's principle was discussed and the principles of Hölder, Suslov and Foss (see, e.g., the work of Capon [30])[14] were considered. However, all these attempts lead only to a formal representation of Hamiltonian analogs of the equations of motion, but such a representation has no dynamical content. In this connection, we mention the work of Eden (1951) [44], communicated by Dirac (before he created general Hamiltonian mechanics), which is erroneous, although it is fairly difficult to explicitly verify the calculations in [44].

Remark. Among formalistic works of later periods in which a fruitless generalization of the Hamiltonian methods in nonholonomic mechanics was performed are [47, 90, 110, 117]. What all these works have in common is that they contain no examples. Just as in the case with the classical problem of a coin rolling on a plane, the result of the study of a particular problem, as carried out in [110], is that the use of the Hamilton – Jacobi method allows one to find only steady motions. A similar point of view is expressed in [101].

In spite of a large number of formal works, we note that in that period various versions of the problem of the dynamics of a wheeled vehicle began to be considered. For example, B. Stückler (1952 and 1955) [105, 106] considered two models of a motor-car with a fixed rear axle. In one of these models, the front axle rotates freely about the normal to the plane, and in the other model, due to the trapezium mechanism (Jeantaud's mechanism), the wheels are always parallel to each other, but are not normal to the axis of the axle. A special feature of these works is analysis of reactions and the application of the Hamel equations.

A stability analysis of the simplest wheeled vehicle consisting of two two-wheelers (later called *roller-racer*) was performed by I. Rocard (1959) in his well-known book [97]. By the way, he found out that the rectilinear motion exhibits asymptotic instability, which is characteristic of nonholonomic systems. In 1964, O. Bottema [23] analyzed the trajectories of a vehicle with a fixed rear axle and presented particularly remarkable solutions.

Remark. Recent studies on the dynamics of wheeled vehicles were initiated mainly by robotic developments (see, e.g., [21, 27]).

Remark. We consider in more detail constraints nonlinear in velocities. The best-known example was given by Appel (1911) [3]; afterwards it was considered by Hamel (1978) [54]. However, this example, due to Appel and Hamel, is not quite convincing, since it arises as a result of a passage to the limit from linear constraints. The realization of nonlinear constraints is discussed in detail in the paper by E. Delassus (1911) [41], in which he drew a not quite convincing conclusion that nonlinear constraints can always be realized using linear constraints.

Later, N. G. Chetaev (1932) [37] posed the problem of agreement between the Gauss principle and the general formalism of nonholonomic mechanics for systems with nonlinear constraints. The point is that in this case the d'Alembert – Lagrange principle and the Gauss principle lead to different results. For this reason, Chetaev introduced the notion of possible displacement so as to simultaneously match both principles.

We note that the constraints nonlinear in velocities arise in a formal way, in postulating the Gauss principle in the Nosé – Hoover mechanics (see, e.g., [93]), which is one of the branches of molecular dynamics.

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[14] We note that attempts are still being made to generalize various facts of Hamiltonian mechanics to nonholonomic systems (Hamilton – Jacobi theory) [89]

4. THE FOURTH PERIOD: 1967–1984 (THE BOOK OF YU. I. NEIMARK AND N. A. FUFAEV)

The next stage in the development of the theories of nonholonomic systems is closely related to the book of Yu. I. Neimark and N. A. Fufaev (1967) [86], which was initiated by various applications in nonholonomic mechanics (for example, by the phenomenon of shimmy wheels in airplanes). For several years the authors of the above-mentioned book held seminars on nonholonomic systems in Nizhny Novgorod, at the well-known school of the Academician Andronov School for Nonlinear Oscillations and Bifurcation Theory. The qualitative analysis methods (which were devised there) were collected in this book, which gained popularity almost immediately (it was published in Russia and in the USA). The book [86], in addition to analysis of elementary problems, outlines new avenues of research and raises questions concerning the correctness of permutation relations, the realization of constraints (analysis of Carathéodory's error concerning the impossibility of realizing constraints using forces of viscous friction), and raises the question of specific equilibria of nonholonomic systems (of genus 2 according to Bottema).

Fig. 7. Yuri Isaakovich Neimark (1920–2011)

Fig. 8. Nikolai Alexeyevich Fufaev (1920–1996)

Perhaps the only shortcoming of the book [86] is that its authors were experts primarily in mechanics, so that from the viewpoint of mathematical formalism some of the problems discussed by the authors are related to the inaccuracy of mathematical definitions. In particular, they discuss the issue of permutation relations, which is simply due to the absence of rigorous definitions. In this sense, their criticism of the work of Volterra is incorrect; although the work of Volterra is erroneous (his equations are inapplicable to nonholonomic systems), but it is erroneous in quite other respects. The Volterra equations hold for Hamiltonian systems written in quasi-velocities, but in the presence of nonholonomic constraints.

The approach of the authors to permutation relations is set forth in detail in Section A.2 in the Appendix. This approach is actually a more modern exposition of the viewpoint of Neimark and Fufaev. On the other hand, we mention the studies [28, 79, 81], in which the representation of permutation relations is too complicated by a redundant formalism.

Remark. We note that the author of the review [74] singles out the paper by A. M. Vershik and L. D. Fadeev (1972) [111] as particularly important. In our opinion, this is not justified, since the paper [111] contains nothing new. In particular, the remark that in the case of homogeneous constraints the energy integral is preserved is well known from classical works and is due, for example, to Hertz. The paper was of importance most probably for the development of formal works rather than for the methods of nonholonomic mechanics.

Historical Comments on the Stability in Nonholonomic Systems

It is well known that the critical points of potential energy are equilibrium positions of both holonomic and nonholonomic systems. They are called equilibrium positions of genus 1. In this case, Korteweg (1899) [64] showed that the analysis of the linear stability of nonholonomic systems actually reduces to the investigation of the stability of holonomic systems obtained by linearizing the constraints. As an illustration, he considered the Kerkhoven – Wythoff problem, in which the convex side of one hemisphere lies on a plane, and the other hemisphere rests on the upper half-plane of the first hemisphere. Afterwards this example was used in the textbook by E. T. Whittaker [118].

V. V. Rumyantsev [60, 100] and his students developed Chetaev's idea about the creation of a theory of stability of nonholonomic systems for equilibrium positions of genus 1. They obtained general results concerning generalizations of the Lyapunov and Routh theorems to nonholonomic systems. However, such studies do not grasp the essential specificity of nonholonomic systems. In this connection, we mention the work of V. V. Kozlov [70], in which the main results concerning inversions of the Legendre – Dirichlet theorem are formulated and the role of the positions of equilibrium of genus 2 in nonholonomic mechanics is pointed out.

We note that the positions of equilibrium of genus 1 are, as a rule, isolated and characteristic of Hamiltonian systems. The specificity of nonholonomic systems manifests itself in positions of equilibrium of genus 2, which were first found by Bottema [22]. They are not related to the minimum of potential energies, and their origin is closely related to the nonintegrability of constraints. We note that the positions of equilibrium of genus 2 are, as a rule, asymptotically stable (or unstable).

The positions of equilibrium of genus 2 were analyzed by Kozlov [69], but without examples. Nevertheless, even nowadays the geometry of these positions of equilibrium is still poorly understood, both from the theoretical point of view and from the point of view of applications.

APPENDIX. THE D'ALEMBERT – LAGRANGE PRINCIPLE AND PERMUTATION RELATIONS

In the literature devoted to nonholonomic mechanics, considerable attention is given to various "theoretical justifications" of the equations of motion. As a rule, all arguments are based on the d'Alembert – Lagrange principle (i.e., the condition of ideal constraints). In this connection, we briefly recall how this principle arises in systems with nonholonomic constraints, namely, that it is a natural generalization (axiomatization) of the model of undeformed bodies rolling on each other without slipping.

In addition, we also discuss the so-called problem of a "correct choice" of permutation relations in nonholonomic mechanics, which seems to play a central role in some publications. It turns out that in this case the problem is not to find out what permutation relations are correct, but to correctly treat the notation appearing in them. The way one interprets the notation determines the meaning and the area of application of permutation relations themselves. We give here a modern description of the historically first interpretation, which was presented by G. Hamel (for other discussions, see [86]).

A.1. Replacement of Constraints with Reaction Forces in the Case of Rolling without Slipping

1. First of all, we recall how the influence of some geometric constraint is taken into account in the dynamics of a material point whose position is given by the radius vector $\boldsymbol{r} = (x_1, x_2, x_3)$:

$$f(\boldsymbol{r}) = 0. \tag{A.1}$$

As is well known, in this case, instead of motion with a constraint, one considers the problem of a free particle acted upon by the reaction force $\boldsymbol{N}$ orthogonal to the surface (A.1). In this case, the Newton equation is written as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{N}, \tag{A.2}$$

where m is the mass of the point and $\boldsymbol{F}$ are the given external forces.

The acceleration of the particle $\ddot{\boldsymbol{r}}$ and the value of the reaction force $|\boldsymbol{N}|$ are found from Eq. (A.2) and the second derivative of the constraint

$$\ddot{f}(\boldsymbol{r}) = \left(\ddot{\boldsymbol{r}}, \frac{\partial f}{\partial \boldsymbol{r}}\right) + \sum_{i,j} \frac{\partial f}{\partial x_i \partial x_j} \dot{x}_i \dot{x}_j = 0. \tag{A.3}$$

This approach implies that the reaction $\boldsymbol{N}$ depends only on $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and the constraint (A.1) acts as an invariant relation of the resulting equations of motion.

In the presence of external forces the choice of the direction of the constraint reaction is, generally speaking, ambiguous. For example, the author of [1] uses a definition of the reaction forces directed along a straight line connected with some projective center. The requirement of orthogonality of the reaction is a natural generalization of the situation where the particle is fastened on a nonstretchable thread (spherical pendulum) so that in this case the reaction is directed along the thread. Generally speaking, the choice of reaction forces can essentially depend on the realization of the constraints [65, 66, 71].

In the form (A.2), (A.3) it is extremely inconvenient to derive equations of motion for many-particle systems and in the presence of many constraints. In this case, to write the equations of motion, one uses, as a rule, the d'Alembert – Lagrange principle. For example, in the case of one particle, Eqs. (A.2) can be represented as

$$(m\ddot{\boldsymbol{r}} - \boldsymbol{F}, \boldsymbol{\tau}) = 0,$$

where $\boldsymbol{\tau}$ is an arbitrary vector tangent to (A.1). This implies, in particular, that *the work of the reaction forces along possible displacements is zero.*

In the general case, when the motion of N points in the presence of geometric constraints is considered, possible configurations are defined by some manifold $\mathcal{M}$ (configuration space) embedded in the space of positions of the free system $\mathbb{R}^{3N}$.

We denote the local coordinates on $\mathcal{M}$ in a standard way as $\boldsymbol{q} = (q_1, \dots, q_n)$, so that the radius vectors of points are defined by them uniquely as $\boldsymbol{r}_i(\boldsymbol{q})$, $i = 1, \dots, N$, and their velocities are given by

$$\dot{\boldsymbol{r}}_i = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_k} \dot{q}_k.$$

The kinetic energy is defined as a function on $T\mathcal{M}$ by the relation

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{\boldsymbol{r}}_i, \dot{\boldsymbol{r}}_i).$$

The generalized forces $\boldsymbol{Q} = (Q_1, \dots, Q_n)$ are found from the relation

$$\sum_{i=1}^{N} (\boldsymbol{F}_i, d\boldsymbol{r}_i) = \sum_{k=1}^{n} Q_k dq_k,$$

where $\boldsymbol{F}_i$ is the force acting on the i-the mass. Then the condition that the work of the reaction forces for possible displacements be equal to zero leads to the equations

$$\left(\frac{\partial T}{\partial \dot{q}_i}\right)^{\boldsymbol{\cdot}} - \frac{\partial T}{\partial q_i} - Q_i = 0, \quad i = 1 \dots n.$$

2. We now pass to the case where one imposes restrictions on the velocity components which cannot be reduced to a set of geometric constraints (i.e., the constraints are nonholonomic). An example on which all possible generalizations are based is the rolling of one body on another. In this case the axiom of replacement of nonholonomic constraints with reaction forces is based on the model of rolling of a rigid body on the surface, under the assumption that the contact occurs at one point (the contact area is equal to zero). At this point the body is acted upon by the reaction force, which in the general case is not orthogonal to the supporting surface.

Consider the problem of a heavy rigid body rolling on a horizontal plane without slipping. The corresponding nonintegrable (nonholonomic) constraint implies that the velocity of the point of contact of the body with the plane is zero. This condition can be written as

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0, \tag{A.4}$$

where $\boldsymbol{r}$ is the vector joining the center of mass G and the point of contact P, and $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are the velocity of the center of mass and the angular velocity of the body, respectively (see Fig. 9). In what follows, all vectors are assumed to be projected onto the axes rigidly attached to the rigid body.

Let us write the equations to change the body's linear momentum and angular momentum relative to the center of mass G (Fig. 9) in the coordinate system attached to the body as follows:

$$\begin{aligned}&\frac{d}{dt}(m\boldsymbol{v}) - m\boldsymbol{v}\times\boldsymbol{\omega} + mg\boldsymbol{\gamma} = \boldsymbol{N},\\ &\frac{d}{dt}(\mathbf{I}\boldsymbol{\omega}) - \mathbf{I}\boldsymbol{\omega}\times\boldsymbol{\omega} = \boldsymbol{r}\times\boldsymbol{N},\end{aligned} \tag{A.5}$$

where $\boldsymbol{N}$ is the reaction force at the point of contact P, m is the mass of the body, g is the free fall acceleration, $\mathbf{I}$ is the tensor of the moments of inertia of the body relative to the center of mass, and $\boldsymbol{\gamma}$ is the normal to the plane.

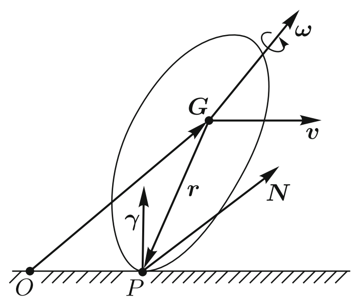

Fig. 9. A rigid body on a plane.

The problem is that when writing the equations of motion (A.5) describing the rolling motion of a rigid body on an absolutely rough surface, one simultaneously uses two independent ideas.

1. Representation of the equations of motion in quasi-velocities, which are the components of the system's velocity in the nonholonomic basis of the vector fields (the fact that the basis is nonholonomic has no direct bearing on the nonholonomicity of constraints, in particular, such a basis is widely used in Hamiltonian mechanics).

2. Replacement of constraints with reaction forces in such a way that the (local) d'Alembert–Lagrange principle is realized, i.e., the work of the reaction forces vanishes.

Consider in detail each of these ideas separately. To do this, we first derive the equations of motion of the rigid body in quasi-velocities without taking constraints into account and then, using them, we find out how the reaction forces can be obtained in the case of rolling.

Remark. We note that the axiom of replacement of constraints with reaction forces is not the only possible axiom. For example, in control theory, the exclusion of constraints gives rise to undetermined multipliers, which are not forces, but momenta (or, in this case, the reaction forces can be said to depend on accelerations).

A.2. Equations of Motion in Quasi-velocities and Permutation Relations

As is well known, for a mechanical system on which only geometric constraints are imposed, the equations of motion can be represented as an equality in which the variational (Lagrangian) derivative of the Lagrangian function is equal to zero:

$$\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{q}_i}\right)^{\cdot} - \frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial q_i} = 0, \quad i = 1,\ldots,n,$$

where $\boldsymbol{q} = (q_1\ldots q_n)$ are the generalized coordinates of the system which uniquely parameterize all possible configurations of the system (taking into account all constraints), and $\dot{\boldsymbol{q}} = (\dot{q}_1,\ldots,\dot{q}_n)$ are the corresponding generalized velocities. That is, $\boldsymbol{q}$ are the local coordinates of configuration space $\mathcal{M}$.

This form of equations implies that the trajectories of the mechanical system $\boldsymbol{q}(t)$ coincide with the extremals of the variational problem

$$\delta \int\limits_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}})dt = 0.$$

We now find out how the variational derivative of the Lagrangian function is written if we parameterize the generalized velocities of the system by quasi-velocities $\boldsymbol{\Omega} = (\Omega_1 \ldots \Omega_n)$, which are related to the initial velocities by the linear transformation:

$$\dot{q}_i = \sum_\alpha \Omega_\alpha E_{\alpha i}(\boldsymbol{q}).$$

Note that in the general case the vector fields $\boldsymbol{E}_\alpha(\boldsymbol{q}) = \big(E_{\alpha 1}(\boldsymbol{q}), \ldots, E_{\alpha n}(\boldsymbol{q})\big)$ do not commute relative to the Lie bracket:

$$[\boldsymbol{E}_\alpha, \boldsymbol{E}_\beta] = \sum_\sigma c^\sigma_{\alpha\beta}(\boldsymbol{q})\boldsymbol{E}_\sigma.$$

In this case, they are called the *nonholonomic basis* of the tangent bundle $T\mathcal{M}$.

Suppose that for some curve $\boldsymbol{q}_0(t)$, $t \in [t_1, t_2]$ we are given a (one-parameter) smooth family of variations $\boldsymbol{q}_\varepsilon(t)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with the fixed ends

$$\boldsymbol{q}_\varepsilon(t_1) = \boldsymbol{q}_0(t_1), \quad \boldsymbol{q}_\varepsilon(t_2) = \boldsymbol{q}_0(t_2)$$

for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We denote the vector field corresponding to differentiation along the parameter t as $\boldsymbol{v}$ and the vector field corresponding to differentiation along ε as $\boldsymbol{u}$ (it is also called the vector field of variation):

$$\boldsymbol{v}\Big(f\big(\boldsymbol{q}_\varepsilon(t)\big)\Big) = \frac{d}{dt}f\big(q_\varepsilon(t)\big), \quad \boldsymbol{u}\Big(f\big(\boldsymbol{q}_\varepsilon(t)\big)\Big) = \frac{d}{d\varepsilon}f\big(q_\varepsilon(t)\big).$$

By construction, these vector fields commute (due to the permutability of differentiations with respect to t and ε):

$$[\boldsymbol{u}, \boldsymbol{v}] = 0. \tag{A.6}$$

If these vector fields are written in the coordinate basis

$$\boldsymbol{v} = \sum_i v_i(\boldsymbol{q})\frac{\partial}{\partial q_i}, \quad \boldsymbol{u} = \sum_i u_i(\boldsymbol{q})\frac{\partial}{\partial q_i},$$

then from (A.6) we obtain

$$\boldsymbol{u}(v_i) - \boldsymbol{v}(u_i) = 0, \quad i = 1, \ldots, n. \tag{A.7}$$

At the same time, in standard physical and mechanical terminology the components of these fields are denoted as

$$dq_i = v_i dt, \quad \delta q_i = u_i d\varepsilon.$$

Therefore, Eqs. (A.7) can be represented as

$$\delta dq_i - d\delta q_i = 0, \quad i = 1 \ldots n. \tag{A.8}$$

If we now write the same fields in the nonholonomic basis

$$\boldsymbol{v} = \sum_\alpha \Omega_\alpha(\boldsymbol{q})\boldsymbol{E}_\alpha, \quad \boldsymbol{u} = \sum_\alpha u_\alpha(\boldsymbol{q})\boldsymbol{E}_\alpha,$$

then relation (A.6) leads to the equalities

$$u(\Omega_\alpha) - v(u_\alpha) + \sum_{\beta,\gamma} u_\beta \Omega_\gamma c^\alpha_{\beta\gamma} = 0, \quad \alpha = 1 \ldots n. \tag{A.9}$$

In mechanics, one uses the notation used by Hamel in [53] for the corresponding components of the vector fields:

$$d\theta_\alpha = \Omega_\alpha dt, \quad \delta\theta_\alpha = u_\alpha d\varepsilon.$$

In this case, we obtain the well-known permutation relations in standard form

$$d\delta\theta_\alpha - \delta d\theta_\alpha = \sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\delta\theta_\beta d\theta_\gamma. \tag{A.10}$$

It should be kept in mind that the quantities q_i in (A.8) have a physical meaning of their own (as generalized coordinates), therefore, d and δ can be regarded as operations of differentiation and variation. In Eqs. (A.10) the symbols θ_α *cannot be regarded as functions in the general sense*, only the quantities $d\theta_\alpha$ and $\delta\theta_\alpha$ have a meaning, therefore, one cannot speak to the full extent of the permutability of differentiation and variation operations. We now express the Lagrangian function in terms of the generalized coordinates $\boldsymbol{q}$ and quasi-velocities:

$$\hat{L}(\boldsymbol{q}, \boldsymbol{\Omega}) = L(\boldsymbol{q}, \dot{\boldsymbol{q}})\Big|_{\dot{\boldsymbol{q}}=\sum \Omega_\alpha E_\alpha}.$$

For the corresponding partial derivatives the following relations hold:

$$\frac{\partial \hat{L}}{\partial q_i} = \frac{\partial L}{\partial q_i} + \sum_{k,\alpha}\frac{\partial L}{\partial \dot{q}_k}\Omega_\alpha\frac{\partial E_{\alpha k}}{\partial q_i}, \quad \frac{\partial \hat{L}}{\partial \Omega_\alpha} = \sum_k \frac{\partial L}{\partial \dot{q}_k}E_{\alpha k}.$$

Using them, we find the formula for variation of the Lagrangian:

$$\delta\hat{L} = \varepsilon\left[\sum_\alpha\left(\boldsymbol{E}_\alpha(\hat{L}) + \sum_{\beta,\gamma}c^\gamma_{\beta\alpha}\Omega^\beta\frac{\partial\hat{L}}{\partial\Omega_\gamma} - \left(\frac{\partial\hat{L}}{\partial\Omega_\alpha}\right)^{\cdot}\right)u_\alpha + \left(\frac{\partial\hat{L}}{\partial\Omega_\alpha}u_\alpha\right)^{\cdot}\right],$$

$$\boldsymbol{E}_\alpha(\hat{L}) = \sum_k\frac{\partial\hat{L}}{\partial q_k}E_{\alpha k}.$$

This yields the equations of motion in nonholonomic (i.e., coordinate-free) basis:

$$\left(\frac{\partial\hat{L}}{\partial\Omega_\alpha}\right)^{\cdot} - \boldsymbol{E}_\alpha(\hat{L}) = \sum_{\beta,\gamma}c^\gamma_{\beta\alpha}\frac{\partial\hat{L}}{\partial\Omega_\gamma}. \tag{A.11}$$

Thus, it can be seen that the permutation relations (A.9) *and* (A.10) *do not have a direct bearing on nonholonomic constraints, but arise when the variational derivative of the Lagrangian is written in the coordinate-free basis of vector fields.*

For example, for the motion of a rigid body in a potential field (without constraints) we choose as generalized coordinates the Cartesian coordinates of the center of mass of the body relative to the fixed axes (x_1, x_2, x_3) and the Euler angles (θ, φ, ψ) defining the rotation of the body relative to the center of mass. As quasi-velocities in Eqs. (A.5) we have chosen the projections of the velocity of the center of mass and the angular velocity onto the axes attached to the body $\boldsymbol{v} = (v_1, v_2, v_3)$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$; they are related to the generalized velocities by

$$\dot{x}_i = \sum_j Q_{ij}v_j,$$

$$\dot{\theta} = \omega_1\cos\varphi - \omega_2\sin\varphi, \quad \dot{\psi} = \omega_1\frac{\sin\varphi}{\sin\theta} + \omega_2\frac{\cos\varphi}{\sin\theta},$$

$$\dot{\varphi} = -\omega_1\frac{\cos\theta\sin\varphi}{\sin\theta} - \omega_2\frac{\cos\theta\cos\varphi}{\sin\theta} + \omega_3,$$

$$||Q_{ij}|| = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\psi\sin\varphi & \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi & \sin\varphi\sin\theta \\ -\sin\varphi\cos\psi - \cos\theta\sin\psi\cos\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\psi\cos\varphi & \cos\varphi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}.$$

The kinetic energy is represented in quasi-velocities as follows:

$$T = \frac{1}{2}m(\boldsymbol{v}, \boldsymbol{v}) + \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}),$$

where $\mathbf{I}$ is the tensor of inertia of the body relative to the center of mass. In addition, we restrict ourselves to the case where the potential of external forces depends only on x_3, θ, φ (i.e., it possesses axial symmetry about the space-fixed axis Ox_3). In this case, using the vector

$$\boldsymbol{\gamma} = (\sin\varphi\sin\theta, \cos\varphi\sin\theta, \cos\theta),$$

we can represent the equations of motion (A.11) as

$$\left(\frac{\partial\hat{L}}{\partial\boldsymbol{v}}\right)^{\boldsymbol{\cdot}} - \frac{\partial\hat{L}}{\partial x_3}\boldsymbol{\gamma} = \frac{\partial\hat{L}}{\partial\boldsymbol{v}} \times \boldsymbol{\omega}, \quad \left(\frac{\partial\hat{L}}{\partial\boldsymbol{\omega}}\right)^{\boldsymbol{\cdot}} - \frac{\partial\hat{L}}{\partial\boldsymbol{\gamma}} \times \boldsymbol{\gamma} = \frac{\partial\hat{L}}{\partial\boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial\hat{L}}{\partial\boldsymbol{v}} \times \boldsymbol{v}, \qquad \text{(A.12)}$$

$$\hat{L} = T - U(x_3, \boldsymbol{\gamma}).$$

These equations are the same as Eqs. (A.5) for $\boldsymbol{N} = 0$ and the potential

$$U = mgx_3. \qquad \text{(A.13)}$$

A.3. The d'Alembert–Lagrange Principle for Systems with Nonholonomic Constraints

The vector constraint equation (A.4) is equavalent to three scalar equations

$$f_\mu(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{\omega}) = 0, \quad i = 1, 2, 3,$$

where f_μ are the components of the vector $\boldsymbol{f} = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r}$ and $\boldsymbol{q} = (x_1, x_2, x_3, \theta, \varphi, \psi)$ are the generalized system's coordinates in terms of which the vector $\boldsymbol{r}$ is expressed. Using the results of the previous section, we can rewrite Eqs. (A.5) in the form

$$\left(\frac{\partial\hat{L}}{\partial\boldsymbol{v}}\right)^{\boldsymbol{\cdot}} - \frac{\partial\hat{L}}{\partial\boldsymbol{v}} \times \boldsymbol{\omega} - \frac{\partial\hat{L}}{\partial x_3}\boldsymbol{\gamma} = \sum_{\mu=1}^{3} N_\mu \frac{\partial f_\mu}{\partial\boldsymbol{v}},$$

$$\left(\frac{\partial\hat{L}}{\partial\boldsymbol{\omega}}\right)^{\boldsymbol{\cdot}} - \frac{\partial\hat{L}}{\partial\boldsymbol{\omega}} \times \boldsymbol{\omega} - \frac{\partial\hat{L}}{\partial\boldsymbol{v}} \times \boldsymbol{v} + \boldsymbol{\gamma} \times \frac{\partial\hat{L}}{\partial\boldsymbol{\gamma}} = \sum_{\mu=1}^{3} N_\mu \frac{\partial f_\mu}{\partial\boldsymbol{\omega}},$$

where $\hat{L}$ is defined by (A.12) with the potential (A.13).
This allows us to draw the following conclusion:

for the model describing the rolling motion of a rigid body without slipping on an undeformable surface, a natural generalization to the case of arbitrary constraints of the form

$$f_\mu(\boldsymbol{q}, \boldsymbol{\Omega}) = 0, \quad \mu = 1, \ldots, m < n$$

is provided by general equations of motion (in the case of potential forces) in the form

$$\left(\frac{\partial\hat{L}}{\partial\Omega_\alpha}\right)^{\boldsymbol{\cdot}} - \boldsymbol{E}_\alpha(\hat{L}) = \sum_{\beta,\gamma=1}^{n} c_{\beta\alpha}^{\gamma} \frac{\partial\hat{L}}{\partial\Omega_\gamma} + \sum_{\mu=1}^{m} N_\mu \frac{\partial f_\mu}{\partial\Omega_\alpha}, \qquad \text{(A.14)}$$

$$\dot{q}_i = \sum_{\alpha=1}^{n} \Omega_\alpha E_{\alpha i}(\boldsymbol{q}).$$

We use the notation proposed in [5] for the variational (Lagrangian) derivative of the function L:

$$[\hat{L}]_\alpha = \left(\frac{\partial\hat{L}}{\partial\Omega_\alpha}\right)^{\boldsymbol{\cdot}} - \boldsymbol{E}_\alpha(\hat{L}) - \sum c_{\beta\alpha}^{\gamma}\Omega^\beta \frac{\partial\hat{L}}{\partial\Omega_\gamma}, \quad \alpha = 1 \ldots n.$$

Remark. In the coordinate basis, i.e., when $\Omega_\alpha = \dot{q}_\alpha$, we obtain accordingly

$$[L]_\alpha = \left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)^{\boldsymbol{\cdot}} - \frac{\partial L}{\partial q_\alpha}.$$

For constraints linear in the velocities

$$\sum_\alpha \hat{a}_{\mu\alpha}(\boldsymbol{q})\Omega_\alpha = 0, \quad \mu = 1\ldots m,$$

the equations of motion (A.14) can be obtained from the d'Alembert – Lagrange principle:

the variational derivative of the Lagrangian function vanishes along the vector field of variations $\boldsymbol{u} = \sum\limits_\alpha u_\alpha \boldsymbol{E}_\alpha$ *satisfying the constraint equation:*

$$\sum_\alpha [\hat{L}]_\alpha u_\alpha = 0, \quad \sum_\alpha \hat{a}_{\mu\alpha}(\boldsymbol{q})u_\alpha = 0, \quad \mu = 1\ldots m. \tag{A.15}$$

In solving the system (A.15) by the method of undetermined multipliers we obtain Eq. (A.14), where the reaction forces N_μ, $\mu = 1\ldots m$ coincide with the undetermined multipliers.

If in Eqs. (A.15) we pass to the coordinate basis and assume that $\boldsymbol{u} = \sum\limits_i (\delta q_i)\frac{\partial}{\partial q_i}$, then we obtain nonholonomic constraints and the d'Alembert – Lagrange principle in standard form

$$\sum_i a_{\mu i}(\boldsymbol{q})\dot{q}_i = 0,$$

$$\sum_i \left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)^{\boldsymbol{\cdot}} - \frac{\partial L}{\partial q_i}\right]\delta q_i = 0, \quad \sum_i a_{\mu i}(\boldsymbol{q})\delta q_i = 0,$$

where $a_{\mu i} = \sum\limits_\alpha \hat{a}_{\mu\alpha} E^{-1}_{\alpha i}$.

Remark. In mechanics the vector $\delta\boldsymbol{q} = (\delta q_1, \ldots, \delta q_n)$ is assumed to be (infinitely) small and is called the virtual displacement of the system. Then the expression

$$\delta A = \sum_i \left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)^{\boldsymbol{\cdot}} - \frac{\partial L}{\partial q_i}\right]\delta q_i$$

has the meaning of work along this displacement. Therefore, the d'Alembert – Lagrange principle is often formulated as follows:

the work of reaction forces along virtual displacements satisfying the conditions $\sum\limits_i a_{\mu i}\delta q_i = 0$ *vanishes.*

As is well known, the system of equations (A.14) or (A.15) (under certain natural conditions of nondegeneracy of the Lagrangian function and the constraints) is consistent and defines the vector field on the submanifold

$$\mathcal{M}^{n-m} = \{(\boldsymbol{q}, \boldsymbol{\Omega}) | f_\mu(\boldsymbol{q}, \boldsymbol{\Omega}) = 0, \quad \mu = 1\ldots m\} \subset T\mathcal{M}.$$

It should also be borne in mind that if no additional restrictions are imposed, then for the variations of the path $\boldsymbol{q}_0(t)$ the "deformed" path

$$\boldsymbol{q}_\varepsilon(t) = \boldsymbol{q}_0(t) + \varepsilon\boldsymbol{u}(t) + \ldots$$

does not satisfy the constraint equations. Hence, *in the general case the solutions of the system* (A.14) *or* (A.15) *are no extremals of the functional*

$$\int\limits_{t_1}^{t_2} L dt.$$

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