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An Acentric Rotation of Two Helical Vortices of the Same Circulations

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Abstract—The aim of this paper is to test the possibility of a secondary solution of the acentric rotation of helical vortex pairs with the same pitch, sign and strength. The investigation addresses the three-dimensional vortex dynamics of thin vortex filaments. As a result of the current investigation, this secondary solution with acentric vortex positions in the helical pairs is found. This fact was not discussed in previous studies, and the existence of the new equilibrium solution for the helical vortex pairs is an original result.

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Dedicated to the memory of Professors Hassan Aref and Vyacheslav Meleshko

1. INTRODUCTION

In 2007, Professors Vyacheslav V. Meleshko and Hasan Aref presented a retrospective of early vortex dynamics [1] and clarified the role of Hermann Helmholtz in the creation of this subfield of fluid mechanics. Later Aref [2] pointed out that the original paper of Helmholtz [3] may rightfully be seen as the initiation of what we today refer to as vortex dynamics. Indeed, in this fundamental article Helmholtz established three famous "laws" of vortex motion which are reproduced now in each textbook on fluid mechanics. Based on these laws, in § 5 of the same article [3], entitled "Straight parallel vortex-filaments", Helmholtz studied possible motions for a set of rectilinear vortex filaments or 2-D point vortices. He established the law of conservation of the center of vorticity of an assembly of such vortices. In particular, as mentioned in [1], Helmholtz noted the following consequence concerning the motion of the two vortices:

"If there be two rectilinear vortex-filaments of indefinitely small section in an unlimited fluid, each will cause the other to move in a direction perpendicular to the line joining them. Thus the length of this joining line will not be altered. They will thus turn about their common centre of gravity at constant distances from it. If the rotation be in the same direction for both (that is, of the same sign) their centre of gravity lies between them. If in opposite directions (that is, of different signs), the centre of gravity lies in the line joining them produced..."

A symmetric vortex pair which consists of two co-rotating vortices of the same sign and strength (circulation) can be considered as an elementary case of these two vortices. According to the consequence mentioned above, for the two rectilinear vortex filaments only a unique solution exists when both vortices rotate in the same direction around a center that is equally distant from them. Usually the centric equilibrium rotation of a symmetric pair of 2-D vortices plays the role of an initial point to investigate more complex phenomena of viscous interaction between the vortices,

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which are of both fundamental and practical significance. Firstly, both vortices rotate and grow due to diffusion without a visual disturbance of the initial stage. Then the size of their vortex cores becomes a certain fraction of the separation distance and the vortices merge rapidly into a single vortex. These interactions play a significant role in the transfer of energy and entropy and the merging of vortex pairs, and are usually referred as the simplest fundamental model to study more complex vortex unstable behavior in jet, wake, turbulence and transitional flows etc. That is why the 2-D vortex pair has been intensively investigated for the last few decades [4]. The interaction of point vortices with a rigid body was considered in [5–7], and in [7] a new conception of mass vortices was proposed.

The generation of rectilinear vortices in flows is a rare event, and helical vortices are commonly observed in engineering and environmental applications [8]. For example, such helical structures are a result of vortex breakdown in swirling flows [9]. The study of equilibrium motion, instability and merging of these basic configurations is also of interest in rotor aerodynamics [10] because tip vortices in near wakes behind propellers or turbines are generated in ring configurations or multiplets and possess perfect helical forms [11]. In this context a study of single and double rings from helical vortices was started in [12–15]. Those multiplets include the helical vortex duplet or symmetrical pair as a simple case. The authors of [16] continued these studies of the two helical vortices and addressed them to the tip vortices behind two-blade rotors. The initial symmetrical condition of the classical problem for a pair of rectilinear vortices with centric equilibrium rotation was replicated in their study to discover new dynamic features of the vortex pair exhibiting helical symmetry (Fig. 1a). The authors of [16] restrict themselves to considering the solitary case of symmetrical rotations of two co-rotating helical vortices with the same sign, strength Γ and pitch because the rectilinear prototype of the 2-D vortex pair deals with the unique solution only. However, each helical vortex is capable of rotating by itself. This additional self-induced motion is able to change the dynamic behavior of the helical vortex pair for which the secondary solution with an acentric position becomes possible (Fig. 1b). This fact was not discussed in previous studies, and the existence of equilibrium motion of acentric helical vortex pairs with equal circulation is a question that needs to be answered.

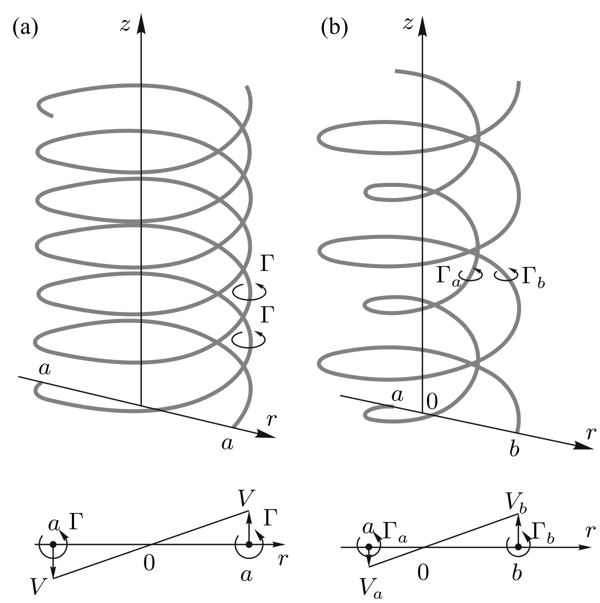

Fig. 1. Two possibilities for equilibrium of helical vortex pairs: (a) centric equilibrium configuration; (b) acentric equilibrium configuration.

The aim of this paper is to test the possibility of secondary solutions of the acentric rotation of helical vortex pairs with the same pitch, sign and strength. The investigation addresses the three-dimensional dynamics of thin vortex filaments in an ideal fluid for which in a natural coordinate system the self-induced vortex-tube motion is expressed by a bi-normal component only. This fact results from the intrinsic equations of vortex filament motion which were strictly derived in the early last century when a conception of self-induced motion was established [17]. In contrast to the different approximations of helical vortices [8, 17] the current study is based on the Kawada – Hardin solution for an infinite thin helical vortex filament [18], which is rewritten in the light of modern developments of helical vortex theory [12–15], and then is summed together with the induction velocity of the second vortex filament of the vortex pair in the same fixed coordinate system.

2. MOTION OF THE HELICAL VORTEX PAIR

In view of the above, the main goal of the current work is to find out conditions for the acentric equilibrium configuration of the helical vortex pair (Fig. 1b). In a general case we consider two slender helical vortices which are placed on two supporting cylinders with radii a and b; a common geometrical pitch h or $l = h/2\pi$, but we should introduce two dimensionless pitches: $\tau_a = l/a$ and $\tau_b = l/b$. We initially put different values of the vortex circulations: Γ_a and Γ_b; because in a limit the rectilinear vortex pair has only one solution ($\Gamma_a/\Gamma_b = a/b$) and we could not prognosticate the acentric case, when $a \neq b$, with $\Gamma_a = \Gamma_b$. Both vortices have circular cores with the same radius (referred to a) $\varepsilon \ll 1$. The vortex cores are a superposition of the helical vortex filaments which are colinear to central helical lines and their vorticity is (instantaneously) uniformly distributed across the core cross-section [12–15].

Let us determine the velocity of the helical vortex pair considered (Fig. 1b), which rotates in an equilibrium with the same angular velocity $\Omega^{(a)} = V_a/a \equiv \Omega^{(b)} = V_b/b$, where $V_x = u_\theta(x, 0)$. Note that in contrast to the rectilinear filaments (point vortices of Helmholtz), where the vortex motion is determined by the velocity induced only by the other vortex of the pair at the point of the vortex position, in the current case for each of the two helical vortices (a) and (b) the total angular velocity around their common center consists of two components:

$$\Omega^{(a)} = \Omega^{(a)}_{Ind} + \Omega^{(a)}_{Sind} \quad \text{and} \quad \Omega^{(b)} = \Omega^{(b)}_{Ind} + \Omega^{(b)}_{Sind}, \tag{2.1}$$

where $\Omega^{(a)}_{Ind}$ or $\Omega^{(b)}_{Ind}$ is the angular velocity of a fixed vortex point of "a" or "b", induced by the other vortex in the position of "b" or "a", respectively, and $\Omega^{(a)}_{Sind}$ or $\Omega^{(b)}_{Sind}$ is the velocity of its self-induced motion of the same vortex placed in "a" or "b" [19–22].

If we choose a cross-section $z = 0$ in a fixed polar coordinate system and neglect the effect of a final core size of another vortex, substituting it by a filament, in accordance with the Kawada – Hardin solution [18], the first component of both angular velocities (2.1) of the helical pair takes the form

$$\frac{4\pi a^2}{\Gamma_a}\Omega^{(a)}_{Ind} = \frac{4}{\tau_b}\sum_{m=1}^{\infty} m I_m\left(\frac{m}{\tau_a}\right)K'_m\left(\frac{m}{\tau_b}\right); \tag{2.2a}$$

$$\frac{4\pi b^2}{\Gamma_b}\Omega^{(b)}_{Ind} = 2 + \frac{4}{\tau_a}\sum_{m=1}^{\infty} m I'_m\left(\frac{m}{\tau_a}\right)K_m\left(\frac{m}{\tau_b}\right), \tag{2.2b}$$

where $I_m(x)$ and $K_m(x)$ are modified Bessel functions and K' and I' are these derivatives. The series of (2.2a) and (2.2b) were simulated in accordance with the procedure of [12]:

$$\begin{aligned}\frac{2\pi a^2}{\Gamma_a}\Omega^{(a)}_{Ind} = -2\frac{\sqrt[4]{1+\tau_b^2}}{\sqrt[4]{1+\tau_a^2}}\Re\Bigg[\frac{1}{E(b,\,a)-1}+ \\ +\frac{1}{24}\left(\frac{3\tau_a^2-2}{(1+\tau_a^2)^{3/2}}+\frac{9\tau_b^2+2}{(1+\tau_b^2)^{3/2}}\right)\ln(1-E(a,\,b))\Bigg],\end{aligned} \tag{2.3a}$$

$$\begin{aligned}\frac{2\pi b^2}{\Gamma_b}\Omega^{(b)}_{Ind} = 2-2\frac{\sqrt[4]{1+\tau_a^2}}{\sqrt[4]{1+\tau_b^2}}\Re\Bigg[\frac{-1}{E(a,\,b)-1}+ \\ +\frac{1}{24}\left(\frac{3\tau_b^2-2}{(1+\tau_b^2)^{3/2}}+\frac{9\tau_a^2+2}{(1+\tau_a^2)^{3/2}}\right)\ln(1-E(b,\,a))\Bigg],\end{aligned} \tag{2.3b}$$

where

$$E(a,\, b) = \frac{\tau_a \left(1 + \sqrt{1 + 1/\tau_b^2}\right) \exp\left(\sqrt{1 + 1/\tau_a^2}\right)}{\tau_b \left(1 + \sqrt{1 + 1/\tau_a^2}\right) \exp\left(\sqrt{1 + 1/\tau_b^2}\right)}.$$

Formulas (2.3a) and (2.3b) can be applied with good approximation for all values of τ_a and τ_b in the fixed polar coordinate system [12].

The determination of the angular velocity of self-induced motion of every single vortex from the helical vortex pair Ω_{Sind} considered here is a more complex problem, which has been examined by many famous experts in hydromechanics in natural coordinates in which any vortex motion results in the binormal direction only [12, 19–22]. Let us consider the solution of the angular velocity Ω_{Sind}, a problem based on the approach of the natural velocity field presented by the binormal velocity component $u_{b_{Sind}}$. The transformation between both coordinate systems permits us to note that

$$\frac{4\pi a^2}{\Gamma_a} \Omega_{Sind}^{(a)} = \frac{2}{1 + \tau_a^2} - \frac{\tau_a^2}{\sqrt{1 + \tau_a^2}} u_{b_{Sind}}^{(a)}, \tag{2.4a}$$

$$\frac{4\pi b^2}{\Gamma_b} \Omega_{Sind}^{(b)} = \frac{2}{1 + \tau_b^2} - \frac{\tau_b^2}{\sqrt{1 + \tau_b^2}} u_{b_{Sind}}^{(b)}, \tag{2.4b}$$

where $u_{b_{Sind}}^{(a)}$ and $u_{b_{Sind}}^{(b)}$ are the binormal velocities of the self-induced motion of the "a" or "b" helical vortices.

We first recall the main points concerning the evaluation of the self-induced motion of any slender vortices in the natural coordinate system based on [22] because sometimes there is some confusion about this problem. The unit vectors of the natural coordinate system are directed along the tangent, the principal normal and the binormal to the filament at some point O. In this system the velocity field induced by a curved vortex filament may be asymptotically presented at a small distance σ from a filament as the sum of the pole, the logarithmic singularity and the regular term [23]. The first of them describes the circulatory motion around the vortex axis and does not cause its displacement. The next two summands should describe the motion of the vortex filament in the direction of the binormal. The dimensionless form scaled to $\Gamma\kappa/4\pi$ of the binormal velocity is

$$\widehat{w}_b^{(Asympt)} = -\frac{2\cos\theta}{\kappa\sigma} + \ln\frac{1}{\kappa\sigma} + C^{(Asympt)}, \tag{2.5}$$

where σ and θ are the polar coordinates of the natural system, and κ is the vortex curvature.

It is well known that the dimensionless self-induced velocity of a helical vortex with a finite-core size of ε after integrating (2.5) is described by the formula

$$\widehat{w}_b^{(Sind)} = \ln\frac{1}{\kappa\varepsilon} + C^{(Sind)}. \tag{2.6}$$

This is an analog of (2.5) without the pole because the integral from this contribution vanishes due to symmetry. In (2.5) the quantity $C^{(Asympt)}$ depends only on the vortex filament geometry and in (2.6) the value of $C^{(Sind)}$ depends also on the same geometric parameters of the vortex as well as on the vorticity distribution inside the vortex core. There is an interrelation between both quantities under the same vortex geometry, which mainly depends on the vorticity distribution in the vortex core [20–22]. For example, for a helical vortex with uniform vorticity distribution in a small vortex core the difference is 1/4 [21], which gives

$$C^{(Sind)} = C^{(Asympt)} + \frac{1}{4}. \tag{2.7}$$

This difference may take different values depending on the form of the vortex core and the vorticity distributions, e.g., the value for a vortex ring was estimated as 3/4 in [21, 24]. Nevertheless, it cannot be equal to zero since in this case ε should also tend to zero and the logarithm term of (2.6) provides infinity of the self-induced motion. According to Ricca's notations [20], in this investigation the self-induced dimensionless binormal velocities of $u^{(a)}_{b_{Sind}}$ and $u^{(b)}_{b_{Sind}}$ are defined as follows:

$$\widehat{u}^{(a)}_{b_{Sind}} = \frac{1}{1+\tau_a^2}\left[\ln\left(\frac{1}{\varepsilon}\right) - \frac{1}{4} + C\left(\tau_a^2\right)\right], \tag{2.8a}$$

$$\text{where} \quad C(\tau_a^2) = \frac{\widehat{v}^{(a)}_{\text{int}} + \widehat{v}^{(a)}_{\text{ext}}}{2} - 2\ln\frac{1}{\varepsilon};$$

$$\widehat{u}^{(b)}_{b_{Sind}} = \frac{1}{1+\tau_b^2}\left[\ln\left(\frac{1}{\varepsilon}\right) - \frac{1}{4} + C(\tau_b^2)\right], \tag{2.8b}$$

$$\text{where} \quad C(\tau_b^2) = \frac{\widehat{v}^{(b)}_{\text{int}} + \widehat{v}^{(b)}_{\text{ext}}}{2} - 2\ln\frac{1}{\varepsilon}.$$

Thus, the solution of the problem is reduced to the determination of $C(\tau)$ using the formulas (2.2a) and (2.2b) for the induction velocity $\widehat{v}_{\text{int}}$ and $\widehat{v}_{\text{ext}}$ on the distance of $\pm\varepsilon$ from the centers of both vortices in the pair. After complex algebra this term, based on the Kawada–Hardin solution (2.2a) and (2.2b) [18] for any torsion τ, was derived in [12]:

$$C(\tau) = \ln\left(\frac{\tau}{(1+\tau^2)^{3/2}}\right) - 4\frac{(1+\tau^2)^{3/2}}{\tau^2} I_1\left(\frac{1}{\tau}\right) K_1'\left(\frac{1}{\tau}\right) - \frac{\sqrt{1+\tau^2}\left(1 - \tau\sqrt{1+\tau^2} + 3\tau^2\right)}{\tau} + \\ + \frac{\tau^2}{(1+\tau^2)^3}\left[\left(\tau^4 - 3\tau^2 + \frac{3}{8}\right)\varsigma(3) - 3\frac{3}{8} - 2\tau^4 - \frac{1}{\tau^2}\right], \tag{2.9}$$

and using (2.4) and (2.8) for both self-induced velocities, we obtain

$$\begin{aligned}\frac{4\pi a^2}{\Gamma_a}\Omega^{(a)}_{Sind} &= 3 - \frac{\tau_a}{\sqrt{1+\tau_a^2}} - \\ &- \frac{\tau_a}{(1+\tau_a^2)^{3/2}}\ln\left(\frac{\tau_a}{(1+\tau_a^2)^{3/2}}\right) + \frac{4}{\tau_a} I_1\left(\frac{1}{\tau_a}\right) K_1'\left(\frac{1}{\tau_a}\right) + \\ &+ \frac{\tau_a^3}{(1+\tau_a^2)^{9/2}}\left[3\frac{3}{8} + 2\tau_a^4 + \frac{1}{\tau_a^2} - \left(\tau_a^4 - 3\tau_a^2 + \frac{3}{8}\right)\varsigma(3)\right] - \\ &- \frac{\tau_a}{(1+\tau_a^2)^{3/2}}\left(\ln\left(\frac{1}{\varepsilon}\right) - \frac{1}{4}\right);\end{aligned} \tag{2.10a}$$

$$\begin{aligned}\frac{4\pi b^2}{\Gamma_b}\Omega^{(b)}_{Sind} &= 3 - \frac{\tau_b}{\sqrt{1+\tau_b^2}} - \\ &- \frac{\tau_b}{(1+\tau_b^2)^{3/2}}\ln\left(\frac{\tau_b}{(1+\tau_b^2)^{3/2}}\right) + \frac{4}{\tau_b} I_1\left(\frac{1}{\tau_b}\right) K_1'\left(\frac{1}{\tau_b}\right) + \\ &+ \frac{\tau_b^3}{(1+\tau_b^2)^{9/2}}\left[3\frac{3}{8} + 2\tau_b^4 + \frac{1}{\tau_b^2} - \left(\tau_b^4 - 3\tau_b^2 + \frac{3}{8}\right)\varsigma(3)\right] - \\ &- \frac{\tau_b}{(1+\tau_b^2)^{3/2}}\left(\ln\left(\frac{1}{\varepsilon}\right) - \frac{1}{4}\right).\end{aligned} \tag{2.10b}$$

The total angular velocity of both helical vortices in the pair can be calculated by summing (2.3a) and (2.10a) or (2.3b) and (2.10b) in accordance with (2.1).

3. EQUILIBRIUM MOTIONS OF THE HELICAL VORTEX PAIR:

RESULTS AND CONCLUSIONS

Clearly, there is an equilibrium in which the pair of helical vortices uniformly rotates with the same angular velocities:

$$\Omega^{(a)} = \Omega^{(b)}. \tag{3.1}$$

Indeed, if we introduce a circulation ratio $\beta = \Gamma_b/\Gamma_a$, then for the equilibrium of the pairs of point vortices a unique solution exists when $\Gamma_b/\Gamma_a = a/b$, and the unique symmetrical condition $\Gamma_a = \Gamma_b$ for the classical problem exists when $a/b = 1$. However, for a helical vortex pair condition (3.1) looks quite different from the point case by the induction part (2.3) and the existence of the additional self-induced contributions (2.10). In Fig. 2 we have studied the influence of a decrease in the helical pitch of the vortex pair on the equilibrium conditions (3.1). Analyzing (3.1) at various values of pitch τ, we determined the circulation ratio $\beta = \Gamma_b/\Gamma_a$ for the stable configurations of the pair as a function of the ratio vortex positions — a/b. Figure 2 shows a rising inclination from the linear behavior of the point solution (solid line). The circulation ratio grows when the helical pitch decreases significantly. Nevertheless, the symmetrical solution with the equal circulation $\Gamma_a = \Gamma_b$ exists for the same vortex position at $a/b = 1$, and it identifies the symmetrical case for the helical vortex pair as it was early found for the point vortex pair by Helmholtz. Moreover, for the small pitch a nonsymmetrical position ($a < b < \infty$) of the helical vortices appears with equal circulations, when $\Gamma_a = \Gamma_b$. The situation of the second solution appears for the small value of the helical pitch and it is impossible for the co-rotating vortex pair of point vortices. Examples of flow streamlines for the new case of the acentric helical vortex pairs are shown in Fig. 3.

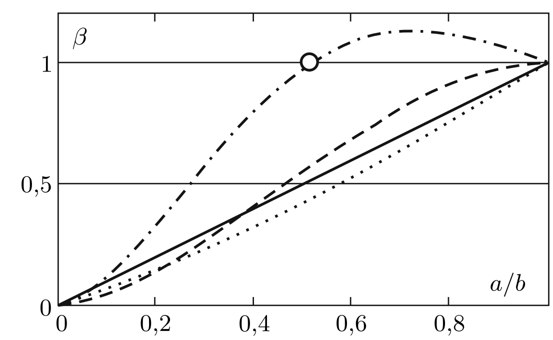

Fig. 2. The circulation ratio β as a function of the helical vortex positions (Fig. 1b) in the equilibrium pair rotation for different values of the helical pitch $h = \infty$(——); 5 (······); 2 (– – –); and 1.5(– · – ·). The core radius $\varepsilon = 0.05b$ was fixed to be the same for both helixes in the pair.

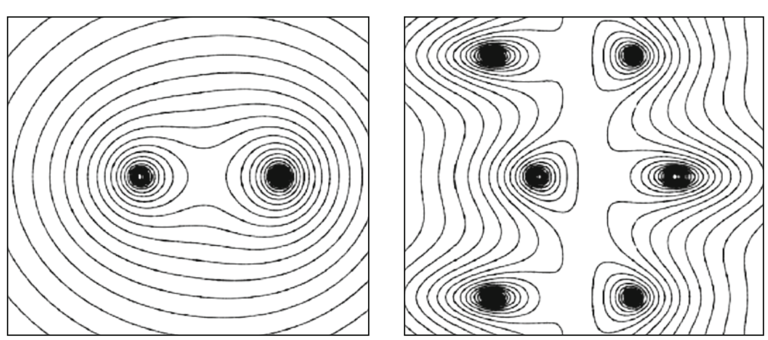

Fig. 3. Cross- and profile-sections of the streamlines for the second solution with the acentric equilibrium of the helical vortex pair.

As a result of this investigation, a secondary solution with an acentric vortex position of the helical pair (Fig. 1b) has been found. This fact was not discussed in previous studies, and the existence of a new equilibrium motion of the acentric helical vortex pairs with equal circulation is an original result.

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