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Local Normal Forms of Smooth Weakly Hyperbolic Integrable Systems

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Abstract—In the smooth (C^∞) category, a completely integrable system near a nondegenerate singularity is geometrically linearizable if the action generated by the vector fields is weakly hyperbolic. This proves partially a conjecture of Nguyen Tien Zung [11]. The main tool used in the proof is a theorem of Marc Chaperon [3] and the slight hypothesis of weak hyperbolicity is generic when all the eigenvalues of the differentials of the vector fields at the non-degenerate singularity are real.

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1. INTRODUCTION

The notion of integrability in the not necessarily Hamiltonian sense has been propounded explicitly during the last fifteen years, see [1], although its history can be traced back to a century ago. In order to study these more general systems, a method is to find their normal forms, as in the case of integrable Hamiltonian systems [5, 7, 8]. For more background on those general integrable systems, one can refer to the excellent survey [10].

In his recent paper [11], Nguyen Tien Zung introduced two important notions for non-Hamiltonian integrable systems: that of a *nondegenerate equilibrium point* and the notion of *geometric linearization* of the system around an equilibrium point. He proved that an integrable non-Hamiltonian system is geometrically linearizable near a nondegenerate singularity in the (real or complex) analytic category. He also conjectured that this remains true in the smooth category and treated some special cases (type $(m, 0)$ in [11] and type $(1, q)$ in [9]).

In the present paper, we prove the conjecture when all the eigenvalues of the differentials at the fixed point of the vector fields are real, under a slightly more restrictive hypothesis (weak hyperbolicity is assumed in addition to nondegeneracy). We provide normal forms as well.

Remark 1. In the hyperbolic Hamiltonian case, integrability does not imply conjugacy to the Birkhoff normal form. This is because the level set of the first integrals may be nonconnected and one can construct a Hamiltonian function whose values are different in different connected components (see [4, 6]).

Remark 2. In [10], Zung gave some ideas for a proof of his conjecture. He claimed that the problem can be reduced to the pure hyperbolic case. The result in this paper could be helpful.

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2. HYPOTHESES AND STATEMENT OF THE THEOREM

2.1. Integrability, Nondegeneracy, Geometric Linearization (*Nguyen Tien Zung*)

Definition 1. A dynamical system given by a vector field X on an m-dimensional manifold M is called **integrable** in the non-Hamiltonian sense if there exist p vector fields $X_1 = X$, $X_2, \ldots, X_p$ and q functions $F_1, \ldots, F_q$ on M, such that the vector fields commute pairwise and $X_1 \wedge \cdots \wedge X_p \neq 0$ almost everywhere, the functions are common first integrals for these vector fields and $\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_q \neq 0$ almost everywhere. The integers p, q satisfy $p \geqslant 1, q \geqslant 0, p + q = m$. We call $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ an integrable system of type (p, q), and the first integrals F of the system, defined by $\mathrm{d}F \wedge \mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_q = 0$, form an algebra $\mathcal{F}_{X_1,\ldots,X_p}$.

Definition 2. Let $(X_1, \ldots, X_p, \mathcal{F})$ be an integrable system of type (p, q) on M with a coordinate system around a common zero O of the vector fields. Denote by $Y_1, \ldots, Y_p$ the linear parts of $X_1, \ldots, X_p$, respectively. If there exist q homogeneous functions $G_1, \ldots, G_q$ which are the homogeneous parts of lowest degree of certain first integrals $F_1, \ldots, F_q \in \mathcal{F}$ such that $(Y_1, \ldots, Y_p, G_1, \ldots, G_q)$ is an integrable system of type (p, q) and the linear vector fields are semisimple, namely, $\mathrm{d}Y_1, \ldots, \mathrm{d}Y_p$ are all diagonalizable over $\mathbb{C}$, then we call O a *nondegenerate* equilibrium point and $(X_1, \ldots, X_p, \mathcal{F})$ is *nondegenerate* around O.

Definition 3. An integrable dynamical system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ of type (p, q) around a nondegenerate equilibrium point in $\mathbb{R}^m$ is geometrically linearizable if there exists a coordinate system in which

$$X_i = \sum_j a_{ij} X_j^{(1)} \ \forall\, i = 1, \ldots, p,$$

where $X_i^{(1)}$ is the linear part of X_i and the a_{ij}'s are common first integrals of $X_1^{(1)}, \ldots, X_p^{(1)}$.

2.2. Statement of the Theorem

Hypothesis 1. *Let $X_1, X_2, \ldots, X_p$ be p vector fields defining an integrable system of type (p, q) in a neighborhood of 0 in $\mathbb{R}^m$, for which 0 is a* nondegenerate *equilibrium point.*

We assume that all the eigenvalues of $\mathrm{d}X_i(0)$ are real for $i = 1, \ldots, p$. *Hence [2, 10], modulo a linear change of coordinates in $\mathbb{R}^m$, the linear parts of the vector fields are diagonal:*

$$X_i^{(1)} = \sum_{k=1}^{m} c_{ik} x_k \frac{\partial}{\partial x_k}, \quad i = 1, 2, \ldots, p,$$

where $c_{ik} \in \mathbb{R}$. These linear vector fields define an action of $\mathbb{R}^p$ on $\mathbb{R}^m$.

We assume this linear action weakly hyperbolic, *which means the following: if c_k denotes the linear form $(t_1, \ldots, t_p) \mapsto c_{1k}t_1 + \cdots + c_{pk}t_p$ on $\mathbb{R}^p$ for $k = 1, 2, \ldots, m$, the convex hull in $(\mathbb{R}^p)^*$ of any p elements of $\{c_1, \ldots, c_m\}$ does not contain the origin* (*the action is called* strongly hyperbolic *if any p elements of $\{c_1, \ldots, c_m\}$ are linearly independent. Obviously,strong hyperbolicity implies both weak hyperbolicity and nondegeneracy in the sense of [10], as the latter means that $c_1, \ldots, c_m$ generate* $(\mathbb{R}^p)^*$).

We can now state the main theorem of this article:

Theorem 1. *Under those hypotheses, the system is geometrically linearizable around* 0.

If $p = m$, i.e., $q = 0$, this means that the vector fields X_i are simultaneously linearizable, a straightforward consequence [2] of Sternberg's linearization theorem for contractions (in that case, nondegeneracy coincides with strong hyperbolicity). *In the sequel, we assume $q > 0$.*

3. PRELIMINARIES: FORMAL FIRST INTEGRALS OF THE LINEAR PART

Note that a nonconstant monomial $x^\gamma := x_1^{\gamma_1}\cdots x_m^{\gamma_m}$, $\gamma=(\gamma_1,\ldots,\gamma_m)\in\mathbb{N}^m$, is a first integral of the linear part of the system if and only if γ lies in

$$P := \{\gamma\in\mathbb{N}^m\setminus\{0\} : \sum_{k=1}^m \gamma_k c_k = 0\}. \tag{3.1}$$

Of course, up to the addition of a constant, a formal first integral of the linear part is an infinite linear combination of such monomials.

Proposition 1. *The subset P of $\mathbb{N}^m$ defined by* (3.1) *is finitely generated in the following sense: there exists a* finite *set $P_0\subset P$ such that every element of P can be expressed* (*in a nonunique way in general*) *as the sum of finitely many elements of P_0.*

The proof will use the following consequence of the fact that $\mathbb{N}$ is well-ordered:

Lemma 1 ([2], Lemma 4, p. 135). *For each integer $m>0$, one defines an ordering of $\mathbb{N}^m$ by "$\gamma := (\gamma_1,\ldots,\gamma_m)\leqslant(\gamma'_1,\ldots,\gamma'_m) =: \gamma'$ if and only if $\gamma_k\leqslant\gamma'_k$ for all k". For every nonempty subset P of $\mathbb{N}^m$, the set $\min P := \{\gamma\in P : \text{if } \gamma'\leqslant\gamma, \text{ then } \gamma=\gamma'\}$ of minimal elements of P for this ordering is finite and every $\gamma\in P$ satisfies $\gamma'\leqslant\gamma$ for some $\gamma'\in\min P$.*

Proof. The proof in [2] being somewhat cryptic, we provide one. If $m=1$, this is the well-orderedness of $\mathbb{N}$. Otherwise, assume the lemma proved for $m-1$, and let $\pi\gamma := (\gamma_1,\ldots,\gamma_{m-1})$ for all $\gamma\in\mathbb{N}^m$. By our induction hypothesis, the set $\pi P\subset\mathbb{N}^{m-1}$ has a minimal set Q_0 and the projection $\pi\gamma$ of every $\gamma\in P$ is comparable to at least one element of Q_0. Clearly, the set $\widetilde{Q}_0$ of all $(\bar\gamma,\gamma'_m)$ with $\bar\gamma\in Q_0$ and $\gamma'_m=\min\{\gamma_m : \gamma\in P\cap\pi^{-1}\bar\gamma\}$ is finite (it has as many elements as Q_0) and included in $\min P$. Moreover, every $\gamma\in P$ with $\gamma_m\geqslant\max\{\gamma'_m : \gamma'\in\widetilde{Q}_0\} =: \mu$ is comparable to some element of $\widetilde{Q}_0$; in other words, every $\gamma\in P$ that is not comparable to any $\gamma'\in\widetilde{Q}_0$ lies in some $P_n := \{\gamma\in P : \gamma_m = n\}$ with $n<\mu$. Now, by our induction hypothesis, each P_n has a finite minimal set and every element of P_n is comparable to some element of $\min P_n$. It follows that $\min P$ is included in the finite set $\widetilde{Q}_0\cup\bigcup_{0\leqslant n<\mu}\min P_n$, of which it is the minimal set, and every element of P is comparable to an element of the finite subset $\min P$. □

Proof of Proposition 1. To prove that $P_0=\min P$ satisfies our requirements, we should show that every $\gamma\in P$ is the sum of finitely many elements of P_0, which we can do using induction on $|\gamma|_1 := \gamma_1+\cdots+\gamma_m$. If $|\gamma|_1=\min\{|\gamma'|_1 : \gamma'\in P\}$, one must have $\gamma\in P_0$, which proves our contention in that case. Otherwise, by Lemma 1, there exists $\bar\gamma^1\in P_0$ with $\bar\gamma^1\leqslant\gamma$, hence either $\bar\gamma^1=\gamma$ or $\gamma^2 := \gamma-\bar\gamma^1\in P$; in the first case, our claim is proved; otherwise, as $|\gamma^2|_1$ is less than $|\gamma|_1$, we can assume inductively that $\gamma^2=\bar\gamma^2+\cdots+\bar\gamma^n$ with $\bar\gamma^2,\ldots,\bar\gamma^n\in P_0$, hence $\gamma=\bar\gamma^1+\cdots+\bar\gamma^n$ with $\bar\gamma^1,\ldots,\bar\gamma^n\in P_0$. □

Corollary 1. *Every monomial first integral x^γ of the linear part of the system is the product of finitely many monomial first integrals x^γ with $\gamma\in P_0$. Thus, every formal first integral of the linear part can be written as a formal power series in the variables x^γ with γ in the finite set P_0.*

4. PROOF OF THE MAIN THEOREM

Lemma 2. *We can put the smooth vector fields $X_1,\ldots,X_p$ of the nondegenerate integrable system $(X_1,\ldots,X_p,F_1,\ldots,F_q)$ around the origin in $\mathbb{R}^m$ into the following normal forms up to some flat terms: more precisely, in some smooth coordinate system, we have*

$$X_i = \sum_{j=1}^p a_{ij}X_j^{(1)} + \mathrm{Flat}_i,\ i=1,\ldots,p, \tag{4.1}$$

where $X_1^{(1)},\ldots,X_p^{(1)}$ are the linear parts of $X_1,\ldots,X_p$, respectively, under the coordinates, the a_{ij}'s are smooth *first integrals of the linear vector fields and the vector fields* $\mathrm{Flat}_1,\ldots,\mathrm{Flat}_p$ *are flat at the origin.*

Proof. Denote by $\hat{X}_i$ the ∞-jet of X_i and by $\hat{F}_j$ the ∞-jet of F_j at the origin. Then we get a formal integrable system $(\hat{X}_1, \ldots, \hat{X}_p, \hat{F}_1, \ldots, \hat{F}_q)$. The procedure in [11] also works in the formal category [10], so that we have a formal coordinate system in which

$$\hat{X}_i = \sum_{j=1}^{p} \hat{a}_{ij} X_j^{(1)},\ i = 1, \ldots, p,$$

where each $X_j^{(1)}$ is identified to its ∞-jet at 0 and the $\hat{a}_{ij}$'s are formal first integrals of $X_1^{(1)}, \ldots, X_p^{(1)}$.

By Corollary 1, there exist monomial first integrals $I_1, \ldots, I_r$ of the linear part and formal power series $\hat{g}_{ij}$ such that $\hat{a}_{ij} = \hat{g}_{ij}(I_1, \ldots, I_r)$. The Borel extension theorem implies that there exist smooth coordinates whose ∞-jets are the formal ones and smooth functions g_{ij}'s whose ∞-jets are $\hat{g}_{ij}$'s, respectively, hence the lemma with $a_{ij} = g_{ij}(I_1, \ldots, I_r)$. □

We now use the following generalization of the Sternberg–Chen theorem:

Theorem 2 (Chaperon [3]). *Two weakly hyperbolic smooth $\mathbb{R}^m$-action germs are smoothly conjugate if and only if they are formally conjugate.*

This implies the existence of a smooth coordinate system in which the flat terms Flat_i's in (4.1) are all zero, i.e.,

$$X_i = \sum_{j=1}^{p} a_{ij} X_j^{(1)},\ i = 1, \ldots, p, \tag{4.2}$$

hence Theorem 1. □

Expanding the linear vector fields $X_j^{(1)}$ in (4.2), we get

Corollary 2. *Let $X_1, \ldots, X_p$ be the p vector fields of an integrable system of type (p, q) near a nondegenerate equilibrium point. If they generate a weakly hyperbolic action germ, then they can be put into a special type of the Poincaré–Dulac normal forms simultaneously by a smooth conjugacy, i.e., in some smooth coordinate system $(x_1, \ldots, x_m)$, we have*

$$X_i = \sum_{k=1}^{m} f_{ik} x_k \frac{\partial}{\partial x_k}, \quad i = 1, \ldots, p,$$

where the f_{ik}'s are smooth first integrals of the vector fields $X_1, \ldots, X_p$. □

Remark 3. These results are not in contradiction with the counterexample we gave in an appendix to [4], as no completely integrable Hamiltonian system in dimension $\geqslant 4$ can have a weakly hyperbolic germ at an equilibrium point.

APPENDIX. A REMARK

Under Hypotheses 1 (weak hyperbolicity is not needed), we start from the linear vector fields

$$X_i^{(1)} = \sum_{k=1}^{m} c_{ik} x_k \frac{\partial}{\partial x_k}, \quad i = 1, \ldots, p.$$

We will show that the c_{ik}'s can be assumed to lie in $\mathbb{Z}$.

These linear vector fields have $q = m - p$ common first integrals which are independent almost everywhere. Moreover, we have q monomials $G_1, \ldots, G_q$ as the common first integrals of the linear vector fields and $\mathrm{d}G_1 \wedge \cdots \wedge \mathrm{d}G_q \neq 0$ almost everywhere. We can assume the monomials to have coefficient 1, i.e.,

$$G_j = x^{\gamma_j}, \quad \gamma_j = (\gamma_{j1}, \ldots, \gamma_{jm}) \in \mathbb{N}^m, \quad j = 1, \ldots, q. \tag{A.1}$$

Then, off the coordinate hyperplanes, we get

$$\begin{pmatrix} \mathrm{d}G_1 \\ \vdots \\ \mathrm{d}G_q \end{pmatrix} = \begin{pmatrix} \gamma_{11}\frac{G_1}{x_1} & \cdots & \gamma_{1m}\frac{G_1}{x_m} \\ \vdots & \vdots & \vdots \\ \gamma_{q1}\frac{G_q}{x_1} & \cdots & \gamma_{qm}\frac{G_q}{x_m} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_m \end{pmatrix};$$

hence (still off the coordinate hyperplanes), we have $\mathrm{d}G_1 \wedge \cdots \wedge \mathrm{d}G_q \neq 0$ if and only if the matrix $\begin{pmatrix} \gamma_{11}\frac{G_1}{x_1} & \cdots & \gamma_{1m}\frac{G_1}{x_m} \\ \vdots & \vdots & \vdots \\ \gamma_{q1}\frac{G_q}{x_1} & \cdots & \gamma_{qm}\frac{G_q}{x_m} \end{pmatrix}$ has rank q, i.e.,

$$\Gamma := \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \vdots & \vdots \\ \gamma_{q1} & \cdots & \gamma_{qm} \end{pmatrix}$$

has rank q. Thus, our hypotheses imply $\operatorname{rank}\Gamma = q$.

Now, as $0 = \mathcal{L}_{X_i^{(1)}} G_j = \left(\sum_{k=1}^m \gamma_{jk} c_{ik}\right) G_j$ by (A.1), we have $\sum_{k=1}^m \gamma_{jk} c_{ik} = 0$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$, i.e., $\sum_{k=1}^m \gamma_{jk} c_k = 0$ for $1 \leqslant j \leqslant q$; this writes

$$\Gamma C = 0, \quad \text{where} \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}, \tag{A.2}$$

and our nondegeneracy hypothesis that $X_1^{(1)} \wedge \cdots \wedge X_p^{(1)} \neq 0$ almost everywhere means that

$$\operatorname{rank} C = p. \tag{A.3}$$

As Γ is a matrix with rational coefficients and has rank q, its kernel as a linear map $\mathbb{R}^m \to \mathbb{R}^q$ has dimension $m - q = p$ and admits a basis $(n_1, \ldots, n_p)$ with $n_j = (n_{j1}, \ldots, n_{jm}) \in \mathbb{Z}^m$ for $1 \leqslant j \leqslant p$. Now, by (A.2)–(A.3), the columns $c_{,1}, \ldots, c_{,p}$ of C form another basis of $\ker\Gamma$, hence there exists an invertible $p \times p$ real matrix $A = (a_{ij})$ such that $\sum_{i=1}^p a_{ij} c_{,i} = n_j$ for $1 \leqslant j \leqslant p$, which proves

Proposition A.1. *There exist an invertible $p \times p$ real matrix (a_{ij}) and integers n_{jk} such that $\sum_{i=1}^p a_{ij} X_i = \sum_{k=0}^m n_{jk} x_k \frac{\partial}{\partial x_k}$ for $1 \leqslant j \leqslant p$. In other words, up to a linear change of coordinates in the group $\mathbb{R}^p$ acting on $\mathbb{R}^m$, the matrix C has integer coefficients, i.e., the eigenvalues of every $X_i^{(1)}$ are integers.* □

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