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Spherical Robot of Combined Type: Dynamics and Control

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Abstract—This paper is concerned with free and controlled motions of a spherical robot of combined type moving by displacing the center of mass and by changing the internal gyrostatic momentum. Equations of motion for the nonholonomic model are obtained and their first integrals are found. Fixed points of the reduced system are found in the absence of control actions. It is shown that they correspond to the motion of the spherical robot in a straight line and in a circle. A control algorithm for the motion of the spherical robot along an arbitrary trajectory is presented. A set of elementary maneuvers (gaits) is obtained which allow one to transfer the spherical robot from any initial point to any end point.

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INTRODUCTION

Over the last twenty years, starting with the work of Halme [1], the development and investigation of robots in the form of a sphere has attracted the attention of both many researchers in the area of mechanics and mechanical engineers. The advantage of such devices is that the spherical robots are more mobile and maneuverable than wheeled vehicles, and the spherical form protects well the fragile and moving parts of the robots against external damage. Also, the behavior of such robots is described by mathematical models providing a detailed study of their dynamics and allowing control strategies for them to be developed.

Indeed, the investigation of the dynamics and controllability of spherical robots is a completely new and popular model problem relating to a rapidly developing field, called the dynamics and control of nonholonomic systems. This problem is a tool for testing the methods of controlling dynamical systems in the presence of nonholonomic constraints. These nonholonomic constraints arise since slipping at the point of contact is neglected (model of an absolutely rough surface).

Elementary control problems concerning the stabilization of motion were discussed as early as 1972 in the classical monography [2]. But with the advent of various mathematical software packages allowing computer simulations to be carried out and with the development of new approaches in dynamical systems theory, new interesting results were obtained in the dynamics and control problem. In particular, we point out that the behavior of nonholonomic systems essentially differs from standard Hamiltonian mechanics. Nonholonomic systems can display properties that are typical of both Hamiltonian systems and dissipative and reversible systems and can exhibit both interesting regular [7–9] and complex chaotic behavior [11, 12]. The free dynamics of the nonholonomic system we consider here are apparently nonintegrable and require a separate study.

To date, a huge number of prototypes of spherical robots have been developed for a variety of applications ranging from different toys to devices for the exploration of other planets. The main difference between the existing designs of spherical robots lies in the internal drive mechanism.

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There have been many literature reviews concerning the description of the existing models and their technical realization (see, e.g., [13–16] and references therein), so we present here only some of the most recent results in this area (as compared to the review [17]). A detailed review of the principles of motion of spherical robots is made in [13]. Here we mention two main types of spherical robots: those moving by displacement of the center of mass [15, 18–22] and those set in motion by producing a variable gyrostatic momentum [17, 23].

We consider in more detail one of the most popular (most studied) forms of locomotion, namely, displacement of the common center of mass of the whole robot, which causes it to move. This mechanism is implemented most commonly using a pendulum [18, 19] or an internal moving platform [15]. In [18], the dynamics of a spherical robot with a spherical pendulum is studied. The authors propose a path planning algorithm for a limited contact area of the spherical shell with the plane. They also consider the controlled motion "acceleration – deceleration" in a straight line. We note that the forcing actions described by the authors for such a maneuver are analogous to the controls obtained in [19] for a ball with the Lagrange gyroscope. The paper [20] is devoted to the path planning for a spherical robot with a pendulum mechanism within the framework of the kinematic and dynamic models. We also mention the recent paper [21], where the local controllability of a mobile spherical robot with a two-dimensional (spherical) pendulum is investigated. In [15], the dynamics and control of the motion of a spherical robot with an internal omniwheel platform is discussed. In the same paper, particular solutions of the system are found, their stability is analyzed and control algorithms are presented for the motion along an arbitrary trajectory and using elementary maneuvers (gaits). Also in [15], the theoretical results obtained are compared with the experimental results.

The dynamics of spherical robots controlled by changing the internal gyrostatic momentum is considered, for example, in [17, 23]. In [17], the controlled motion of a dynamically asymmetric balanced ball by means of three noncoplanar rotors is investigated and the construction of an explicit algorithm for controlling the ball along a given trajectory is presented. Related problems for the case of shortage of control actions (planning of paths realized by means of two rotors placed on orthogonal axes) are discussed in [23]. Unfortunately, in practice this model exhibits unsatisfactory dynamical behavior due to the influence of spinning friction and rolling friction.

Despite a large number of models of spherical robots and their technical realizations, the question remains open as to what type of propulsion device is best-suited to ensure a simple control and efficiency of maneuvers. Experimental investigations of the dynamics of spherical robots with different internal propulsion devices (pendulum, rotors, omniwheel platform) have shown that a mechanism combining the above-mentioned effects may become the most promising for controlled motion. Motivated by this, we study the dynamics and controllability of a spherical robot of combined type that uses for its motion both the displacement of the center of mass and the change of gyrostatic momentum.

1. EQUATIONS OF MOTION AND FIRST INTEGRALS

Consider the motion of a spherical robot of combined type rolling without slipping on a horizontal, absolutely rough plane (Fig. 1). The spherical robot is a spherical shell of radius R_s to the center of which an axisymmetric pendulum (gyroscopic pendulum) is attached. We shall model the gyroscopic pendulum by a weightless rod at the end of which a massive rotor is installed. The rotor is an axisymmetric body (disk) rotating about the symmetry axis coinciding with the rod (see Fig. 1). The technical design of the spherical robot has been realized so that the pendulum is capable of executing oscillations only in a given plane related to the shell, which we shall call the *plane of rotation of the pendulum* in what follows. The spherical robot is set in motion by forced oscillations of the pendulum and by the rotation of the rotor by means of two motors.

To describe the dynamics of the spherical robot, we define two coordinate systems. The first one, $O\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}$, is a fixed (inertial) coordinate system with the basis vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$. The second one, $C\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$, is a moving coordinate system with the basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, the axes of which are attached to the pendulum so that the basis vector $\boldsymbol{e}_1$ is perpendicular to the plane of rotation of the pendulum and the basis vector $\boldsymbol{e}_3$ is directed along its symmetry axis. The origin of the moving coordinate system coincides with the geometric center of the shell, C (see Fig. 1). In what follows, unless otherwise stated, we shall refer all vectors to the moving coordinate system $C\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$.

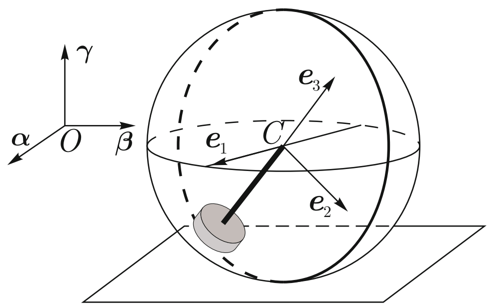


Fig. 1. Schematic model of a spherical robot of combined type.

We shall specify the position of the system by the coordinates of the center of the sphere $\boldsymbol{r} = (x, y, 0)$, by the angles of rotation, θ and φ, of the pendulum relative to the axes $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$, respectively, and by the matrix $\mathbf{Q}$ of transition from the fixed coordinate system to the moving coordinate system, the columns of which contain the coordinates of the fixed vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}, \boldsymbol{\gamma}$ referred to the moving coordinate system $C\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}.$$

Thus, the configuration space of the system considered is the product $\mathcal{N} = \{(\boldsymbol{r}, \theta, \varphi, \mathbf{Q})\} = \mathbb{R}^2 \times \mathbb{T}^2 \times SO(3)$.

The absence of slipping at the point of contact of the shell with the plane is described by the nonholonomic constraint

$$\boldsymbol{F} = \boldsymbol{v} - R_s \boldsymbol{\Omega} \times \boldsymbol{\gamma} = 0, \tag{1.1}$$

where $\boldsymbol{v}$ and $\boldsymbol{\Omega}$ are, respectively, the velocity of the center and the angular velocity of rotation of the shell.

The kinetic energy of the sphere+pendulum system can be represented as

$$T = \frac{1}{2} m_s \boldsymbol{v}^2 + \frac{1}{2} I_s \boldsymbol{\Omega}^2 + \frac{1}{2} m_b \boldsymbol{v}_b^2 + \frac{1}{2} (\boldsymbol{\omega}, \mathbf{I}_{bc} \boldsymbol{\omega}),$$

where m_s and I_s are the mass and the moment of inertia of the spherical shell, m_b and $\mathbf{I}_{bc} = \operatorname{diag}(I_{bc1}, I_{bc1}, I_{bc3})$ are the mass and the central tensor of inertia of the pendulum, and the velocity of the center of mass of the pendulum $\boldsymbol{v}_b$ and its angular velocity $\boldsymbol{\omega}$ are given by

$$\boldsymbol{v}_b = \boldsymbol{v} - R_b \boldsymbol{\omega} \times \boldsymbol{e}_3, \qquad \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\theta} \boldsymbol{e}_1 + \dot{\varphi} \boldsymbol{e}_3, \tag{1.2}$$

where R_b is the distance from the center of the sphere to the center of mass of the pendulum.

Let us write the dynamical equations of the system in the form of the D'Alembert – Lagrange equations of genus 2 in quasi-velocities with undetermined multipliers and forcing actions (for a detailed derivation, see the Appendix)

$$\begin{gathered} \left(\frac{\partial L}{\partial \dot{\varphi}}\right)^{\cdot} - \frac{\partial L}{\partial \varphi} = K_\varphi, \\ \left(\frac{\partial L}{\partial \dot{\theta}}\right)^{\cdot} - \frac{\partial L}{\partial \theta} + \left(\boldsymbol{e}_1, \boldsymbol{\Omega} \times \frac{\partial L}{\partial \boldsymbol{\Omega}}\right) + \left(\boldsymbol{e}_1, \boldsymbol{v} \times \frac{\partial L}{\partial \boldsymbol{v}}\right) + \left(\boldsymbol{e}_1, \boldsymbol{\gamma} \times \frac{\partial L}{\partial \boldsymbol{\gamma}}\right) = K_\theta, \\ \left(\frac{\partial L}{\partial \boldsymbol{\Omega}}\right)^{\cdot} + \left(\boldsymbol{\Omega} + \dot{\theta} \boldsymbol{e}_1\right) \times \frac{\partial L}{\partial \boldsymbol{\Omega}} + \boldsymbol{v} \times \frac{\partial L}{\partial \boldsymbol{v}} + \boldsymbol{\gamma} \times \frac{\partial L}{\partial \boldsymbol{\gamma}} = \left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\Omega}}\right)^T \boldsymbol{\lambda}, \\ \left(\frac{\partial L}{\partial \boldsymbol{v}}\right)^{\cdot} + \left(\boldsymbol{\Omega} + \dot{\theta} \boldsymbol{e}_1\right) \times \frac{\partial L}{\partial \boldsymbol{v}} = \left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{v}}\right)^T \boldsymbol{\lambda}, \end{gathered} \tag{1.3}$$

where $\boldsymbol{\lambda}$ is the vector of undetermined multipliers, K_φ and K_θ are the moments of external forces (control actions) applied to the point of attachment of the pendulum to the ball and to the point of attachment of the rotor to the rod of the pendulum, $L = T - U$ is the Lagrangian function, and $U = -m_b R_b g(\boldsymbol{\gamma}, \boldsymbol{e}_3)$ is the potential energy of the system. For some aspects of the theory of nonholonomic systems, see [2].

Supplementing Eqs. (1.3) with the kinematic relations describing the motion of the center of the spherical robot and the rotation of the moving axes relative to the fixed axes

$$\dot{\mathbf{Q}} = \widetilde{\boldsymbol{\Omega}}\mathbf{Q} + \dot{\theta}\mathbf{A}\mathbf{Q}, \quad \dot{\boldsymbol{r}} = \mathbf{Q}^\top \boldsymbol{v}, \tag{1.4}$$

where the matrices $\widetilde{\boldsymbol{\Omega}}$ and $\mathbf{A}$ have the form

$$\widetilde{\boldsymbol{\Omega}} = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

we obtain a closed system of equations completely describing the rolling motion of the spherical robot on the plane.

The resulting system admits obvious geometric integrals

$$\boldsymbol{\alpha}^2 = 1, \quad \boldsymbol{\beta}^2 = 1, \quad \boldsymbol{\gamma}^2 = 1, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\beta}, \boldsymbol{\gamma}) = (\boldsymbol{\gamma}, \boldsymbol{\alpha}) = 0.$$

After reduction to these integrals, Eqs. (1.3) and (1.4) define the flow in the twelve-dimensional phase space with two control actions. Thus, the system under consideration is an underactuated system. We recall that systems in which the number of controls is smaller than the number of degrees of freedom are called *underactuated systems*. A description of control with similar systems can be found, for example, in [23, 24].

The explicit expression for the Lagrangian of the system considered has the form

$$\begin{aligned} L = \frac{1}{2}(m_s + m_b)\boldsymbol{v}^2 - m_b R_b(\boldsymbol{v}, \boldsymbol{\Omega} \times \boldsymbol{e}_3) + \frac{1}{2}(\boldsymbol{\Omega}, (\mathbf{I}_s + \mathbf{I}_b)\boldsymbol{\Omega}) + (\boldsymbol{\Omega}, \boldsymbol{e}_1) I_{b1}\dot{\theta} + (\boldsymbol{\Omega}, \boldsymbol{e}_3) I_{b3}\dot{\varphi} \\ + \frac{1}{2} I_{b1}\dot{\theta}^2 + \frac{1}{2} I_{b3}\dot{\varphi}^2 - m_b R_b(\boldsymbol{v}, \boldsymbol{e}_1 \times \boldsymbol{e}_3)\dot{\theta} + m_b R_b g(\boldsymbol{\gamma}, \boldsymbol{e}_3), \end{aligned} \tag{1.5}$$

where $\mathbf{I}_b = \operatorname{diag}(I_{b1}, I_{b1}, I_{b3}) = \operatorname{diag}(I_{bc1} + m_b R_b^2, I_{bc1} + m_b R_b^2, I_{bc3})$ is the tensor of inertia of the pendulum relative to the center of the sphere.

Since the Lagrangian L is independent of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{r}$, the equations of motion for $\dot{\varphi}$, $\dot{\theta}$, $\boldsymbol{\Omega}$, $\boldsymbol{\gamma}$ decouple from the complete system after substituting (1.5) into (1.3) and eliminating the undetermined multipliers, and take the following form:

$$\begin{aligned} &\big(\boldsymbol{e}_3, \mathbf{I}_b(\dot{\boldsymbol{\Omega}} + \ddot{\varphi}\boldsymbol{e}_3)\big) = K_\varphi, \\ &\Big(\boldsymbol{e}_1, \mathbf{I}_b(\dot{\boldsymbol{\Omega}} + \ddot{\theta}\boldsymbol{e}_1) - m_b R_b R_s \boldsymbol{e}_3 \times (\dot{\boldsymbol{\Omega}} \times \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \dot{\boldsymbol{\gamma}})\Big) - \Big(\boldsymbol{e}_1, m_b R_b R_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \big((\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) \times \boldsymbol{e}_3\big)\Big) \\ &\quad + \Big(\boldsymbol{e}_1, \boldsymbol{\Omega} \times \Big(m_b R_b R_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\boldsymbol{\Omega} + I_{b1}\dot{\theta}\boldsymbol{e}_1 + I_{b3}\dot{\varphi}\boldsymbol{e}_3\Big)\Big) + m_b R_b g\,(\boldsymbol{e}_1, \boldsymbol{\gamma} \times \boldsymbol{e}_3) = K_\theta, \\ &m_b R_b R_s(\dot{\boldsymbol{\Omega}} \times \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \dot{\boldsymbol{\gamma}}) \times \boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\dot{\boldsymbol{\Omega}} + I_{b1}\ddot{\theta}\boldsymbol{e}_1 + I_{b3}\ddot{\varphi}\boldsymbol{e}_3 - m_b R_b R_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \big((\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) \times \boldsymbol{e}_3\big) \\ &\quad + (\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) \times \Big(m_b R_b R_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\boldsymbol{\Omega} + I_{b1}\dot{\theta}\boldsymbol{e}_1 + I_{b3}\dot{\varphi}\boldsymbol{e}_3\Big) + m_b R_b g \boldsymbol{\gamma} \times \boldsymbol{e}_3 \\ &\quad = R_s\Big((m_s + m_b)R_s(\dot{\boldsymbol{\Omega}} \times \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \dot{\boldsymbol{\gamma}}) - m_b R_b(\dot{\boldsymbol{\Omega}} + \ddot{\theta}\boldsymbol{e}_1) \times \boldsymbol{e}_3\Big) \times \boldsymbol{\gamma} \\ &\quad + R_s\Big((\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) \times \Big((m_s + m_b)R_s\boldsymbol{\Omega} \times \boldsymbol{\gamma} - m_b R_b(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) \times \boldsymbol{e}_3\Big)\Big) \times \boldsymbol{\gamma}, \\ &\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times (\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1). \end{aligned} \tag{1.6}$$

Equations (1.6) govern the dynamics of the reduced system and define the flow in the eight-dimensional phase space $\mathcal{M} = \{(\boldsymbol{\Omega}, \boldsymbol{\gamma}, \dot{\theta}, \dot{\varphi})\}$. The dynamics of the complete system can be reconstructed from the solution of Eqs. (1.6) with the help of the quadratures (1.4) and the no-slip condition (1.1).

2. DYNAMICS OF THE FREE SYSTEM

Consider the dynamics of the free motion of the spherical robot. By the free motion we mean the motion in the absence of control actions ($K_\theta = K_\varphi = 0$).

The free system admits, along with the geometric integral $\boldsymbol{\gamma}^2 = 1$, another two integrals of motion

– the integral linear in the angular velocities

$$F_1 = (\boldsymbol{\Omega} + \dot{\varphi}\boldsymbol{e}_3, \boldsymbol{e}_3) = \Omega_3 + \dot{\varphi}; \tag{2.1}$$

– the energy integral

$$E = \frac{1}{2}m_s\boldsymbol{v}^2 + \frac{1}{2}I_s\boldsymbol{\Omega}^2 + \frac{1}{2}m_b\boldsymbol{v}_b^2 + \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}_b\boldsymbol{\omega}) - m_bR_bg(\boldsymbol{\gamma}, \boldsymbol{e}_3). \tag{2.2}$$

The reduced system (1.6) needs three integrals and an invariant measure to become completely integrable by the Euler–Jacobi theorem[1)]. It can be shown that the system does not admit the existence of an invariant measure depending only on the positional variables $\boldsymbol{\gamma}$ and additional first integrals linear in the velocities. Hence, apparently, the system is integrable. Moreover, the absence of an invariant measure can lead to complex chaotic behavior. In particular, the system may exhibit strange attractors, as, for example, in rattleback dynamics [10].

In this paper we restrict the study of the dynamics of the free system to analysis of its simplest particular solutions, namely, fixed points of the reduced system (1.6). Fixed points of the reduced system correspond to steady-state solutions of the complete system (1.3)–(1.4), which can be of practical interest. The experiments conducted with different models of spherical robots have shown that such steady-state solutions can be technically realized by means of constant forcing actions. For this to be done, constant velocities of rotation of the control elements must be given and the ball must be brought to a certain initial position by means of the motors.

Let us find fixed points of the reduced system. Setting in (1.6) the derivatives $\dot{\boldsymbol{\Omega}}$, $\dot{\boldsymbol{\gamma}}$, $\ddot{\theta}$, $\ddot{\varphi}$ and K_θ, K_φ to be equal to zero, we obtain the following system of algebraic equations:

$$\begin{gathered}
\Big(\boldsymbol{e}_1, \boldsymbol{\Omega} \times \Big(m_bR_bR_s\,(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\boldsymbol{\Omega} + I_{b1}\dot{\theta}\boldsymbol{e}_1 + I_{b3}\dot{\varphi}\boldsymbol{e}_3\Big)\Big) \\
- \Big(\boldsymbol{e}_1, m_bR_bR_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \Big((\boldsymbol{\Omega} + \dot{\theta}\,\boldsymbol{e}_1) \times \boldsymbol{e}_3\Big)\Big) + m_bR_bg\,(\boldsymbol{e}_1, \boldsymbol{\gamma} \times \boldsymbol{e}_3) = 0, \\
m_bR_bR_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \Big((\boldsymbol{\Omega} + \dot{\theta}\,\boldsymbol{e}_1) \times \boldsymbol{e}_3\Big) - m_bR_bg\,\boldsymbol{\gamma} \times \boldsymbol{e}_3 \\
+ \Big(m_bR_bR_s(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \times \boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\boldsymbol{\Omega} + \mathbf{I}_{b1}\dot{\theta}\boldsymbol{e}_1 + \mathbf{I}_{b3}\dot{\varphi}\boldsymbol{e}_3 \\
-((m_s + m_b)R_s^2(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) - m_bR_bR_s(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) \times \boldsymbol{e}_3) \times \boldsymbol{\gamma}\Big) \times \Big(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1\Big) = 0, \\
\boldsymbol{\gamma} \times (\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1) = 0.
\end{gathered} \tag{2.3}$$

We present all possible solutions of this system.

1) Two two-parameter families of fixed points

$$\boldsymbol{\gamma} = \pm\boldsymbol{e}_3, \quad \boldsymbol{\Omega} = \frac{v_0}{R_s}\boldsymbol{e}_1, \quad \dot{\theta} = -\frac{v_0}{R_s}, \quad \dot{\varphi} = \dot{\varphi}_0, \tag{2.4}$$

[1)] We recall that the function $\rho(\boldsymbol{x})$ that satisfies the equation $\operatorname{div}(\rho(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})) = 0$ is called the *invariant measure* for the system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$.

where v_0 and $\dot{\varphi}_0$ are the free parameters of the family. In absolute space, to fixed points of this family there correspond motions of the ball in a straight line with constant velocity v_0 where the pendulum is in a vertical position and the rotor rotates with constant angular velocity $\dot{\varphi}_0$. For $\boldsymbol{e}_3$ the sign is chosen depending on whether the pendulum is in the upper or lower position. The parameters of the family are related to the values of the first integrals by

$$\dot{\varphi}_0 = F_1, \quad v_0^2 = \frac{2E - I_{b3}F_1^2 - 2m_bR_bg}{I_s + (m_s + m_b)R_s^2}R_s^2.$$

2) Three-parameter family of fixed points

$$\begin{gathered}\dot{\theta} = \frac{v_0}{R_s\gamma_3}, \qquad \boldsymbol{\gamma} = (\sin\xi, 0, \cos\xi), \qquad \boldsymbol{\Omega} = \frac{v_0}{\rho}\boldsymbol{\gamma} - \frac{v_0}{R_s\gamma_3}\boldsymbol{e}_1, \\ \dot{\varphi} = -\frac{\big(-R_s^2((m_s+m_b)\rho + m_bR_b\gamma_1) - R_s\gamma_3(m_bR_b\rho + \gamma_1(I_{b1} - I_{b3})) + I_s\rho\big)v_0^2 + m_bR_bgR_s\rho^2\gamma_1}{\rho R_s v_0 I_{b3}\gamma_1},\end{gathered} \tag{2.5}$$

where ρ, v_0, ξ are the parameters of the family. In absolute space, to this family there corresponds a motion in a circle of radius ρ with velocity v_0 where the plane of rotation of the pendulum is inclined through the angle ξ relative to the vertical.

Thus, the reduced system of equations (1.6) *describing the free motion of the spherical robot* ($K_\theta = K_\varphi = 0$) *admits three families of fixed points:*

1) *two two-parameter families* (2.4)*, which in absolute space correspond to the motion of the ball with an arbitrary velocity in a straight line with an arbitrary angular velocity of rotation of the rotor;*
2) *the three-parameter family* (2.5)*, to which in absolute space there corresponds the motion of the spherical robot with an arbitrary velocity v_0 in a circle of arbitrary radius ρ. In this case the inclination angle of the plane of rotation of the pendulum is constant and can be arbitrary too.*

Remark. It follows from the relations $v_0 = R_s\dot{\theta}\gamma_3$ and $|\gamma_3| \leqslant 1$ that the angular velocity of the pendulum, $\dot{\theta}$, is larger than or equal to the quantity v_0/R_s. As $\gamma_3 \to 0$ (as the plane of rotation of the pendulum tends to the horizontal position), the angular velocity of the pendulum tends to infinity. This means that the larger the angle of inclination of the plane of rotation of the pendulum, the faster the pendulum must rotate relative to the shell to ensure that the spherical robot moves with the same velocity v_0 in a circle of the same radius ρ. Also, the absolute angular velocity of the moving coordinate system $\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1$ remains constant.

3. CONTROLLED MOTION

3.1. Control along an Arbitrary Trajectory

Consider the following version of the problem of controlled motion of the spherical robot.

Suppose that at the initial instant of time we are given the position of the spherical robot $\boldsymbol{r}(0)$, $\mathbf{Q}(0)$*, the velocity of rotation of the shell about the vertical* $\Omega_\gamma(0)$ *and the angular velocities of rotation of the pendulum* $\dot{\theta}(0)$, $\dot{\varphi}(0)$*. Can the control actions* K_θ *and* K_φ *be chosen such that during the interval* $t \in [0, T]$ *the spherical robot* (*its center*) *moves along a given trajectory* $\boldsymbol{r}(t) = (x(t), y(t))$*?*

A similar problem in the case where the projection of the angular velocity $\Omega_\gamma(t)$ is given along with the trajectory was solved in [17] for the Chaplygin ball with rotors and in [15] for a ball with an internal omniwheel platform. In the case we consider here the general solution of the control problem with an arbitrarily given function $\Omega_\gamma(t)$ apparently cannot be obtained. This is due to a smaller number of control actions in the model considered. Therefore, in this section we restrict ourselves to controlling only a part of the variables, that is, we specify only the dependence $x(t)$, $y(t)$, and the projection of the angular velocity onto the vertical $\Omega_\gamma(t)$ remains an unknown function of time.

Below we present an algorithm that allows us to solve this problem.

1. Write the the angular velocity of the shell $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \Omega_\alpha(t)\boldsymbol{\alpha} + \Omega_\beta(t)\boldsymbol{\beta} + \Omega_\gamma\boldsymbol{\gamma}, \tag{3.1}$$

where $\Omega_\alpha(t)$ and $\Omega_\beta(t)$ are expressed from (1.4) using the constraint equation (1.1) as

$$\Omega_\alpha(t) = \frac{\dot{y}(t)}{R_s}, \qquad \Omega_\beta(t) = -\frac{\dot{x}(t)}{R_s},$$

and are known functions of time.

2. Substitute the expression for the angular velocity (3.1) into the third equation of (1.6). The resulting equation, combined with the kinematic relations describing the rotation of the moving axes (1.4), forms a closed, explicitly time-dependent system of equations in the variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}, \Omega_\gamma$, $\dot{\theta}$, $\dot{\varphi}$:

$$\begin{gathered}
m_bR_bR_s(\dot{\boldsymbol{\Omega}}\times\boldsymbol{\gamma} + \boldsymbol{\Omega}\times\dot{\boldsymbol{\gamma}})\times\boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\dot{\boldsymbol{\Omega}} + I_{b1}\ddot{\theta}\boldsymbol{e}_1 + I_{b3}\ddot{\varphi}\boldsymbol{e}_3 - m_bR_bR_s(\boldsymbol{\Omega}\times\boldsymbol{\gamma})\times\Big((\boldsymbol{\Omega} + \dot{\theta}\,\boldsymbol{e}_1)\times\boldsymbol{e}_3\Big)\\
+(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1)\times\Big(m_bR_bR_s\,(\boldsymbol{\Omega}\times\boldsymbol{\gamma})\times\boldsymbol{e}_3 + (\mathbf{I}_s + \mathbf{I}_b)\boldsymbol{\Omega} + I_{b1}\dot{\theta}\boldsymbol{e}_1 + I_{b3}\dot{\varphi}\boldsymbol{e}_3\Big) + m_bR_bg\,\boldsymbol{\gamma}\times\boldsymbol{e}_3\\
= R_s\Big((m_s + m_b)R_s(\dot{\boldsymbol{\Omega}}\times\boldsymbol{\gamma} + \boldsymbol{\Omega}\times\dot{\boldsymbol{\gamma}}) - m_bR_b(\dot{\boldsymbol{\Omega}} + \ddot{\theta}\,\boldsymbol{e}_1)\times\boldsymbol{e}_3\Big)\times\boldsymbol{\gamma}\\
+R_s\Big((\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1)\times\Big((m_s + m_b)R_s\boldsymbol{\Omega}\times\boldsymbol{\gamma} - m_bR_b(\boldsymbol{\Omega} + \dot{\theta}\,\boldsymbol{e}_1)\times\boldsymbol{e}_3\Big)\Big)\times\boldsymbol{\gamma},\\
\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\times(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1), \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta}\times(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1), \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times(\boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_1),
\end{gathered} \tag{3.2}$$

where $\boldsymbol{\Omega}$ should be viewed as the function $\boldsymbol{\Omega}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \Omega_\gamma, t)$.

3. Solve this system numerically under the given initial conditions $\boldsymbol{\alpha}(0)$, $\boldsymbol{\beta}(0)$, $\boldsymbol{\gamma}(0)$, $\Omega_\gamma(0)$, $\dot{\theta}(0)$, $\dot{\varphi}(0)$. This yields the time dependence of the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, of the vertical projection of the angular velocity of the ball, Ω_γ, and of the angular velocities of the pendulum, $\dot{\theta}$ and $\dot{\varphi}$.

4. Substitute the dependencies obtained into the first two equations of (1.6) and find the explicit time dependence of the control torques K_φ, K_θ.

Example. As an example, we consider the motion of a ball in a circle of radius R_0 in time T. We specify the law of motion of the center of the ball along the trajectory in the form

$$\begin{gathered}
x(t) = R_0\cos s(t), \qquad y(t) = R_0\sin s(t),\\
s(t) = wt - \sin wt,
\end{gathered} \tag{3.3}$$

where $w = \frac{2\pi}{T}$. Figure 2 shows the resulting dependencies of the components of the vector $\boldsymbol{\gamma}$, of the angular velocities $\dot{\theta}$ and $\dot{\varphi}$ and of the control torques K_θ and K_φ for the motion of the ball in a circle of radius $R_0 = 0.5$ in time $T = 78$. The dependencies of the control torques and angular velocities have the form of oscillations near zero with pronounced maxima and minima, and so Figs. 2b and 2c are represented on a modified (signed) logarithmic scale, i.e., instead of the functions $f(t)$ the graphs $\operatorname{arcsh}(5\cdot f(t))$ are presented. Here and in the sequel, the numerical simulation was carried out for the following system parameters:

$$m_b = 1,\ m_s = 0.1,\ R_s = 1,\ R_b = R_s/4,\ I_{b1} = 0.3,\ I_{b3} = 0.47,\ I_s = 0.07. \tag{3.4}$$

Remark. The problem of construction of control actions is not solvable for all trajectories of the spherical robot. Indeed, equations (3.2) can be reduced to the form

$$\mathbf{A}(\dot{\boldsymbol{z}}) = \boldsymbol{f}(\boldsymbol{z}, t),$$

where $\boldsymbol{z} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \Omega_\gamma, \dot{\theta}, \dot{\varphi})$, and $\mathbf{A}(\boldsymbol{z})$ and $\boldsymbol{f}(\boldsymbol{z}, t)$ are, respectively, a matrix and a vector that depend on the variables $\boldsymbol{z}$ and time. The system under consideration contains a singularity on the manifold given by the degeneracy condition of the matrix $\mathbf{A}$

$$\det\mathbf{A}(\boldsymbol{z}) = 0.$$

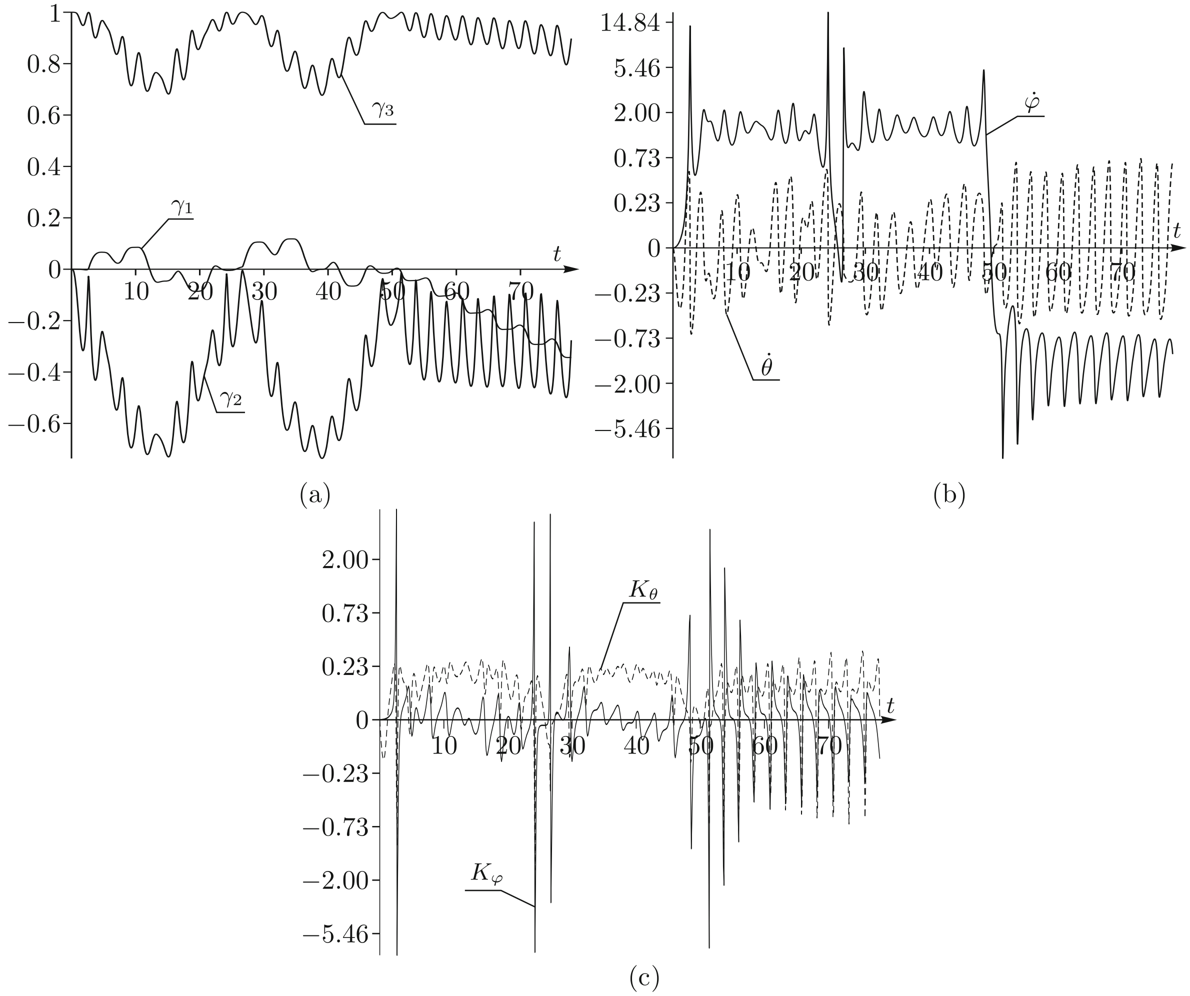


Fig. 2. Time dependence of the components of the vector $\boldsymbol{\gamma}$ (a), of the angular velocities $\dot\theta$ and $\dot\varphi$ (b) and of the control torques K_θ and K_φ (c) for the motion of the ball in a circle (3.3). The dependence graphs (b) and (c) are represented on a modified logarithmic scale.

If the trajectory of the system with given initial conditions passes near this manifold, then the velocities $\dot{\boldsymbol{z}}$ take large values. Such a motion is technically not realizable since the velocities of rotation of the rotor and the pendulum are bounded by the parameters of the motors. But if the trajectory enters this manifold in finite time, then the construction of control for larger times is impossible even theoretically. Thus, the presence of the above-mentioned singularities imposes restrictions on possible trajectories of the system and on the maneuverability of the model considered.

3.2. Control Using Gaits

Above we have shown that the system under consideration admits two kinds of steady-state solutions: the motion in a straight line (2.4) and that in a circle (2.5). Obviously, combining these solutions, one can realize the motion from any initial point to any end point. Moreover, since the steady-state solutions (particularly the stable ones) are the most applicable in practice, we consider here the problem of definition of control torques for the realization of transition from one steady-state solution to another. Using standard terminology, we shall refer to the elementary maneuvers realizing such transitions as *gaits*.

Rotation about a fixed point. The special features of the design of the spherical robot we consider here allow it to move in a straight line only in a direction parallel to the plane of rotation of the pendulum. Consequently, before the start of the motion (acceleration), it is necessary to execute rotation about the vertical axis so that the direction of motion coincides with the plane of rotation of the pendulum. Therefore, as the first gait we consider the rotation of the spherical robot by a given angle ψ_0 about the vertical axis passing through the fixed point in time T. Obviously, this gait connects two steady-state solutions corresponding to the state of rest.

The rotation about the vertical lies on the invariant manifold of Eqs. (1.4) and (1.6), which is given by

$$\boldsymbol{\Omega} = (0, 0, \Omega_3), \quad \dot{\theta} = 0, \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0), \quad \boldsymbol{\beta} = (\beta_1, \beta_2, 0), \quad \boldsymbol{\gamma} = (0, 0, 1), \tag{3.5}$$

where $\Omega_3 = \dot{\psi}$ with ψ being the precession angle of the spherical shell. On this invariant manifold, Eqs. (1.6) take the simple form

$$\ddot{\varphi} = -\frac{I_s + I_{b3}}{I_{b3}}\ddot{\psi}, \qquad K_\varphi = -I_s\ddot{\psi}. \tag{3.6}$$

Integrating the first equation of (3.6) and noting the initial conditions corresponding to the state of rest

$$\varphi(0) = \dot{\varphi}(0) = 0, \quad \psi(0) = \dot{\psi}(0) = 0, \tag{3.7}$$

we obtain

$$\varphi = -\frac{I_s + I_{b3}}{I_{b3}}\psi. \tag{3.8}$$

Now, defining $\psi(t)$ in the form of an arbitrary function satisfying the initial conditions (3.7) and the required boundary conditions

$$\psi(T) = \psi_0, \qquad \dot{\psi}(T) = 0,$$

we obtain, from (3.8) and the second equation of (3.6), the dependencies $\varphi(t)$ and $K_\varphi(t)$ which realize the rotation by the angle ψ_0.

Example. As an example, we consider the case where the angle ψ is given as

$$\psi = \frac{\psi_0}{2\pi}(wt - \sin wt),$$

where $w = \frac{2\pi}{T}$. In this case, it is easy to find an explicit form of the control

$$K_\varphi = -\frac{I_s\psi_0 w^2}{2\pi}\sin wt.$$

Acceleration in a straight line. We now consider the acceleration (deceleration) maneuver in the case of motion in a straight line. Since the acceleration is possible only in a direction parallel to the plane of rotation of the pendulum, we assume in what follows that this plane now coincides with the required direction of motion. Suppose that at the initial instant of time the ball moves in a straight line according to the solution (2.4) with initial velocity v_0. After completing the maneuver (at time $t = T$) it keeps moving according to the steady-state solution (2.4) in the same straight line, but now with velocity v_T. Let us find the control torque K_θ connecting these two solutions.

The motion under consideration lies on the invariant manifold of (1.4) and (1.6), which is given by

$$\boldsymbol{\Omega} = (\Omega_1, 0, 0), \quad \dot{\varphi} = 0, \quad \boldsymbol{\alpha} = (1, 0, 0), \quad \boldsymbol{\beta} = (0, \beta_2, \beta_3), \quad \boldsymbol{\gamma} = (0, \gamma_2, \gamma_3). \tag{3.9}$$

After parameterization of the vector $\boldsymbol{\gamma}$ by the angle ϑ

$$\boldsymbol{\gamma} = (0, \sin\vartheta, \cos\vartheta), \tag{3.10}$$

Eqs. (1.6) on the manifold (3.9) take the form

$$\begin{gathered}
\dot{\Omega}_1 = -\frac{m_bR_bR_s\sin\vartheta\left(I_{b1}(\dot{\theta}+\Omega_1)^2 + m_bR_bg\cos\vartheta\right) - K_\theta\left(m_bR_bR_s\cos\vartheta - I_{b1}\right)}{I_{b1}I_0 - m_b^2R_b^2R_s^2\cos^2\vartheta},\\
\ddot{\theta} = \frac{m_bR_b\sin\vartheta\Big(R_s(\Omega_1+\dot{\theta})^2(I_{b1} - m_bR_bR_s\cos\vartheta) + g(m_bR_bR_s\cos\vartheta - I_0)\Big) + K_\theta(I_0 + I_{b1} - 2m_bR_bR_s\cos\vartheta)}{I_{b1}I_0 - m_b^2R_b^2R_s^2\cos^2\vartheta},\\
\dot{\vartheta} = \dot{\theta} + \Omega_1,
\end{gathered} \tag{3.11}$$

where the notation $I_0 = I_s + (m_s + m_b)R_s^2$ is used for brevity.

When $t < 0$ and $t > T$, the spherical robot moves in a straight line according to the steady-state solution (2.4), hence, at the initial and the final instant of time the pendulum must be in the lower vertical position, and its velocity must be equal to zero. The corresponding boundary conditions for the system (3.11) have the form

$$\begin{gathered}\vartheta(0) = \vartheta(T) = 0,\\ \Omega(0) = -\dot{\theta}(0) = \frac{v_0}{R_s}, \qquad \Omega(T) = -\dot{\theta}(T) = \frac{v_T}{R_s}.\end{gathered} \tag{3.12}$$

Moreover, it follows from (3.12) and the third equation of (3.11) that the function $\vartheta(t)$ must satisfy the additional condition

$$\dot{\vartheta}(0) = \dot{\vartheta}(T) = 0. \tag{3.13}$$

Eliminating $\dot{\theta}$ from the first two equations of (3.11) using the third one, we obtain

$$\begin{gathered}K_\theta = \frac{\ddot{\vartheta}\Big(m_b^2 R_b^2 R_s^2 \cos^2\vartheta + I_0 I_{b1}\Big) + m_b R_b \sin\vartheta \Big(m_b R_b R_s^2 \cos\vartheta \ddot{\vartheta} + g I_0\Big)}{m_b R_b R_s \cos\vartheta - I_0},\\ \dot{\Omega}_1 = \frac{m_b R_b \sin\vartheta (g + R_s \dot{\vartheta}^2) - \ddot{\vartheta}(m_b R_b R_s \cos\vartheta - I_{b1})}{m_b R_b R_s \cos\vartheta - I_0}.\end{gathered} \tag{3.14}$$

We now choose some dependence $\vartheta(t, p)$ satisfying the conditions (3.12) and (3.13), where p is some parameter of the maneuver. Substituting this dependence into (3.14), we obtain an explicit expression for $K_\theta(t)$ and $\dot{\Omega}_1(t)$, which realize the maneuver we consider here.

Integrating the second equation of (3.14) over time, we obtain the value of the angular velocity of the spherical robot $\Omega(T)$ at the final instant of time T. Now, using the constraint equation (1.1), it is easy to obtain the value of linear velocity v_T at the final instant of time and hence the change in the velocity $\Delta v(p)$ depending on the maneuver parameter. Inverting this function, we obtain the dependence $p(\Delta v)$, which will allow us to determine the maneuver parameters for acceleration to a given velocity.

Example. As an example, we consider a controlled motion where the spherical robot with parameters (3.4) accelerates from rest with the initial conditions

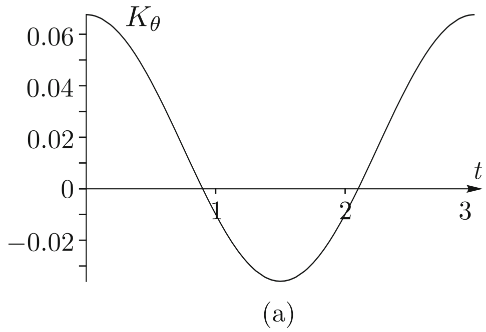
$$\boldsymbol{\Omega}(0) = 0, \quad \boldsymbol{\alpha}(0) = (1, 0, 0), \quad \boldsymbol{\beta}(0) = (0, 1, 0), \quad \boldsymbol{\gamma}(0) = (0, 0, 1), \quad \dot{\varphi}(0) = 0$$

to a given velocity v_T. Let us choose the dependence $\vartheta(t)$ in the form

$$\vartheta(t) = p \sin^2\left(\frac{\pi t}{T}\right), \tag{3.15}$$

where p is the oscillation amplitude of the pendulum. Assume that it is necessary to accelerate the ball in time $T = 3$ to the velocity $v_T = 0.04$. Integrating the second equation of (3.14) using (3.15), we obtain the (numerical) dependence $v_T(p)$. Inverting this dependence, we find that the acceleration to $v_T = 0.04$ is performed with the amplitude $p = 0.1$. Substituting the resulting value of p and the dependence (3.15) into the first equation of (3.14), we obtain an explicit form of the control torque $K_\theta(t)$. Figure 3 shows this time dependence of the control torque K_θ and the projections of the vector $\boldsymbol{\gamma}$ onto the axis $\boldsymbol{e}_3$ and of velocity v. The figure shows that after completion of the maneuver $\boldsymbol{\gamma}$ returns to the vertical position, and the further motion is a rolling motion in a straight line with new velocity v_T.

Combining the gaits that realize acceleration (deceleration), rotation about a fixed point and steady motion in a straight line (2.4), one can execute the motion of the spherical robot along an arbitrary trajectory. But in this case a stop is necessary before each change in the direction of motion. To eliminate this limitation, it would be interesting to construct other gaits, for example, those connecting the motions along straight lines having different directions, without an intermediate stop. Examples of such gaits for a ball with the Lagrange pendulum are given in [19]. In the case at hand the construction of such controls is much more complicated and is beyond the scope of this paper.



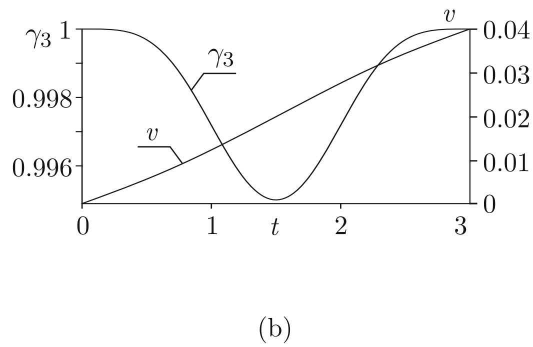


Fig. 3. Example of the time dependence K_θ (a) and γ_3, v (b) during the acceleration of the ball in a straight line to a given velocity.

CONCLUSION

In conclusion, we highlight some open problems which we believe to be of particular interest.

It would be interesting to examine in more detail the possibility (or impossibility) of controlled motion along some curve $\boldsymbol{r}(t)$ from the point of view of solvability of the system (3.2). This would allow one to determine the maneuverability of the model considered and the class of "permissible" trajectories along which a controlled motion can be executed.

Another open problem is that of construction of gaits realizing the rotation without a stop, in particular, the construction of gaits connecting steady motions in straight lines located at an angle to each other, and of gaits connecting the motions in a straight line and in a circle.

Also of interest is an experimental testing of the results obtained: realization of steady-state solutions by specifying constant angular velocities of the control elements; realization of controlled motion in a straight line and of rotation about a fixed point using the gaits obtained. Based on such an experimental testing, it would be possible, in particular, to determine the scope of applicability of the model considered.

APPENDIX

We present here a derivation of equations that are a generalization of the Poincaré – Hamel equations [25] in quasi-coordinates to systems with nonholonomic constraints.

Consider the equations of motion of a Lagrangian dynamical system defined by the generalized coordinates $q_i = \{\mathbf{Q}, \boldsymbol{r}, \theta, \varphi\}$ and the quasi-velocities $w_i = \{\boldsymbol{\Omega}, \boldsymbol{v}, \dot{\theta}, \dot{\varphi}\}$, which are expressed in terms of the generalized velocities $\dot{q}_i$ by the formulas

$$\dot{q}_i = \sum_{s=1}^{k} v_i^s(\boldsymbol{q}) w_s, \quad i = 1 \ldots 8,$$

or in the explicit form

$$\dot{\mathbf{Q}} = \widetilde{\boldsymbol{\Omega}}\mathbf{Q} + \dot{\theta}\mathbf{A}\mathbf{Q}, \qquad \dot{\boldsymbol{r}} = \mathbf{Q}^\top \boldsymbol{v}, \qquad \dot{\theta} = \dot{\theta}, \qquad \dot{\varphi} = \dot{\varphi}.$$

To derive the equations of motion, it is necessary to find the velocity components of the system in the nonholonomic basis of the vector fields

$$\boldsymbol{v}^s = \sum_i v_i^s(\boldsymbol{q}) \frac{\partial}{\partial q_i}. \tag{A.1}$$

To do so, we write explicitly the total time derivative of some function f:

$$\dot{f} = \left(\dot{Q}_{ij}\frac{\partial}{\partial Q_{ij}} + \dot{r}_i \frac{\partial}{\partial r_i} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\varphi}\frac{\partial}{\partial \varphi}\right) f = \left((\widetilde{\boldsymbol{\Omega}} + \mathbf{A}\dot{\theta})_{ik} Q_{kj} \frac{\partial}{\partial Q_{ij}} + Q_{ji} v_j \frac{\partial}{\partial r_i} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\varphi}\frac{\partial}{\partial \varphi}\right) f$$
$$= \left(Q_{kj}\widetilde{\Omega}_{ik}\frac{\partial}{\partial Q_{ij}} + Q_{ji} v_j \frac{\partial}{\partial r_i} + \dot{\theta}\left(\frac{\partial}{\partial \theta} + A_{ik} Q_{kj}\frac{\partial}{\partial Q_{ij}}\right) + \dot{\varphi}\frac{\partial}{\partial \varphi}\right) f = (\Omega_i \boldsymbol{\xi}_i + v_i \boldsymbol{\zeta}_i + \dot{\theta}\boldsymbol{\eta} + \dot{\varphi}\boldsymbol{\nu}) f,$$

where the vector fields have the form

$$\begin{gathered}\boldsymbol{\xi}_i = -\varepsilon_{ijk}\left(\alpha_j\frac{\partial}{\partial\alpha_k} + \beta_j\frac{\partial}{\partial\beta_k} + \gamma_j\frac{\partial}{\partial\gamma_k}\right), \qquad \boldsymbol{\zeta}_i = \left(\alpha_i\frac{\partial}{\partial r_1} + \beta_i\frac{\partial}{\partial r_2} + \gamma_i\frac{\partial}{\partial r_3}\right),\\ \boldsymbol{\eta} = \frac{\partial}{\partial\theta} - \varepsilon_{1jk}\left(\alpha_j\frac{\partial}{\partial\alpha_k} + \beta_j\frac{\partial}{\partial\beta_k} + \gamma_j\frac{\partial}{\partial\gamma_k}\right) = \frac{\partial}{\partial\theta} + \boldsymbol{\xi}_1, \qquad \boldsymbol{\nu} = \frac{\partial}{\partial\varphi},\end{gathered} \tag{A.2}$$

here ε_{ijk} is the Levi-Civita symbol, $i, j, k = 1\ldots 3$. Then the nonholonomic basis of the vector fields has the components $\boldsymbol{v} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3, \boldsymbol{\eta}, \boldsymbol{\nu})$.

The equations of motion in quasi-velocities with undetermined Lagrange multipliers have the form [26]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial w_i}\right) = \sum_{r,s} c^s_{ri} w_r \frac{\partial L}{\partial w_s} + \boldsymbol{v}^i(L) + \sum_j \lambda_j \frac{\partial F_j}{\partial w_i}, \qquad i = 1\ldots 8, \tag{A.3}$$

where the differentiation along the vector fields $\boldsymbol{v}^i$ is defined using (A.1) and c^s_{ri} are the commutators of the vector fields (Poincaré parameters):

$$[\boldsymbol{v}^r, \boldsymbol{v}^i] = c^s_{ri}(\boldsymbol{q})\boldsymbol{v}^s,$$

where $[\,\cdot\,,\cdot\,]$ is the Lie bracket of the vector fields. The commutation relations of the vector fields (A.2) have the form

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \varepsilon_{ijk}\boldsymbol{\xi}_k, \quad [\boldsymbol{\xi}_i, \boldsymbol{\zeta}_j] = \varepsilon_{ijk}\boldsymbol{\zeta}_k, \quad [\boldsymbol{\xi}_i, \boldsymbol{\eta}] = -\varepsilon_{1ik}\boldsymbol{\xi}_k, \quad [\boldsymbol{\zeta}_i, \boldsymbol{\eta}] = -\varepsilon_{1ik}\boldsymbol{\zeta}_k,$$

$$[\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j] = [\boldsymbol{\eta}, \boldsymbol{\eta}] = [\boldsymbol{\xi}_i, \boldsymbol{\nu}] = [\boldsymbol{\zeta}_i, \boldsymbol{\nu}] = [\boldsymbol{\nu}, \boldsymbol{\nu}] = 0.$$

Adding the forcing actions K_θ, K_φ to the explicit form of (A.3), we obtain Eqs. (1.3).

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