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On the Motion of a Heavy Material Point on a Rotating Sphere (Dry Friction Case)

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Abstract—The problem of motion of a heavy particle on a sphere uniformly rotating about a fixed axis is considered in the case of dry friction. It is assumed that the angle of inclination of the rotation axis is constant. The existence of equilibria in an absolute coordinate system and their linear stability are discussed. The equilibria in a relative coordinate system rotating with the sphere are also studied. These equilibria are generally nonisolated. The dependence of the equilibrium sets of this kind on the system parameters is also considered.

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On the occasion of the 65th anniversary of Acad. V. V. Kozlov,
a prominent scientist and teacher

Based on the results of computer simulation (see [1, 2]), it is possible to investigate the dynamics of systems with a large number of moving parts. However, the output of such a simulation usually does not represent any qualitative results. That is why it is reasonable to consider some simple problems such as the motion of a material particle on some surface under the action of friction force. In [3] the problem of motion of a heavy bead on a circular hoop rotating about its vertical diameter was studied. A similar problem for a circular hoop rotating about some other vertical axis was investigated in [4].

In these papers the dependence of the nonisolated equilibrium sets of a bead on system parameters was considered. The fact of existence of these sets for systems with dry friction is well known (see [5, 6]). The investigation of the existence and stability of nonisolated equilibria in gyro systems with friction [7, 8] laid a foundation for the development of the stability theory for systems with dry friction [9]. Methods of stability analysis of equilibria of this kind based on the general theory of systems with discontinuous right-hand sides were later developed in [10–12]. An original approach to studying the dependence of nonisolated equilibrium sets on the parameters of a system for two-dimensional and three-dimensional problems was suggested by A. P. Ivanov [13, 14]. The same kind of bifurcations, as well as sufficient conditions for stability of equilibrium sets, were considered in [15].

Remark 1. The motion of *rigid bodies* on moving surfaces is generally investigated in the presence of nonholonomic constraints and under the assumption of no slippage (see, for example, [16]). The same problems for nonholonomic systems under the assumption of dry friction are still very poorly understood.

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1. FORMULATION OF THE PROBLEM

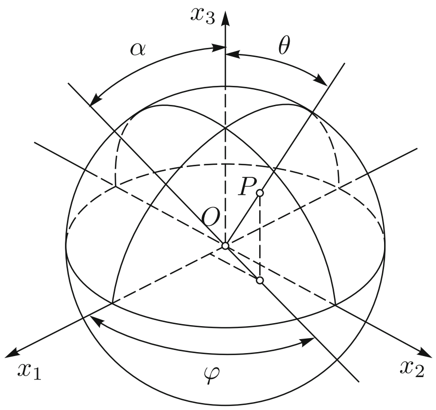

Fig. 1. A point on a rotating sphere.

Let P be a heavy material point of mass m. This point moves on the surface of a two-dimensional sphere of radius ℓ under the action of the dry friction force, with the coefficient of friction being μ. The sphere is rotating with a constant angular velocity $\vec{\omega}$ about a fixed axis. It is assumed that the axis passes through the center of the sphere O and its unit vector is $\vec{e}$. Denote the angle of inclination of the axis by α, $0 \leqslant \alpha \leqslant \pi/2$.

Let $Ox_1x_2x_3$ be an absolute coordinate system such that the plane Ox_1x_2 is horizontal and the axis Ox_3 is directed along the upward vertical. Assume that the rotation axis belongs to the plane Ox_1x_3. Denote the spherical angles that specify the position of the point on the sphere by θ and φ, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$ (Fig. 1).

Using these coordinates, the absolute velocity of the point P and its relative velocity can be written as

$$\begin{aligned}
\vec{v} &= v_\theta \vec{e}_\theta + v_\varphi \vec{e}_\varphi = \ell\dot{\theta}\vec{e}_\theta + \ell\sin\theta\dot{\varphi}\vec{e}_\varphi, \\
\vec{v_1} &= v_{1\theta}\vec{e}_\theta + v_{1\varphi}\vec{e}_\varphi \\
&= (\ell\dot{\theta} + \ell\omega\sin\alpha\sin\varphi)\vec{e}_\theta \\
&\quad + (\ell\sin\theta\dot{\varphi} + \ell\omega\sin\alpha\cos\theta\cos\varphi - \ell\omega\sin\theta\cos\alpha)\vec{e}_\varphi.
\end{aligned}$$

Assume that the point is slipping, then the equations of motion can be written as

$$\begin{cases}
m\ell(\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2) = mg\sin\theta - F\dfrac{v_{1\theta}}{|\vec{v_1}|}, \\
m\ell(\sin\theta\ddot{\varphi} + 2\dot{\varphi}\dot{\theta}\cos\theta) = -F\dfrac{v_{1\varphi}}{|\vec{v_1}|},
\end{cases} \tag{1.1}$$

where F is a friction force.

Equations (1.1) are obtained by projecting the Newton equation of motion on the axes $\vec{e_\theta}$ and $\vec{e_\varphi}$. In the case of slippage of the point the friction force can be found from the following equation:

$$F = \mu|N|, \quad N = m\tilde{N}, \quad \tilde{N} = \ell(-\sin^2\theta\dot{\varphi}^2 - \dot{\theta}^2) + g\cos\theta, \tag{1.2}$$

where $\vec{N}$ is a normal reaction. It is also assumed that the constraint is bilateral, so the point always remains on the surface of the sphere.

Now consider two different cases of the inclination angle of the rotation axis.

2. THE CASE OF AN INCLINED ROTATION AXIS

We first consider the case of an inclined rotation axis. The system for equilibria can be obtained by putting $\dot{\theta} = \dot{\varphi} = 0$, $\ddot{\theta} = \ddot{\varphi} = 0$ in Eqs. (1.1) and (1.2). Dropping tildes, one arrives at the equations

$$\begin{cases}
0 = g\sin\theta - \mu|N|\sin\alpha\sin\varphi \cdot R^{-1}(\theta,\varphi,\alpha), \\
0 = -\mu|N|(\sin\alpha\cos\theta\cos\varphi - \sin\theta\cos\alpha) \cdot R^{-1}(\theta,\varphi,\alpha), \\
N = g\cos\theta,
\end{cases} \tag{2.1}$$

where $R(\theta,\varphi,\alpha) = \sqrt{(\sin\alpha\sin\varphi)^2 + (\sin\alpha\cos\theta\cos\varphi - \sin\theta\cos\alpha)^2}$.

Now there are two cases for the disposition of a point on a sphere.

Case 1. Let the point be placed on the upper hemisphere. Then $\cos\theta > 0$. Using Eq. (2.1), one obtains the solution

$$\begin{cases}
\cos\varphi = \mu\operatorname{ctg}\alpha, \\
\operatorname{tg}\theta = \mu,
\end{cases} \tag{2.2}$$

when $\sin\varphi > 0$.

Let μ be fixed and α be varying from $\pi/2$ to 0. For every allowed value of α the point is placed on a circle $\mathcal{C}$ defined by the angle $\theta = \operatorname{arctg}\mu$. If $\alpha = \pi/2$, i.e., the rotation axis is horizontal, from Eq. (2.2) one obtains $\varphi = \pi/2$. Thus, the point is placed on the intersection of this circle and the vertical great circle, perpendicular to the rotation axis. If the angle α decreases from $\pi/2$ to $\alpha_* = \operatorname{arctg}\mu$, then the equilibrium is moving along $\mathcal{C}$, and when $\alpha = \alpha_*$, one obtains $\varphi = 0$. There are no equilibria if $\alpha < \alpha_*$ or $\sin\varphi < 0$.

Case 2. Similarly for the point located on the lower hemisphere. In this case $\cos\theta < 0$ and if $\sin\varphi > 0$, the solution becomes

$$\begin{cases} \cos\varphi = -\mu \operatorname{ctg}\alpha, \\ \operatorname{tg}\theta = -\mu, \end{cases} \tag{2.3}$$

and there are no solutions if $\sin\varphi < 0$.

Thus, if α varies from $\pi/2$ to $\alpha_* = \operatorname{arctg}\mu$ with μ being fixed, the equilibria move along the circle given by $\theta = \pi - \operatorname{arctg}\mu$. If the rotation axis is horizontal, then $\cos\varphi = 0$, i.e, $\varphi = \pi/2$, so the point is placed on the vertical great circle, perpendicular to the rotation axis. With decreasing angle α the equilibrium is moving along the circle, and when $\alpha = \alpha_*$, one obtains $\varphi = \pi$. There are also no equilibria if $\alpha < \alpha_*$.

Figure 2 illustrates the dependence of the coordinate φ on the angle of inclination of the rotation axis in cases of different magnitudes of the coefficient of friction μ. The coordinate θ remains constant on each of the curves. Thus, if the angle α is increasing, then the upper and the lower equilibria are moving on the horizontal plane from $\varphi = \pi/2$ to $\varphi = 0$ and from $\varphi = \pi/2$ to $\varphi = \pi$, respectively. Figure 3 illustrates the displacement of equilibria on a sphere with the parameter μ increasing from 0 to 1 for $\alpha = \pi/4$.

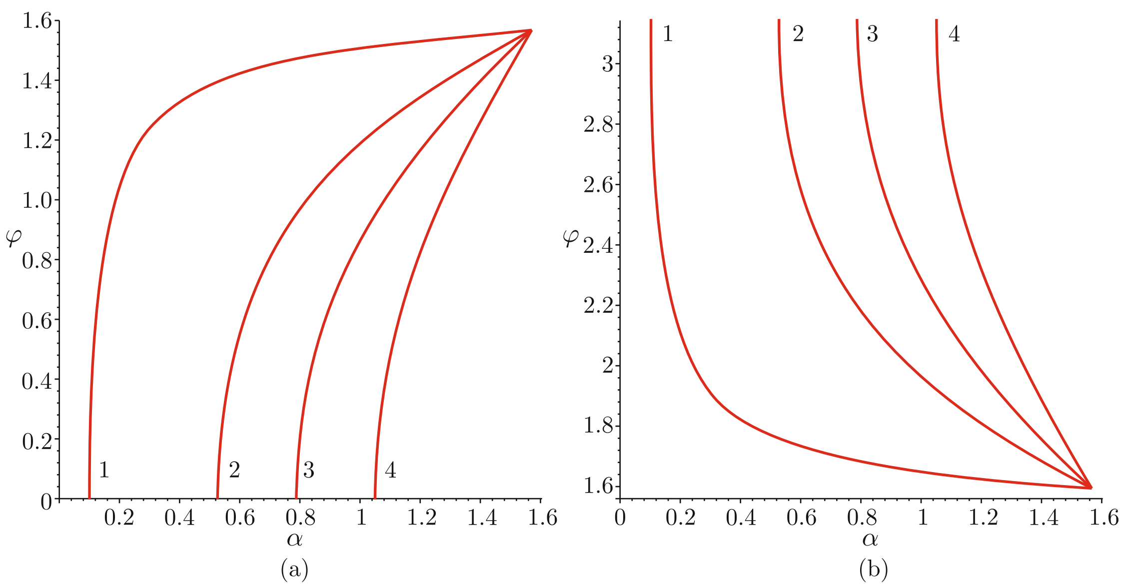

Fig. 2. The dependence of the coordinate φ on the inclination angle α increasing from 0 to $\frac{\pi}{2}$ at different values of the friction coefficient μ: 1 — 0.1, 2 — $\frac{1}{\sqrt{3}}$, 3 — 1, 4 — $\sqrt{3}$: (a) for the upper equilibrium, (b) for the lower equilibrium.

In order to discuss the stability of the equilibria, we linearize the equations of motion near the equilibria by introducing small perturbations $\theta = \theta_0 + \hat{\theta}$, $\varphi = \varphi_0 + \hat{\varphi}$. Since these perturbations are assumed small, the perturbed normal reaction $\hat{N} = g\cos\theta_0 - g\hat{\theta}\sin\theta_0$ for $\theta_0 \neq \pi/2$ is of the same sign as the normal reaction for equilibrium. Introducing the parameter $\varepsilon = \pm 1$, one obtains the

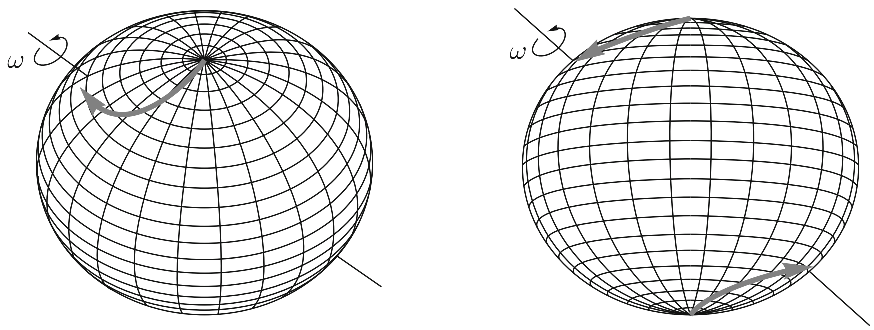

Fig. 3. The displacement of equilibria on a sphere with increasing μ: (a) top view, (b) side view.

linearized equations on the equilibria found:

$$\begin{cases} \ell\ddot{\hat{\theta}} - g\cos\theta_0\hat{\theta} - \varepsilon\mu g\sin\theta_0\hat{\theta} = 0, \\ \ell\omega\sin\theta_0\sin\varphi_0\sin\alpha\ddot{\hat{\varphi}} + \varepsilon\mu g\cos\theta_0\sin\theta_0\dot{\hat{\varphi}} - \varepsilon\mu g\omega\cos^2\theta_0\sin\varphi_0\sin\alpha\hat{\varphi} \\ -\varepsilon\mu g\omega(\sin\theta_0\cos\theta_0\cos\varphi_0\sin\alpha + \cos^2\theta_0\cos\alpha)\hat{\theta} = 0 \end{cases} \quad (2.4)$$

The characteristic equation of the system is

$$\begin{aligned} \ell^2\omega\sin\theta_0\sin\varphi_0\sin\alpha P_1(\sigma) &= (\ell\sigma^2 - g\cos\theta_0 - \varepsilon\mu g\sin\theta_0) \\ \times(\ell\omega\sin\theta_0\sin\varphi_0\sin\alpha\sigma^2 &+ \varepsilon\mu g\cos\theta_0\sin\theta_0\sigma - \varepsilon\mu g\omega\sin\varphi_0\cos^2\theta_0\sin\alpha), \end{aligned} \quad (2.5)$$

where θ_0 and φ_0 can be found from Eqs. (2.2) and (2.3).

The expression inside the first parentheses of Eq. (2.5) is equal to zero when

$$\sigma_{1,2} = \sqrt{\frac{\varepsilon g}{\ell}\sqrt{1+\mu^2}}.$$

If $\varepsilon = 1$, then the characteristic equation has a root with a positive real part and the equilibrium is unstable.

The expression inside the second parentheses is equal to zero when

$$\sigma_{3,4} = \frac{-\mu^2 g \pm \sqrt{\mu^4 g^2 + 4\varepsilon\mu^2\sqrt{1+\mu^2}g\ell\omega^2(1-\mu^2\operatorname{ctg}^2\alpha)\sin^2\alpha}}{2\ell\omega\mu\sin\alpha\sqrt{1+\mu^2}\sqrt{1-\mu^2\operatorname{ctg}^2\alpha}}.$$

If $\varepsilon = -1$, i.e., the lower equilibrium is considered, then the characteristic equation has two pure imaginary roots and two roots with negative real parts. Thus, this equilibrium is linearly stable.

Remark 2. In [17] a similar problem for the viscous friction force was discussed. The set of equilibria in this case differs from that obtained above, but in both of them it is necessary to consider the case of two pure imaginary roots to conduct a full investigation of stability (see, e.g., [18]).

3. THE CASE OF A VERTICAL ROTATION AXIS

In the case of a vertical rotation axis the spherical coordinate system degenerates in equilibrium positions. That is why in this case the problem should be considered using the Cartesian coordinates $Ox_1x_2x_3$.

Let $\vec{r}$ be the vector $\vec{OP}$ and let $\vec{\omega} = \omega\vec{e_3}$ be the angular velocity of a sphere. The point is slipping, so the equations of motion are

$$m\ddot{\vec{r}} = -mg\vec{e_3} - \mu|\lambda\vec{r}|\frac{\dot{\vec{r}} - \vec{\omega}\times\vec{r}}{|\dot{\vec{r}} - \vec{\omega}\times\vec{r}|} + \lambda\vec{r},$$

where λ is the Lagrange multiplier. These equations should be considered together with the constraint equation

$$\frac{1}{2}((\vec{r},\vec{r}) - \ell^2) = 0.$$

In this case the solution of the equations of motion is

$$r_{10} = 0, \quad r_{20} = 0, \quad r_{30} = \pm\ell, \quad \lambda_0 = \pm\frac{mg}{\ell}, \quad \dot{\vec{r}} = 0,$$

i.e., the equilibria are placed on the poles of the sphere.

Since these solutions do not specify the slippage of the point on the sphere, it is difficult to analyze the stability of these equilibria using the linearized equations of motion. However, these equilibria are also the equilibria in a relative coordinate system that is rotating with the sphere. In the case of a vertical axis of rotation the problem is spherically symmetric. That is why the investigation of the existence of solutions of this kind can be reduced to the one conducted in [3]. Other sets of relative equilibria are discussed in the next section of this paper.

4. SETS OF RELATIVE EQUILIBRIA

Let us now introduce a relative coordinate system $Oy_1y_2y_3$ with the axis Oy_3 coinciding with the rotation axis. The position of the point in this system will be specified by two spherical angles ξ and η. Let ξ be the angle between the axis Oy_3 and $\vec{OP}$ and let η be the angle between the axis Oy_1 and $\vec{OP'}$, where P' is a projection of the point P on the plane Oy_1y_2 (Fig. 4).

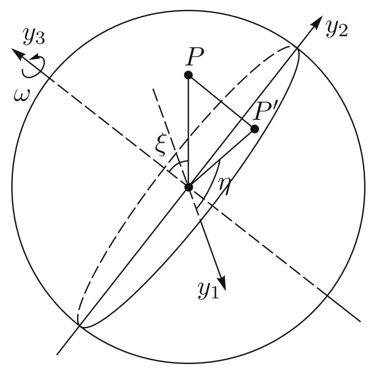

Fig. 4. Spherical angles ξ and η specifying the position of the point P.

Now introduce dimensionless time $t \mapsto \sqrt{\frac{l}{g}}t$ and dimensionless a parameter $\Omega^2 = \frac{l}{g}\omega^2$. The derivative with respect to the new time will be denoted by a stroke. Then the system of motion in the relative coordinate system can be rewritten as

$$\begin{cases} (-\sin^2\xi\eta'^2 - \xi'^2) = -(\sin\xi\cos\eta\sin\omega t\sin\alpha + \sin\xi\sin\eta\cos\omega t\sin\alpha \\ +\cos\xi\cos\alpha) + \tilde{N}_r + \Omega^2\sin^2\xi + 2\Omega\sin^2\xi\eta', \\ (\xi'' - \sin\xi\cos\xi\eta'^2) = -(\cos\xi\cos\eta\sin\omega t\sin\alpha + \cos\xi\sin\eta\cos\omega t\sin\alpha \\ -\sin\xi\cos\alpha) - \tilde{F}_\xi + \Omega^2\cos\xi\sin\xi + 2\Omega\sin\xi\cos\xi\eta', \\ (\sin\xi\eta'' + 2\xi'\eta'\cos\xi) = (\sin\omega t\sin\eta\sin\alpha - \cos\omega t\cos\eta\sin\alpha) - \tilde{F}_\eta \\ -2\Omega\cos\xi\xi', \end{cases} \tag{4.1}$$

where $\tilde{F}_\xi = F_\xi/mg$, $\tilde{F}_\eta = F_\eta/mg$ are dimensionless projections of the friction force on the coordinate axes $\vec{e_\xi}$ and $\vec{e_\eta}$, and $\tilde{N}_r = N_r/mg$ is a dimensionless normal reaction. By introducing an angle $\gamma = \eta - \dfrac{\pi}{2} + \omega t$ and omitting tildes the system becomes

$$\begin{cases} (-\sin^2\xi\gamma'^2 - \xi'^2) = -(\sin\xi\sin\alpha\cos\gamma + \cos\xi\cos\alpha) + N_r + \Omega^2\sin^2\xi \\ +2\Omega\sin^2\xi\gamma', \\ (\xi'' - \sin\xi\cos\xi\gamma'^2) = -(\cos\xi\sin\alpha\cos\gamma - \sin\xi\cos\alpha) - F_\xi \\ +\Omega^2\cos\xi\sin\xi + 2\Omega\sin\xi\cos\xi\gamma', \\ (\sin\xi\gamma'' + 2\xi'\gamma'\cos\xi) = \sin\alpha\sin\gamma - F_\eta - 2\Omega\cos\xi\xi'. \end{cases} \tag{4.2}$$

The equilibria can be found from these equations by supposing $\xi' = 0$, $\gamma' = 0$, $\xi'' = 0$, $\gamma'' = 0$. If the point is in a state of equilibrium, then

$$\sqrt{F_\xi^2 + F_\eta^2} \leqslant \mu N_r,$$

or

$$F_\xi^2 + F_\eta^2 \leqslant \mu^2 N_r^2, \tag{4.3}$$

where

$$F_\xi = -\cos\xi\sin\alpha\cos\gamma + \sin\xi\cos\alpha + \Omega^2\cos\xi\sin\xi, \quad F_\eta = \sin\alpha\sin\gamma,$$
$$N_r = \sin\xi\sin\alpha\cos\gamma - \cos\xi\cos\alpha - \Omega^2\sin^2\xi.$$

Using these expressions and inequality (4.3), one obtains

$$\begin{gathered}(-\cos\xi\sin\alpha\cos\gamma + \sin\xi\cos\alpha + \Omega^2\cos\xi\sin\xi)^2 + \sin^2\alpha\sin^2\gamma \\ \leqslant \mu^2(\sin\xi\sin\alpha\cos\gamma - \cos\xi\cos\alpha - \Omega^2\sin^2\xi)^2.\end{gathered} \tag{4.4}$$

Now introduce

$$P(x) = ax^2 + bx + c, \quad x = \cos\gamma,$$

where

$$\begin{gathered} a = -(\mu^2+1)\sin^2\xi\sin^2\alpha, \\ b = 2\sin\xi\sin\alpha(g\cos\xi\cos\alpha(1-\mu^2) - \Omega^2(\cos^2\xi - \mu^2\sin^2\xi)), \\ c = \cos^2\alpha(\sin^2\xi - \mu^2\cos^2\xi) + \sin^2\alpha + \Omega^4\sin^2\xi(\cos^2\xi - \mu^2\sin^2\xi) \\ -2\Omega^2\sin^2\xi\cos\xi\cos\alpha(1+\mu^2), \end{gathered}$$

and $a < 0$ if $\alpha \neq 0, \pi$. Then inequality (4.4) reads

$$P(x) \leqslant 0, \tag{4.5}$$

where $x \in [-1, 1]$.

The boundary of the area of relative equilibrium is the parabola $y = ax^2 + bx + c$. The coordinates of the vertex A of this parabola are

$$\begin{gathered} x_A = -\frac{b}{2a} = \frac{\cos\xi\cos\alpha(1-\mu^2) - \Omega^2(\cos^2\xi - \mu^2\sin^2\xi)}{(\mu^2+1)\sin\xi\sin\alpha}, \\ y_A = -\frac{b^2-4ac}{4a} = \frac{(\cos\xi\cos\alpha(1-\mu^2) - \Omega^2(\cos^2\xi - \mu^2\sin^2\xi))^2}{\mu^2+1} \\ +\cos^2\alpha(\sin^2\xi - \mu^2\cos^2\xi) + \sin^2\alpha + \Omega^4\sin^2\xi(\cos^2\xi - \mu^2\sin^2\xi) \\ -2\Omega^2\sin^2\xi\cos\xi\cos\alpha(1+\mu^2). \end{gathered}$$

If $y_A \leqslant 0$, then inequality (4.5) is fulfilled. If $y_A > 0$, then the inequality is fulfilled when: 1) the condition holds true for the right boundary $x = 1$, i.e., $P(1) = a + b + c \leqslant 0$ and $x_A \geqslant 1$ or 2) the condition holds true for the left boundary $x = -1$, i.e., $P(-1) = a - b + c \leqslant 0$ and $x_A \leqslant -1$.

Thus, bifurcation diagrams can be drawn as conjugations of these three sets. Figures 5–9 represent the bifurcation diagrams for different values of the inclination angle α and $\mu = 0.7$. The equilibrium sets can be obtained by a rotation through angle 2π about an axis that coincides with the rotation axis. When $\alpha = 0$, the diagram represents a half of a "fat fork" F and an equilibrium set in the form of a needle G near the axis $\xi = \pi$, which converges as $\omega \to \infty$ (Fig. 5, see also [3]). With increasing angle α the area G and the "bridge" between the "jags" of the fork are getting thinner (Fig. 6), then the "bridge" disappears (Fig. 7), and when $\alpha > \arctan \mu$, there is only the bigger jag left (Fig. 8), which straightens itself with further increase in α (Fig. 9). When $\omega \to \infty$, the bifurcation diagram for every α is a strip with the boundaries $\xi = \arctan(1/\mu)$ and $\xi = \pi - \arctan(1/\mu)$.

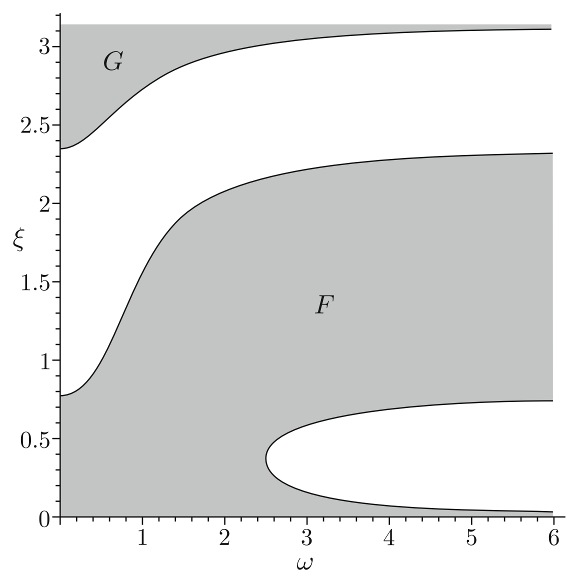

Fig. 5. Bifurcation diagram for $\alpha = 0$.

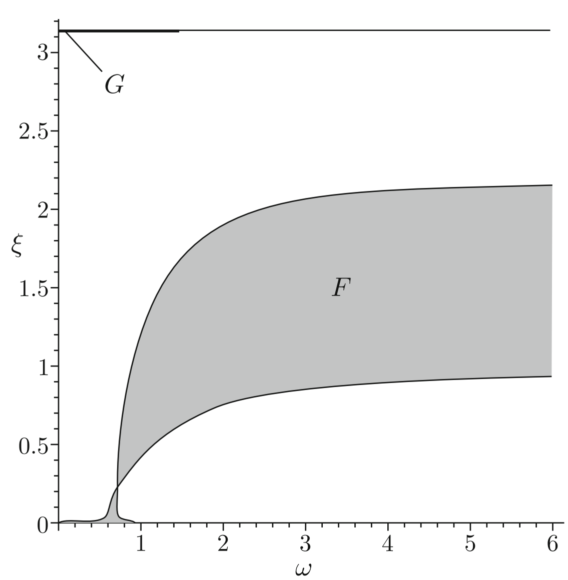

Fig. 6. Bifurcation diagram for $\alpha = 0.6077$.

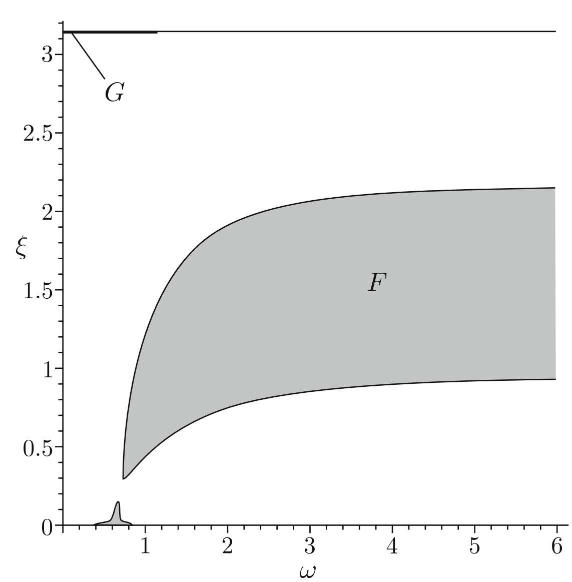

Fig. 7. Bifurcation diagram for $\alpha = 0.6087$.

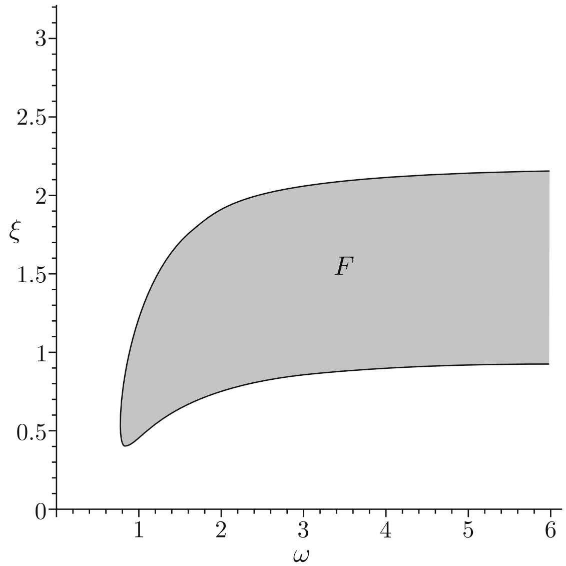

Fig. 8. Bifurcation diagram for $\alpha = \arctan(0.7) + 0.01$.

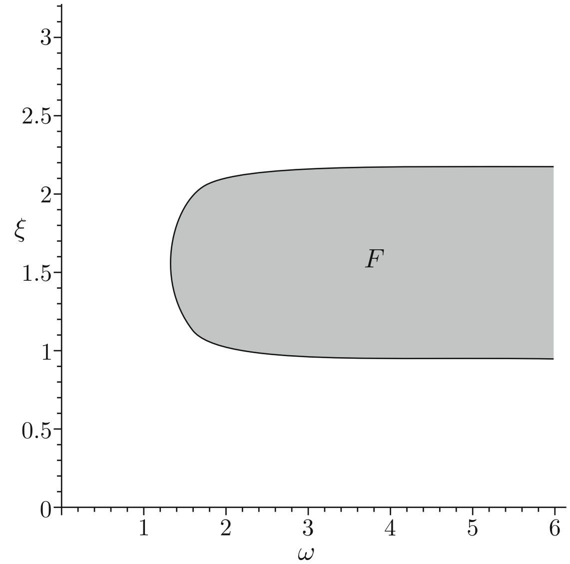

Fig. 9. Bifurcation diagram for $\alpha = \pi/2$.

Remark 3. In [19, 20] an approach to the question of the nature of dry friction, which is based on Painlevé's investigations [21], was developed. The question of using this approach to solving the problem of existence and stability of equilibria of systems with dry friction is of interest.

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