
PHYSICS OF ELEMENTARY PARTICLES
AND ATOMIC NUCLEI. THEORY

Gerstein–Greiner–Zeldovich Effect: Induced Charge Density and Vacuum Energy

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Abstract—The Gerstein–Greiner–Zeldovich effect—the spontaneous emission of vacuum positrons under conditions of Coulomb supercriticality—has been studied in detail based on the first principles of quantum electrodynamics within the framework of an essentially nonperturbative approach based on a special combination of analytical methods, computer algebra, and numerical calculations. Particular attention is paid to the vacuum energy \mathcal{E}_{VP} , considered a function of the parameters of the external Coulomb source: charge Z and radius R . The specific contribution to \mathcal{E}_{VP} arising due to the direct Coulomb interaction of the vacuum charge densities $\rho_{\text{VP}}(\vec{r})$ has been studied in detail. It is shown that, with correct renormalization, this contribution becomes negative after the first discrete levels descend into the lower continuum. Therefore, it is a purely quantum effect, not observed in classical electrodynamics. The problem of lepton number conservation during spontaneous emission is also discussed.

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1. INTRODUCTION

Currently, the behavior of the electron–positron vacuum in supercritical Coulomb fields is the subject of active research [1–12]. The interest in the problem is due to the fact that quantum electrodynamics (QED) predicts a nonperturbative restructuring of the vacuum state caused by diving of the discrete spectrum levels into the lower continuum, which should be accompanied by a number of nontrivial effects, including the Gerstein–Greiner–Zeldovich effect—the spontaneous emission of vacuum positrons and the appearance of corresponding vacuum shells (see [1, 13–16] and the literature cited there). In dimensionality $3 + 1$, similar effects should be observed in Coulomb fields generated by localized extended sources with a charge $Z > Z_{\text{cr},1} \approx 170$. Such sources can be obtained as part of heavy ion collision experiments at new accelerator complexes, such as FAIR (Darmstadt), NICA (Dubna), and HIAF (Lanzhou) [17–19].

The problem of spontaneous positron emission in supercritical Coulomb fields has a long history, starting with the pioneering works of V. Greiner and Ya. Zeldovich et al. in the early 1970s [20–25] (see also [1, 13, 16, 26] and literature cited there). However, in experiments carried out at the GSI heavy ion complex (Darmstadt) and subsequently repeated at the Argonne National Laboratory, no evidence of the cre-

ation of vacuum positrons was found [27]. The next generation of heavy ion complexes should take these studies to a new level [17–19], which makes it necessary to deepen the theoretical understanding of the spontaneous emission effect.

This paper examines the nonperturbative effects of electron–positron vacuum polarization in the field of a supercritical quasi-static Coulomb source with a charge $Z > Z_{\text{cr},1}$ and radius R . Particular attention is paid to the vacuum polarization energy \mathcal{E}_{VP} . A key role in the field of supercriticality is played by the vacuum energy \mathcal{E}_{VP} , especially in the effect of creation of vacuum positrons, since this process must occur solely due to vacuum polarization, without any additional energy transfer channels. In particular, it is the decrease in vacuum energy \mathcal{E}_{VP} that should provide spontaneously produced positrons with additional repulsive energy, necessary for their emission from the vicinity of the source. Considered as a function of the source charge Z , the vacuum energy \mathcal{E}_{VP} turns out to be a rapidly decreasing function with increasing Z , reaching large negative values [28]. This decrease is accompanied by negative jumps, the magnitude of which exactly coincides with the rest mass of the electron, occurring every time the next discrete level descends into the lower continuum. If we consider the vacuum energy as a function of the source radius R at a fixed Z , then its

behavior adequately describes all nonperturbative effects of vacuum polarization that arise during low-energy collisions of heavy ions. Due to the active interest in the problem under consideration [1, 4, 7, 12, 17, 19], these issues require separate study.

Until now, most attention in this issue has been paid to the polarization energy of the Dirac sea [13–15, 28]

$$\mathcal{E}_{D,VP} = \frac{1}{2} \left(\sum_{\epsilon_n < \epsilon_F} \epsilon_n - \sum_{\epsilon_n \geq \epsilon_F} \epsilon_n \right), \quad (1)$$

where $\epsilon_F = -m_e c^2$ is the Fermi level, which in problems with an external Coulomb source is chosen at the threshold of the lower continuum, and ϵ_n are the eigenvalues of the corresponding spectral Dirac–Coulomb (DC) problem. Essentially, $\mathcal{E}_{D,VP}$ is the Casimir energy of the electron–positron system in an external Coulomb field [6, 14].

However, the total vacuum energy of the system is not limited by this term. Namely, in the Coulomb gauge, which is the most natural for such problems, within the framework of QED, there appears an additional contribution to the total vacuum energy of the system, corresponding to the direct Coulomb interaction of vacuum charge densities $\rho_{VP}(\vec{r})$

$$\mathcal{E}_{C,VP} = \frac{1}{2} \int d\vec{r} d\vec{r}' \frac{\rho_{VP}(\vec{r}) \rho_{VP}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (2)$$

This contribution is a direct consequence of the corresponding operator expression that arises when quantizing the electromagnetic field in the Coulomb gauge (see, for example, [29], Chapter 15; an even more detailed discussion of this issue is given in the monograph [30]). Thus, the correct expression for the total vacuum energy of the system in the problem under consideration has the form

$$\mathcal{E}_{VP} = \mathcal{E}_{D,VP} + \mathcal{E}_{C,VP}. \quad (3)$$

This work aims at studying the total vacuum energy (3) of the system in the supercritical region for $Z > Z_{cr,1}$, paying special attention to the contribution of the Coulomb term $\mathcal{E}_{C,VP}$. Interest in this term is due to two factors. First, in the supercritical region $Z > Z_{cr,1}$, the change in the vacuum charge density $\rho_{VP}(\vec{r})$ occurs mainly due to the formation of vacuum shells, contributions from which change the total vacuum charge Q_{VP} . Without taking into account these contributions, weak effects of changes in the density of states of the lower continuum and the evolution of discrete levels remain, which cannot significantly change the overall picture of the processes occurring. Second, the behavior of the contribution from vacuum shells differs significantly from what would be expected from general concepts. The vacuum shell that appears after the diving of the next discrete level into the lower continuum becomes negatively charged only after the

decay of the resonance, accompanied by the positron emission [13–15].¹ Until this moment, the vacuum shell remains vacant and does not contribute to $\rho_{VP}(\vec{r})$. After renormalization, the contribution to the total vacuum energy due to the vacuum shells charged as a result of the emission of positrons turns out to be negative, which is a purely quantum effect not observed in classical electrodynamics. This result is similar to the Juhling effect, in which the spatial distribution of charge density in vacuum, induced by a point Coulomb source, contradicts the classical picture [13–15]. Although this contribution turns out to be significantly less than the main component of the vacuum energy $\mathcal{E}_{D,VP}$, the fact that the properly renormalized term $\mathcal{E}_{C,VP}^{ren}$ is negative plays an important role in the mechanism of spontaneous emission, especially for values of the charge Z slightly exceeding the first critical charge $Z_{cr,1}$, since it is this negative contribution that provides vacuum positrons with the additional kinetic energy necessary for their emission with a non-zero probability. It is to be noted that, although the nonrenormalized contribution of $\mathcal{E}_{C,VP}$ to the total vacuum energy also contains negative jumps that arise with each diving of a discrete level into the lower continuum, in general it turns out to be significantly positive and increases with increasing Z , which would only further complicate the conditions of the energy balance necessary for spontaneous emission.

To proceed further, without loss of generality, it is sufficient to consider the DC problem in an external spherically symmetric Coulomb field created by a uniformly charged sphere

$$V(r) = -Q \left(\frac{1}{R} \theta(R-r) + \frac{1}{r} \theta(r-R) \right), \quad (4)$$

or a ball

$$V(r) = -Q \left(\frac{3R^2 - r^2}{2R^3} \theta(R-r) + \frac{1}{r} \theta(r-R) \right). \quad (5)$$

Hereinafter,

$$Q = Z\alpha, \quad (6)$$

and the source radius R is restricted is limited below by the value

$$R_{\min}(Z) \simeq 1.2(2.5Z)^{1/3} \text{ fm}, \quad (7)$$

approximately corresponding to the size of a super-heavy nucleus with the charge Z ; from above, it is lim-

¹ At this stage, the problem of lepton number conservation is deliberately not discussed. It is assumed that, upon emission of a positron, the corresponding positive lepton number must remain in the system in the form of spatial density, localized in vacuum shells. If this is not the case, then either the conservation law of the lepton number in such processes is violated or the emission of positrons turns out to be strictly forbidden. However, there is currently no evidence that the lepton number can exist as a spatial density.

ited by the value R_{\max} on the order of the Compton wavelength of the electron, since at such distances the vacuum polarization effects become small corrections of the order of $O(1/R)$.

It should be especially noted that, in the vacuum polarization effects under consideration, the parameter Q plays the role of an effective coupling constant. In addition, for the models of a uniformly charged sphere and a ball (the latter is more suitable for the role of a superheavy nucleus or a cluster consisting of heavy ions), the difference in the considered effects of vacuum polarization turns out to be insignificant. It manifests itself mainly in relation to vacuum energies $\mathcal{E}_{\text{VP,ball}}/\mathcal{E}_{\text{VP,sphere}} \approx 6/5$ for the same values of Z and R , provided that the source radius R is sufficiently close to the value $R_{\min}(Z)$ [28]. Moreover, the model of a uniformly charged sphere makes it possible to carry out most of the calculations in analytical form, which is an obvious advantage. For the model of a uniformly charged ball, this is impossible, since in this case there are no exact analytical solutions to the DC problem, so it is necessary to use numerical methods or special approximations [28].

As in other works on the topic under consideration [1, 7, 10, 25, 27], the contribution of processes involving the exchange of virtual photons is omitted. Further, everywhere, if specified separately, the relativistic system of units $\hbar = m_e = c = 1$ and the standard representation of Dirac matrices are used. Specific calculations illustrating the general conclusions are carried out for the value $\alpha = 1/137.036$.

2. GENERAL PROBLEM STATEMENT

The most effective nonperturbative approach in calculating the vacuum charge density $\varrho_{\text{VP}}(\vec{r})$ is based on the Wichmann–Kroll (WK) method [31, 32, 36]. The starting point is the expression for the vacuum charge density

$$\begin{aligned} & \varrho_{\text{VP}}(\vec{r}) \\ &= -\frac{|e|}{2} \left(\sum_{\epsilon_n < \epsilon_F} \Psi_n(\vec{r})^\dagger \Psi_n(\vec{r}) - \sum_{\epsilon_n \geq \epsilon_F} \Psi_n(\vec{r})^\dagger \Psi_n(\vec{r}) \right). \end{aligned} \quad (8)$$

with $\epsilon_F = -1$ being the Fermi level, which in such problems with an external Coulomb source is chosen at the threshold of the lower continuum, and ϵ_n and $\Psi_n(\vec{r})$ are the eigen values and eigen functions of the corresponding spectral DC problem. Expression (8) for the vacuum charge density is a direct consequence of the Schwinger prescription for the fermionic operator of current

$$j_\mu(\vec{r}, t) = -|e|/ [2\bar{\Psi}(\vec{r}, t), \gamma_\mu \Psi(\vec{r}, t)]. \quad (9)$$

In the case under consideration, a very important fact is that $\varrho_{\text{VP}}(\vec{r})$ turns out to be not the average, but the eigenvalue of the fermionic charge density opera-

tor $j_0(\vec{r}, t)$ acting on the vacuum state. To do this, it is sufficient to use Farry's picture, i.e., expand fermion fields in a single-particle basis,

$$\Psi(\vec{r}, t) = \sum_{\epsilon_n > \epsilon_F} b_n \Psi_n(\vec{r}) e^{-i\epsilon_n t} + \sum_{\epsilon_n < \epsilon_F} d_n^\dagger \Psi_n(\vec{r}) e^{-i\epsilon_n t}, \quad (10)$$

with

$$\{b_n, b_{n'}^\dagger\} = \delta_{nn'}, \quad \{d_n, d_{n'}^\dagger\} = \delta_{nn'}. \quad (11)$$

All other commutators disappear, and the complete set $\{\Psi_n(\vec{r})\}$ is chosen according to the above definitions. Writing the charge density operator as the sum of its normally ordered form and the vacuum charge density determined by expression (8), we obtain

$$j_0(\vec{r}, t) =: j_0(\vec{r}, t) : + \varrho_{\text{VP}}(\vec{r}). \quad (12)$$

Since the following relations are satisfied in Farry's picture for the vacuum state $|\text{vac}\rangle$,

$$b_n |\text{vac}\rangle = d_n |\text{vac}\rangle = 0, \quad (13)$$

the action of operator (12) on it leads to

$$j_0(\vec{r}, t) |\text{vac}\rangle = \varrho_{\text{VP}}(\vec{r}) |\text{vac}\rangle. \quad (14)$$

Thus, $\varrho_{\text{VP}}(\vec{r})$ is indeed an eigenvalue of the charge density operator on the vacuum state. It is appropriate to compare this result with the properties of a classical wave packet, which is the average of the electromagnetic field operator over a coherent state with an indefinite number of photons generated by a classical external source. In this case, both the electric and magnetic components of the wave packet have nonzero dispersion created by fluctuations in the number of photons in each mode.

For models quadratic in fermion fields, such as QED, QCD, and other gauge theories, a similar statement is true for vacuum energy. In QED, this is a consequence of the fact that the current structure (9) requires the following form of the Dirac Hamiltonian in an external electromagnetic field [6, 14]

$$\begin{aligned} \mathcal{H}_D(\vec{r}, t) &= \frac{1}{4i} \{ [\Psi(\vec{r}, t)^\dagger, \vec{\alpha} \vec{\nabla} \Psi(\vec{r}, t)] \\ &+ [\Psi(\vec{r}, t), \vec{\nabla} \Psi(\vec{r}, t)^\dagger \vec{\alpha}] \} + \frac{1}{2} [\Psi(\vec{r}, t)^\dagger, \beta \Psi(\vec{r}, t)] \\ &+ j_\mu(\vec{r}, t) A^\mu(\vec{r}, t). \end{aligned} \quad (15)$$

Carrying out calculations completely analogous to those given above for the current, it is not difficult to arrive at the relation

$$\mathcal{H}_D(\vec{r}, t) |\text{vac}\rangle = h_{\text{D,VP}}(\vec{r}) |\text{vac}\rangle, \quad (16)$$

where

$$\begin{aligned} h_{\text{D,VP}}(\vec{r}) &= \frac{1}{2} \left(\sum_{\epsilon_n < \epsilon_F} \epsilon_n \Psi_n(\vec{r})^\dagger \Psi_n(\vec{r}) \right. \\ &\left. - \sum_{\epsilon_n \geq \epsilon_F} \epsilon_n \Psi_n(\vec{r})^\dagger \Psi_n(\vec{r}) \right). \end{aligned} \quad (17)$$

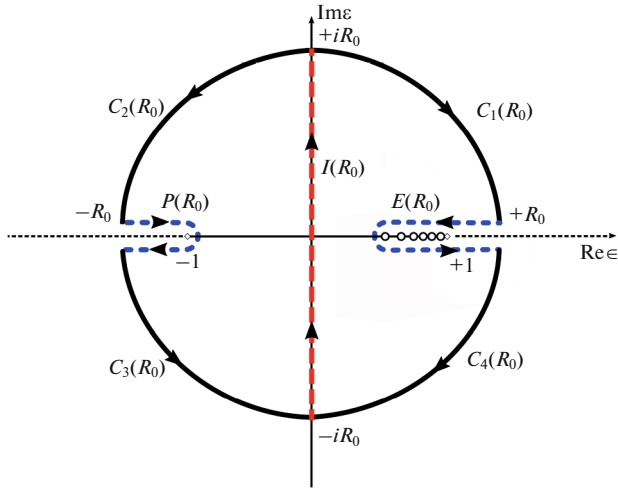


Fig. 1. Special contours on the first sheet of the Riemann energy surface, used to represent vacuum charge density (8) as contour integrals. The direction of bypassing of contours in accordance with (25).

However, within the framework of the approach we use, the density $h_{D,VP}(\vec{r})$ does not allow us to directly apply the WK method, because working with the density requires significantly more advanced calculation techniques.

Integrating relation (17) over the entire space leads to an expression for the main component of vacuum energy

$$\mathcal{E}_{D,VP} = \frac{1}{2} \left(\sum_{\epsilon_n < \epsilon_F} \epsilon_n - \sum_{\epsilon_n \geq \epsilon_F} \epsilon_n \right), \quad (18)$$

which is the eigen value of the Dirac Hamiltonian, acting on the vacuum state.

The second component of vacuum energy—the Coulomb term (2)—arises due to the structure of the electromagnetic field Hamiltonian

$$H_{EM} = \frac{1}{8\pi} \int d\vec{r} \left[\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2 \right], \quad (19)$$

where

$$\begin{aligned} \vec{\mathcal{E}}(\vec{r}, t) &= -\vec{\nabla} A^0(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t), \\ \vec{\mathcal{H}}(\vec{r}, t) &= \vec{\nabla} \times \vec{A}(\vec{r}, t). \end{aligned} \quad (20)$$

Writing the electric field as a sum of the longitudinal and transverse components

$$\vec{\mathcal{E}}(\vec{r}, t) = \vec{\mathcal{E}}_{\parallel}(\vec{r}, t) + \vec{\mathcal{E}}_{\perp}(\vec{r}, t), \quad (21)$$

where $\vec{\mathcal{E}}_{\parallel}(\vec{r}, t) = -\vec{\nabla} A^0(\vec{r}, t)$ and $\vec{\mathcal{E}}_{\perp}(\vec{r}, t) = -\partial \vec{A}(\vec{r}, t) / \partial t$ and integrating by parts with allowance for the gauge condition $\vec{\nabla} \cdot \vec{A} = 0$, one obtains the expression

$$H_{EM} = \frac{1}{8\pi} \int d\vec{r} \vec{\mathcal{E}}_{\parallel}^2 + \frac{1}{8\pi} \int d\vec{r} \left[\vec{\mathcal{E}}_{\perp}^2 + \vec{\mathcal{H}}^2 \right], \quad (22)$$

the first term of which is simply transformed into operator expression

$$\mathcal{E}_{Coul} = \frac{1}{2} \int d\vec{r} d\vec{r}' \frac{j_0(\vec{r}, t) j_0(\vec{r}', t)}{|\vec{r} - \vec{r}'|}. \quad (23)$$

Combining it with (14), we arrive at the final expression for the second component of the total vacuum energy (3)—Coulomb term (2).

The purpose of this work is to study the behavior of the total vacuum energy (3) of the system in the supercritical region for $Z > Z_{cr,1}$, paying special attention to the contribution of the Coulomb term $\mathcal{E}_{C,VP}$. To do this, first and foremost, we should consider in detail vacuum charge density (8) and the contribution to it from vacuum shells.

3. VACUUM CHARGE DENSITY IN LHC FORMALISM

The essence of the WK method is to use the vacuum density representation in the form of contour integrals in the complex plane over the energy variable from the trace of Green's function of the corresponding spectral problem of the WK. In the problem under consideration, Green's function satisfies the equation

$$[-i\alpha \vec{\nabla}_r + \beta + V(r) - \epsilon] G(\vec{r}, \vec{r}'; \epsilon) = \delta(\vec{r} - \vec{r}'). \quad (24)$$

The formal solution of Eq. (24) is written in the form

$$G(\vec{r}, \vec{r}'; \epsilon) = \sum_n \frac{\Psi_n(\vec{r}) \Psi_n(\vec{r}')^\dagger}{\epsilon_n - \epsilon}. \quad (25)$$

Following [31], the vacuum charge density can be expressed through integrals along the contours $P(R_0)$ and $E(R_0)$ on the first sheet of the Riemann energy surface (Fig. 1)

$$\begin{aligned} \varrho_{VP}(\vec{r}) &= -\frac{|e|}{2} \lim_{R_0 \rightarrow \infty} \left(\frac{1}{2\pi i} \int_{P(R_0)} d\epsilon \text{Tr} G(\vec{r}, \vec{r}'; \epsilon) \Big|_{\vec{r}' \rightarrow \vec{r}} \right. \\ &\quad \left. + \frac{1}{2\pi i} \int_{E(R_0)} d\epsilon \text{Tr} G(\vec{r}, \vec{r}'; \epsilon) \Big|_{\vec{r}' \rightarrow \vec{r}} \right). \end{aligned} \quad (26)$$

Note that the Green's function included in this expression must be properly regularized in order for the limit $\vec{r}' \rightarrow \vec{r}$ to exist and the integrals by $d\epsilon$ to converge. This procedure is discussed in detail below. At this stage, we will assume that all expressions contain only regularized Green's functions. The main consequence of this assumption is the uniform asymptotic behavior of integrands in (26) on arcs of a great circle for $|\epsilon| \rightarrow \infty$. This allows one to deform the contours $P(R_0)$ and $E(R_0)$, reducing them to a segment $I(R_0)$ of

the imaginary axis. After this we can pass to the limit $R_0 \rightarrow \infty$ and obtain the expression

$$\varrho_{\text{VP}}(\vec{r}) = |e| \left[\sum_{-1 \leq \epsilon_n < 0} |\Psi_n(\vec{r})|^2 + \frac{1}{2\pi} \times \int_{-\infty}^{\infty} dy \text{Tr} G(\vec{r}, \vec{r}'; iy) \Big|_{\vec{r}' \rightarrow \vec{r}} \right], \quad (27)$$

where $\{\Psi_n(\vec{r})\}$ are the normalized eigen functions of discrete levels $-1 \leq \epsilon_n < 0$, for which hereinafter the following notation is adopted:

$$\Psi_n(\vec{r})^\dagger \Psi_n(\vec{r}) \equiv |\Psi_n(\vec{r})|^2. \quad (28)$$

Representing Green's function (25) as a partial expansion in $k = \pm(j + 1/2)$ [31], [32],

$$\text{Tr} G(\vec{r}, \vec{r}'; \epsilon) \Big|_{\vec{r}' \rightarrow \vec{r}} = \sum_k \frac{|k|}{2\pi} \text{Tr} G_k(r, r'; \epsilon) \Big|_{r' \rightarrow r}, \quad (29)$$

where the radial Green function $G_k(r, r'; \epsilon)$ is defined as

$$\mathcal{H}_k(r) G_k(r, r'; \epsilon) = \delta(r - r') / rr' \quad (30)$$

and the radial Hamiltonian of DC problem has the form

$$\mathcal{H}_k(r) = \left(V(r) + 1 - \epsilon - \frac{1}{r} \frac{d}{dr} r - \frac{k}{r} \right), \quad (31)$$

we obtain the following expressions for the partial expansion terms $\varrho_{\text{VP}}(r)$:

$$\varrho_{\text{VP},k}(r) = \frac{|e||k|}{2\pi} \left[\sum_{-1 \leq \epsilon_{n,k} < 0} |\Psi_{n,k}(r)|^2 + \frac{1}{2\pi} \times \int_{-\infty}^{\infty} dy \text{Tr} G_k(r, r'; iy) \Big|_{r' \rightarrow r} \right], \quad (32)$$

where $\Psi_{n,k}(r)$ are the normalized radial wavefunctions with eigen values k and $\epsilon_{n,k}$ of the corresponding radial DC problem. Using the symmetry properties of $G_k(r, r'; \epsilon)$, reported in [32] and expression (32), we obtain the sum $\varrho_{\text{VP},|k|}(r)$ of two partial vacuum charge densities with opposite signs of k :

$$\varrho_{\text{VP},|k|}(r) = \frac{|e||k|}{2\pi} \left\{ \sum_{k=\pm|k|} \sum_{-1 \leq \epsilon_{n,k} < 0} |\Psi_{n,k}(r)|^2 + \frac{2}{\pi_0} \int dy \text{Re} [\text{Tr} G_k(r, r'; iy)] \Big|_{r' \rightarrow r} \right\}, \quad (33)$$

which, by construction, is a known real quantity and odd in Z (in full accordance with Farry's theorem).

4. VACUUM DENSITY RENORMALIZATION: MOTIVATION AND MAIN CONSEQUENCES

The general result found in [32] through the expansion of $\varrho_{\text{VP}}(r)$ in powers of the parameter $Q = Z\alpha$ (but for a fixed value of the source radius R)

$$\varrho_{\text{VP}}(\vec{r}) = \sum_{n=\text{odd}} Q^n \varrho_{\text{VP}}^{(n)}(\vec{r}), \quad (34)$$

is that all divergences of $\varrho_{\text{VP}}(\vec{r})$ are contained only in the fermionic loop with two outer ends, and all subsequent orders of expansion in Q are already free from divergences (see also [36] and the cited literature). This statement is always true in dimensions $1 + 1$ and $2 + 1$ and, in the three-dimensional case, for a spherically symmetric external potential. It was verified by direct calculations within the framework of the non-perturbative approach for $1 + 1$ in [5, 6] and for $1 + 2$ in [37–40]. This approach can be transferred with minimal changes to the $3 + 1$ dimension, since the structure of the partial Green's functions in 2D and 3D cases is the same, with the exception of the additional factor $1/r$, and the replacement $m_j \rightarrow k$. In this case, the spherical symmetry of the Coulomb potential plays a key role, since the above statement can be proven using a partial expansion $\varrho_{\text{VP}}(r)$ for each term $\varrho_{\text{VP},|k|}(r)$, but not immediately for the entire series as a whole.

This circumstance was already discussed earlier in [32], where it was shown that the main reason for this difference is the difference in the properties of individual partial radial Green's functions G_k and the entire series as a whole. First, there is a limit of $G_k(r, r'; \epsilon)$ for $r' \rightarrow r$. At the same time, the complete Green's function $G(\vec{r}, \vec{r}'; \epsilon)$ diverges for $\vec{r}' \rightarrow \vec{r}$. In this case, for the vacuum charge density $\varrho_{\text{VP},|k|}^{(1)}(r)$, corresponding to the first order of perturbation theory, or linear in Q , the permutation of operation of taking the limit $r' \rightarrow r$ and contour integration in (33) leads to different final results.

On the other hand, attempts to directly calculate the vacuum density $\varrho_{\text{VP}}^{(3)}(\vec{r})$ by means (27) lead to uncertainty arising when permutating the operation of taking the limit $\vec{r}' \rightarrow \vec{r}$ and integrating along the imaginary axis. At the same time, when calculating the partial terms $\varrho_{\text{VP},|k|}^{(3)}(r)$, no such problem occurs. Thus, for regular spherically symmetric external potentials, the renormalization of $\varrho_{\text{VP}}^{(3)}(r)$ is carried out by calculating it in the form of a sum of partial contributions of $\varrho_{\text{VP},|k|}^{(3)}(r)$ without any additional manipulations.

As a result, the procedure for renormalizing vacuum density (8) is actually the same for all three spatial dimensions and is carried out as follows. First, it is necessary to separate in expression (27) the terms linear in the external field and replace them with the renormalized

perturbative density $\varrho_{\text{VP}}^{\text{PT}}(r)$ (corresponding to the first order of diagrammatic perturbation theory (PT)), calculated for the same value of the source radius R . For this purpose, a partial component $\varrho_{\text{VP},|k|}^{(3+)}(r)$ of vacuum density is introduced, defined as follows

$$\varrho_{\text{VP},|k|}^{(3+)}(r) = \frac{|e||k|}{2\pi} \left\{ \sum_{k=\pm|k|-1 \leq \epsilon_{n,k} < 0} \sum_{n,k} |\Psi_{n,k}(r)|^2 + \frac{2}{\pi} \int_0^\infty dy \text{Re}[\text{Tr}G_k(r, r; iy) - \text{Tr}G_k^{(1)}(r; iy)] \right\}, \quad (35)$$

where $G_k^{(1)}(r; iy)$ is the linear in Q component of the partial Green's function $G_k(r, r; iy)$, coinciding with the first term of the Born series

$$G_k^{(1)} = G_k^{(0)}(-V)G_k^{(0)}, \quad (36)$$

where $G_k^{(0)}$ is the free Green's function of the radial Dirac equation with the same k and ϵ . Components $\varrho_{\text{VP},|k|}^{(3+)}(r)$ by construction contain only odd powers of Q , starting from $n = 3$, and therefore are free from divergences. At the same time, just they are responsible for all the nonlinear effects that arise when the discrete levels descend into the lower continuum.

The Born term (36), being necessary to separate the nonlinear component from the total vacuum density, for the external potential $V(r)$ of the Coulomb type, has the form

$$\begin{aligned} \text{Tr}G_k^{(1)}(r; iy) = & -\frac{1}{r} \left[(1-y^2) \left(K_{k-1/2}^2(\gamma r) \int_0^r dr' r' V(r') I_{k-1/2}^2(\gamma r') + K_{k+1/2}^2(\gamma r) \right. \right. \\ & \times \left. \int_0^r dr' r' V(r') I_{k+1/2}^2(\gamma r') + I_{k-1/2}^2(\gamma r) \int_r^\infty dr' r' V(r') K_{k-1/2}^2(\gamma r') + I_{k+1/2}^2(\gamma r) \right. \\ & \times \left. \left. \int_r^\infty dr' r' V(r') K_{k+1/2}^2(\gamma r') \right) + (1+y^2) \left(K_{k-1/2}^2(\gamma r) \int_0^r dr' r' V(r') I_{k+1/2}^2(\gamma r') \right. \right. \\ & \left. \left. + K_{k+1/2}^2(\gamma r) \int_0^r dr' r' V(r') I_{k-1/2}^2(\gamma r') + I_{k-1/2}^2(\gamma r) \int_r^\infty dr' r' V(r') K_{k+1/2}^2(\gamma r') + I_{k+1/2}^2(\gamma r) \int_r^\infty dr' r' V(r') K_{k-1/2}^2(\gamma r') \right) \right], \end{aligned} \quad (37)$$

where $I_\nu(z)$ and $K_\nu(z)$ are the Infeld's and MacDonald's functions, respectively, and

$$\gamma = \sqrt{1+y^2}. \quad (38)$$

For both external potentials (4), (5) under consideration, all integrals included in expression (37) are calculated in analytical form, but the corresponding expressions for the Born term turn out to be quite cumbersome. An explicit expression in the case of potential (4) for s -channel, which will be needed later, has the form given below. First, we introduce a number of auxiliary functions defined by the relations

$$\begin{aligned} a_1(y, r) &= \frac{\sinh[2\gamma r] - 2\gamma r}{2\pi\gamma^2}, \quad a_2(y, r) = \frac{2 + 2(\gamma r)^2 - 2 \cosh[2\gamma r] + \gamma r \sinh[2\gamma r]}{2\pi r \gamma^3}, \\ c_1(y, r) &= \frac{\text{Chi}[2\gamma r] - \text{Chi}[2\gamma R] + \ln[R/r]}{\pi\gamma}, \\ c_2(y, r) &= \frac{1}{2\pi\gamma^3} [1/r^2 - 1/R^2 + \cosh[2\gamma R]/R^2 - \cosh[2\gamma r]/r^2] \\ &+ 2\gamma^2 (\text{Chi}[2\gamma R] - \text{Chi}[2\gamma r] - \ln[\gamma R] + \ln[\gamma r]) + 2\gamma (\sinh[2\gamma r]/r - \sinh[2\gamma R]/R), \\ A_1(y, r) &= \theta(R-r)a_1(y, r)/R + (a_1(y, R)/R + c_1(y, r))\theta(r-R), \\ A_2(y, r) &= \theta(R-r)a_2(y, r)/R + (a_2(y, R)/R + c_2(y, r))\theta(r-R), \\ b_1(y, r) &= \pi[e^{-2\gamma r} - e^{-2\gamma R}]/4\gamma^2, \quad b_2(y, r) = \frac{Re^{2\gamma R}(2+\gamma r) - \pi re^{-2\gamma R}(2+\gamma R)}{4Rr\gamma^3}, \\ d_1(y, r) &= \pi\Gamma[0, 2\gamma r]/2\gamma, \quad d_2(y, r) = \frac{\pi e^{-2\gamma r}(1+2\gamma r)/2r^2 + \gamma^2 Ei[-2\gamma r]}{2\gamma^3}, \\ B_1(y, r) &= \theta(R-r)(b_1(y, r)/R + d_1(y, r)) + d_1(y, r)\theta(r-R), \\ B_2(y, r) &= \theta(R-r)(b_2(y, r)/R + d_2(y, r)) + d_2(y, r)\theta(r-R). \end{aligned} \quad (39)$$

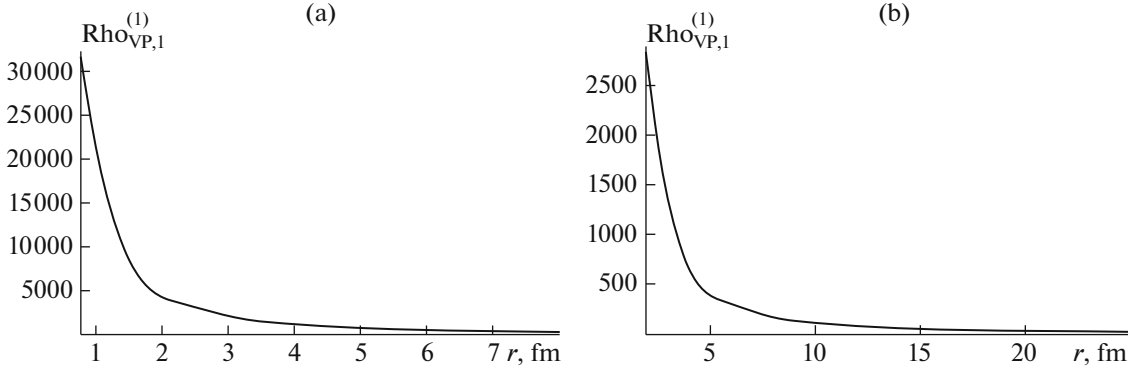


Fig. 2. Function $\varrho_{\text{VP},1}^{(1)}(Z,r)$ for (a) $Z = 200$ and $R = 0.1$; (b) $Z = 280$ and $R = 0.25$.

As a result,

$$\begin{aligned} \text{Tr}G_1^{(1)}(r; iy) = & \frac{Q}{r}[(1-y^2)(K_{1/2}^2(\gamma r)A_1(y,r) \\ & + K_{3/2}^2(\gamma r)A_2(y,r) + I_{1/2}^2(\gamma r)B_1(y,r) \\ & + I_{3/2}^2(\gamma r)B_2(y,r)) + (1+y^2)(K_{1/2}^2(\gamma r)A_2(y,r) \\ & + K_{3/2}^2(\gamma r)A_1(y,r) + I_{1/2}^2(\gamma r)B_2(y,r) \\ & + I_{3/2}^2(\gamma r)B_1(y,r))]. \end{aligned} \quad (40)$$

In a completely similar way, one can perform integration over dr' in the Born term (37) for any value $k > 1$ and obtain the corresponding explicit expression; however, as k increases, these calculations become increasingly cumbersome. In this work, the analysis is limited to the s -channel, which makes a decisive contribution for the considered range of source parameters. Although explicit expressions (39)–(40) include the integral exponential and hyperbolic cosine, taking into account the currently available computational capabilities, their numerical integration over the variable dy , required to allocate the nonlinear component $\varrho_{\text{VP},|k|}^{(3+)}(r)$ from the total partial vacuum density, does not meet with particular difficulties.

The behavior of $\text{Tr}G_k^{(1)}(r; iy)$ for various asymptotic regimes for both its arguments is discussed in detail in [41]. From these results, the component $\varrho_{\text{VP},|k|}^{(1)}(r)$ of vacuum density linear in Q is given by

$$\varrho_{\text{VP},|k|}^{(1)}(r) = \frac{|e||k|}{\pi^2} \int_0^\infty dy \text{Tr}G_k^{(1)}(r; iy) \quad (41)$$

and

$$\varrho_{\text{VP},|k|}^{(1)}(r) \rightarrow C_1(k)/r, \quad r \rightarrow 0, \quad (42)$$

$$\varrho_{\text{VP},|k|}^{(1)}(r) \rightarrow C_2(k)/r^3, \quad r \rightarrow \infty, \quad (43)$$

where C_1, C_2 are certain positive functions of the quantum number k of angular momentum. It is to be noted that $\varrho_{\text{VP},|k|}^{(1)}(r)$ does not contain the contribution of dis-

crete levels, since the latter is essentially nonlinear and manifests itself only in components $\varrho_{\text{VP},|k|}^{(3+)}(r)$. A direct consequence of asymptotics (43), in particular, is the fact that linear in Q component of the nonrenormalized integral vacuum charge

$$Q_{\text{VP}}^{(1)} = \int d\vec{r} \varrho_{\text{VP}}^{(1)}(\vec{r}) = 4\pi \int_0^\infty dr r^2 \varrho_{\text{VP}}^{(1)}(r) \quad (44)$$

diverges.

The next fact that should be noted is the significant differences in behavior of the component of $\varrho_{\text{VP},|k|}^{(1)}(r)$ and the renormalized perturbative density $\varrho_{\text{VP}}^{\text{PT}}(r)$ as functions of r . In particular, the calculation of $\varrho_{\text{VP},1}^{(1)}(r)$ using formulas (39), (40) shows that this function is positive for all r , while the perturbative density has a positive peak in the vicinity of the Coulomb source and is negative outside it (see, for example, [15]). The behavior of $\varrho_{\text{VP},1}^{(1)}(r)$ in the s -channel is shown in Fig. 2 for two values of source charge $Z = 200, 280$. The corresponding radii of the Coulomb source are chosen large enough so that the details of the nonlinear behavior of the charge density in the intermediate range (in which the transition occurs from asymptotics of the form $O(1/r)$ at $r \rightarrow 0$ to asymptotics $O(1/r^3)$ at $r \rightarrow \infty$) are clearly distinguishable.

As a result, the correctly renormalized vacuum charge density $\varrho_{\text{VP}}^{\text{ren}}(r)$ has the form²

$$\varrho_{\text{VP}}^{\text{ren}}(r) = \varrho_{\text{VP}}^{\text{PT}}(r) + \sum_{k=1}^\infty \varrho_{\text{VP},|k|}^{(3+)}(r). \quad (45)$$

² For a point source, the convergence of the partial expansion in (45) was proven in the original work by Wichmann and Kroll [31]. The effect of finite source size on this convergence is discussed in detail in [32]. For the problem under consideration, it is a consequence of the convergence of the partial expansion for $\mathcal{C}_{\text{VP}}^{\text{ren}}$, which was shown in [28].

Perturbative vacuum density $\varrho_{\text{VP}}^{\text{PT}}(r)$ is obtained from the first order perturbation theory according to the commonly accepted scheme [15, 42, 44]

$$\varrho_{\text{VP}}^{\text{PT}}(\vec{r}) = -\frac{1}{4\pi} \Delta A_{\text{VP},0}^{\text{PT}}(\vec{r}), \quad (46)$$

where

$$A_{\text{VP},0}^{\text{PT}}(\vec{r}) = \frac{1}{(2\pi)^3} \int d\vec{q} e^{i\vec{q}\vec{r}} \Pi_R(-\vec{q}^2) \tilde{A}_0(\vec{q}), \quad (47)$$

$$\tilde{A}_0(\vec{q}) = \int d\vec{r}' e^{-i\vec{q}\vec{r}'} A_0^{\text{ext}}(\vec{r}').$$

Polarization function $\Pi_R(q^2)$ in (47) is defined by the relation $\Pi_R^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_R(q^2)$, being dimensionless. In the considered case, $q^0 = 0$, and and the explicit expression for $\Pi_R(-\vec{q}^2)$ takes the form

$$\Pi_R(-\vec{q}^2) = 2\alpha/\pi \int_0^1 d\beta \beta(1-\beta) \quad (48)$$

$$\times \ln[1 + \beta(1-\beta)\vec{q}^2/m^2 - i\epsilon] = \alpha/\pi S(|\vec{q}|/m),$$

where

$$S(x) = -5/9 + 4/3x^2 + (x^2 - 2)\sqrt{x^2 + 4} \quad (49)$$

$$\times \ln[(\sqrt{x^2 + 4} + x)/(\sqrt{x^2 + 4} - x)]/3x^3,$$

with the following IR asymptotics

$$S(x) \rightarrow x^2/15 - x^4/140 + O(x^6), \quad x \rightarrow 0. \quad (50)$$

A direct consequence of renormalization procedure (45) is that it guarantees zeroing the total vacuum charge

$$Q_{\text{VP}}^{\text{ren}} = \int d\vec{r} \varrho_{\text{VP}}^{\text{ren}}(\vec{r}) \quad (51)$$

in the subcritical range $Z < Z_{\text{cr},1}$. In fact, this confirms the assumption that, with an external field uniformly decreasing at spatial infinity, without special boundary conditions or nontrivial topology of the field manifold in the subcritical range $Z < Z_{\text{cr},1}$, the correctly renormalized integral vacuum charge must be zero, and vacuum polarization effects can only distort its spatial distribution. It should be noted, however, that this is not a theorem, but only a plausible statement, which in each specific case must be double-checked by direct calculation.

For the problem under consideration, the validity of this statement can be shown as follows. First and foremost, it should be noted that

$$Q_{\text{VP}}^{\text{PT}} \equiv 0. \quad (52)$$

This relation is a direct consequence of the general QED renormalization condition for the polarization function $\Pi_R(q^2) \sim q^2$ for $q \rightarrow 0$ (in the case under consideration, this directly follows from asymptotics (50)). To prove (52), consider the static equation for the

potential $A_0(\vec{q})$ created by the external charge density $\varrho_{\text{ext}}(\vec{q})$ in momentum space (up to factors of type 2π and common sign)

$$(\vec{q}^2 - \tilde{\Pi}_R(-\vec{q}^2))A_0(\vec{q}) = \varrho_{\text{ext}}(\vec{q}), \quad (53)$$

where $\tilde{\Pi}_R(q^2)$ is the polarization operator in the normal form with dimensionality $[q^2]$, introduced by the relation $\Pi_R^{\mu\nu}(q) = (g^{\mu\nu} - q^\mu q^\nu/q^2)\tilde{\Pi}_R(q^2)$. Within the framework of perturbation theory, we should assume that $\tilde{\Pi}_R(q) \ll q^2$; then potential $A_0(\vec{q})$ can be written in the form of the expansion

$$A_0(\vec{q}) = A^{(0)}(\vec{q}) + A^{(1)}(\vec{q}) + \dots \quad (54)$$

The classical part of potential $A^{(0)}(\vec{q})$ satisfies the equation

$$\vec{q}^2 A^{(0)}(\vec{q}) = \varrho_{\text{ext}}(\vec{q}), \quad (55)$$

and defines the external potential $A_0^{\text{ext}}(\vec{r})$, while the first quantum correction, correspondingly, satisfies the equation

$$\vec{q}^2 A^{(1)}(\vec{q}) = \tilde{\Pi}_R(-\vec{q}^2)A^{(0)}(\vec{q}). \quad (56)$$

The right-hand side of (56), up to a factor $1/4\pi$, represents the perturbative vacuum density $\varrho_{\text{VP}}^{\text{PT}}(\vec{q})$. Passing to the coordinate representation, we obtain (up to factors of type 2π)

$$\varrho_{\text{VP}}^{\text{PT}}(\vec{r}) = \int d\vec{q} e^{i\vec{q}\vec{r}} \tilde{\Pi}_R(-\vec{q}^2)A^{(0)}(\vec{q}) \quad (57)$$

$$= \int d\vec{r}' \tilde{\Pi}_R(\vec{r} - \vec{r}')A_0^{\text{ext}}(\vec{r}').$$

Integrating (57) over the entire space leads to the following expression for the integral perturbative vacuum charge $Q_{\text{VP}}^{\text{PT}}$

$$Q_{\text{VP}}^{\text{PT}} = \int d\vec{q} \delta(\vec{q}) \tilde{\Pi}_R(-\vec{q}^2)A^{(0)}(\vec{q}). \quad (58)$$

Taking into account the renormalization condition for $\tilde{\Pi}_R(q^2)$, from (58) it follows that $Q_{\text{VP}}^{\text{PT}} = 0$ in those cases when the external potential $A_0^{\text{ext}}(\vec{q})$ in momentum space for $|\vec{q}| \rightarrow 0$ has a singularity no stronger than $O(1/|\vec{q}|^4)$ (in the 3D case) or $O(1/|\vec{q}|^3)$ (in the 2 + 1 case). For the considered Coulomb-type potentials in dimensions 2 + 1 and 3 + 1, the $A_0^{\text{ext}}(\vec{q})$ behaves as $O(1/|\vec{q}|)$ and $O(1/\vec{q}^2)$, respectively. In dimension 1 + 1 for potentials similar to (4), $A_0^{\text{ext}}(\vec{q})$ has only a logarithmic singularity. Thus, for potentials of the form (4), (5), the identity $Q_{\text{VP}}^{\text{PT}} \equiv 0$ holds in all the cases.

However, beyond the framework of the first order perturbation theory and for the entire subcritical region, when, due to the presence of negative discrete

levels, the dependence $\varrho_{\text{VP}}(\vec{r})$ on the external field can no longer be described by a perturbative expansion series similar to (54), the question of whether $Q_{\text{VP}}^{\text{ren}}$ disappears requires significantly more detailed analysis.

Direct verification shows that the contribution from $\varrho_{\text{VP}}^{(3+)}$ to $Q_{\text{VP}}^{\text{ren}}$ for $Z < Z_{\text{cr},1}$ is also zero. In dimension 1+1, this statement can be proven analytically, but in dimension 2 + 1, due to the complexity of the expressions included in $\rho_{\text{VP},|m_j|}^{(3+)}(r)$, it is no longer possible to carry out such a check in a purely analytical form. However, it can be performed quite reliably using a combination of analytical and numerical methods (see [37], Appendix B). This approach can be transferred to dimension 3 + 1 with minimal changes, since the structure of the partial Green's functions in two- and three-dimensional cases is the same, with the exception of the additional factor $1/r$ and the replacement $m_j \rightarrow k$. Moreover, it is sufficient to verify the disappearance of the total vacuum charge $Q_{\text{VP}}^{\text{ren}}$ not in the entire subcritical region, but only in the absence of negative discrete levels. If they are present, the disappearance of the total vacuum charge at $Z < Z_{\text{cr},1}$ follows from arguments independent of the model, which are based on the original expression for vacuum density (8). From (8) it follows that a change in the integral charge $Q_{\text{VP}}^{\text{ren}}$ is possible only at $Z > Z_{\text{cr},1}$, when discrete levels reach the lower continuum. One of the possible ways to prove this statement is based on a detailed analysis of the behavior of the integral along the imaginary axis included in the original expression (32) for $\varrho_{\text{VP},k}(r)$, namely,

$$I_k(r) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d \int \text{Tr} G_k(r, r; \epsilon), \quad (59)$$

with such an infinitesimal variation of the parameters of the external source, when the initially positive discrete level $\epsilon_{n,k}$, being located infinitely close to the origin of coordinates, becomes negative. Then the corresponding pole of the Green's function undergoes an infinitesimal displacement along the real axis and also intersects the ordinate axis, which corresponds to a change in $I_k(r)$ equal to the residue value at the point $\epsilon = 0$

$$\Delta I_k(r) = -|\psi_{n,k}(r)|^2 \Big|_{\epsilon_{n,k}=0}. \quad (60)$$

For details of the calculation, see Fig. 3, taking into account definition (25) of the Green's function.

Thus, taking into account the multiplicity of degeneracy of discrete levels in the spherically symmetric case, the contribution $I_k(r)$ to the total vacuum charge decreases by $2|e||k|$ with each appearance of a negative discrete level $\epsilon_{n,k}$. While this negative level

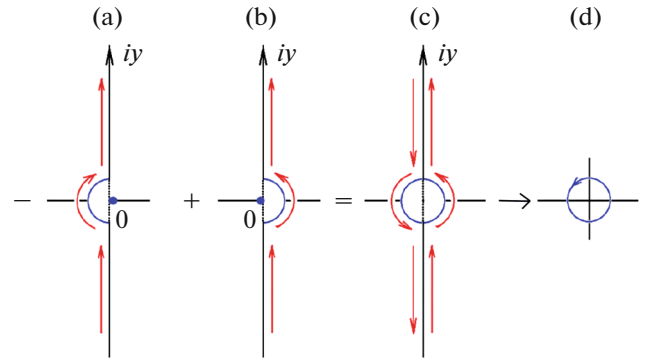


Fig. 3. Demonstration of the behavior of integral (59) taken along the imaginary axis when the pole of the Green's function lying on the real axis passes through the origin.

exists, the vacuum charge jump is compensated by the corresponding term included in the sum over all negative discrete levels in expression (32). The latter, in addition, ensures the continuity of the vacuum charge density when level $\epsilon_{n,k}$ crosses the zero point $\epsilon = 0$. However, once this level reaches the lower continuum, the total vacuum charge will change by exactly the amount $(-2|e||k|)$.

It should be specifically noted that this effect is essentially nonperturbative and is entirely contained in $\varrho_{\text{VP}}^{(3+)}$, while $\varrho_{\text{VP}}^{\text{PT}}$ does not participate in any way and still makes a purely zero contribution to the total charge. Thus, in the problem under consideration, the induced vacuum charge after renormalization turns out to be nonzero only at $Z > Z_{\text{cr},1}$, due to nonperturbative effects of vacuum polarization caused by the diving of discrete levels into the lower continuum. This is in full agreement with [1, 13–15, 36].

As a result, the behavior of the vacuum density renormalized using expression (45) in the nonperturbative region turns out to be exactly what should be expected from general ideas about the structure of the electron–positron vacuum at $Z > Z_{\text{cr}}$. Moreover, it becomes the reference point for the entire renormalization procedure, and, along with the principle of minimal subtraction, allows one to eliminate the inevitable ambiguity in identifying the divergent part of the original expression for $\varrho_{\text{VP}}(\vec{r})$. This fact will be used later in the renormalization of vacuum energy.

A more detailed picture of changes in the value of $\varrho_{\text{VP}}^{\text{ren}}(r)$ in the supercritical region $Z > Z_{\text{cr},1}$ is quite similar to the picture considered in [1, 13–15] based on the Fano formalism for autoionization processes in atomic physics [15], [45]. The main result is that, when the level $\psi_v(\vec{r})$ crosses the boundary of the lower continuum, the change in vacuum density has the form

$$\Delta \varrho_{\text{VP}}^{\text{ren}}(\vec{r}) = -|e| \times |\psi_v(\vec{r})|^2. \quad (61)$$

It should be noted here that the original approach [45] directly considers the change in the density of states $n(\vec{r})$. Due to this, the change in the induced charge density (61) is simply a consequence of the relation $\varrho(\vec{r}) = -|e|n(\vec{r})$. Such a jump in the vacuum charge density occurs for each discrete level that descends into the lower continuum, having a unique set of quantum numbers $\{v\}$. In particular, in the problem under consideration, each $2|k|$ -fold degenerate discrete level, when diving into the lower continuum, causes a jump in the total vacuum charge by amount $(-2|e||k|)$. Other quantities, including the lepton number, should behave similarly. However, it should be taken into account that the Fano formalism uses a number of approximations. In reality, formula (61) is accurate only in the immediate vicinity of the corresponding Z_{cr} . This is clearly demonstrated by specific examples for $1 + 1$ and $2 + 1$ systems in [5, 37, 39, 46] and below for the $3 + 1$ case.

As in [28], we present the results for $\varrho_{VP}^{ren}(r)$ in the model of a charged sphere (4) in the range $100 < Z < 300$. Moreover, the numerical value of the coefficient in relation (7) is chosen to be ³

$$R_{min}(Z) = 1.228935(2.5Z)^{1/3} \text{ fm.} \quad (62)$$

For potential (4), the corresponding partial Green's functions can be written explicitly in terms of Bessel functions and degenerate hypergeometric functions, which significantly simplifies the calculation of integrals along the imaginary axis in expression (33). The corresponding explicit analytical formulas for $\text{Tr}G_k(r, r'; iy)$ are given in [32].

In the range $100 < Z < 300$, four discrete levels from the s -channel descend into the lower continuum: $1s_{1/2}$ for $Z = 173.61$, $2p_{1/2}$ for $Z = 188.55$, $2s_{1/2}$ for $Z = 244.256$, and $3p_{1/2}$ for $Z = 270.494$. The corresponding values of the critical charges for both parities (\pm) are found from the equation

$$\begin{aligned} & 2\sqrt{Q^2 - 2QR}K_{2i\eta_k}(\sqrt{8QR})J_{\pm} \\ & \mp [\sqrt{2QR}(K_{1+2i\eta_k}(\sqrt{8QR}) + K_{1-2i\eta_k}(\sqrt{8QR})) \\ & \pm 2kK_{2i\eta_k}(\sqrt{8QR})]J_{\mp} = 0, \end{aligned} \quad (63)$$

in which $K_\nu(z)$ is the McDonald function and

$$\eta_k = \sqrt{Q^2 - k^2}, \quad J_{\pm} = J_{k\pm 1/2}(\sqrt{Q^2 - 2QR}). \quad (64)$$

Figures 4 and 5 demonstrate the behavior of the nonperturbative component of the vacuum charge density $\varrho_{VP,1}^{(3+)}(r)$, in particular, the jumps in the vacuum

³ At such a choice, the lower level $1s_{1/2}$ for the model of a charged ball at $Z = 170$ has the energy $\epsilon_{1s} = -0.99999$. In addition, the chosen value of the coefficient is quite close to the value 1.23, which is usually used in models of heavy nuclei.

density that occur with each discrete level diving into the lower continuum. Direct calculations confirm that the corresponding jumps in the total vacuum charge are exactly equal to $(-2|e||k|)$. Moreover, the spatial distribution of vacuum density jumps exactly coincides with the profile $|\psi_v(r)|^2$ of discrete levels at the boundary of the lower continuum, multiplied by the factor $(-2|e||k|)$. Figure 6 shows the plots of radial functions $|\psi_v(r)|^2$ normalized to unity for one of two possible spin projections.

Thus, the most correct way to calculate $\varrho_{VP}^{ren}(\vec{r})$ for all domains of Z is to use the original renormalized expression (45) with subsequent verification of the expected integer value of the induced charge Q_{VP}^{ren} through the direct integration of $\varrho_{VP}^{ren}(\vec{r})$.

5. VACUUM ENERGY RENORMALIZATION

Calculation of the renormalized polarization energy of the Dirac sea for such problems with an external spherically symmetric Coulomb potential is considered in detail in [6, 28, 47, 48]. The original expression for $\mathcal{E}_{D,VP}$, in accordance with density $\varrho_{VP}(\vec{r})$, which disappears in the absence of external fields, has the form

$$\begin{aligned} \mathcal{E}_{D,VP} = & \frac{1}{2} \left(\sum_{\epsilon_n < -1} \epsilon_n - \sum_{\epsilon_n \geq -1} \epsilon_n + \sum_{-1 \leq \epsilon_n < 1} 1 \right)_A \\ & - \frac{1}{2} \left(\sum_{\epsilon_n < -1} \epsilon_n - \sum_{\epsilon_n \geq 1} \epsilon_n \right)_0, \end{aligned} \quad (65)$$

where the lower index A denotes a nonzero external field A_{ext} and index 0 corresponds to the free case $A_{ext} = 0$.

Next, in (65), one should separate out the individual contributions from the discrete and continuous spectra, and then, for the difference of integrals over the continuous spectrum, $\left(\int d\bar{q} \sqrt{q^2 + 1} \right)_A - \left(\int d\bar{q} \sqrt{q^2 + 1} \right)_0$, use the well-known technique that represents this difference in the form of an integral of the elastic scattering phase $\delta_k(q)$ (see [6, 28, 49–51] and references therein). The final result for $\mathcal{E}_{D,VP}(Z, R)$ in the form of a partial expansion by angular number k has the form [28]

$$\mathcal{E}_{D,VP}(Z, R) = \sum_{k=1}^{\infty} \mathcal{E}_{D,VP,k}(Z, R), \quad (66)$$

where

$$\begin{aligned} \mathcal{E}_{D,VP,k}(Z, R) = & k \left(\frac{1}{\pi} \int_0^{\infty} \frac{q dq}{\sqrt{q^2 + 1}} \delta_{tot}(k, q) \right. \\ & \left. + \sum_{-1 \leq \epsilon_{n,\pm k} < 1} (1 - \epsilon_{n,\pm k}) \right). \end{aligned} \quad (67)$$

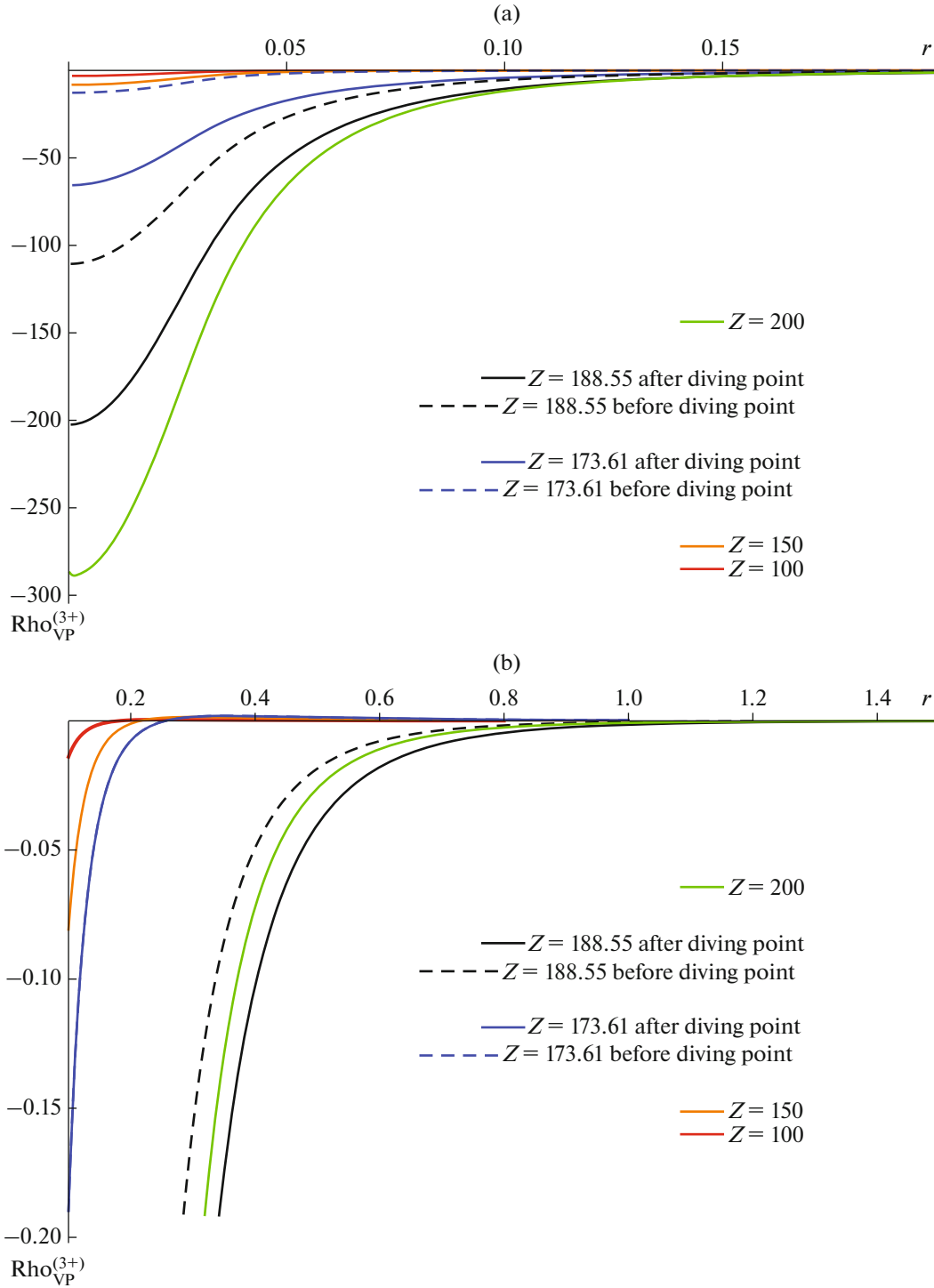


Fig. 4. Dominating behavior of the function $\rho_{\text{VP}}^{(3+)}(Z, r)$ in the s -channel at given Z , yielding a contribution of more than 99% into all vacuum polarization effects: (a) in the domain $0 < r < 0.2$, where it changes most rapidly; (b) in the domain $0.1 < r < 1.5$, where its behavior nearly coincides with the asymptotes.

The quantity $\delta_{\text{tot}}(k, q)$ in expression (67) is the total phase shift at a given value of the wave number q and angular number $\pm k$, including contributions from scattering states from the upper and lower continua of

both parities for the radial DK problem with Hamiltonian (31). The additional sum \sum_{\pm} in the term corresponding to the contribution of the discrete spectrum

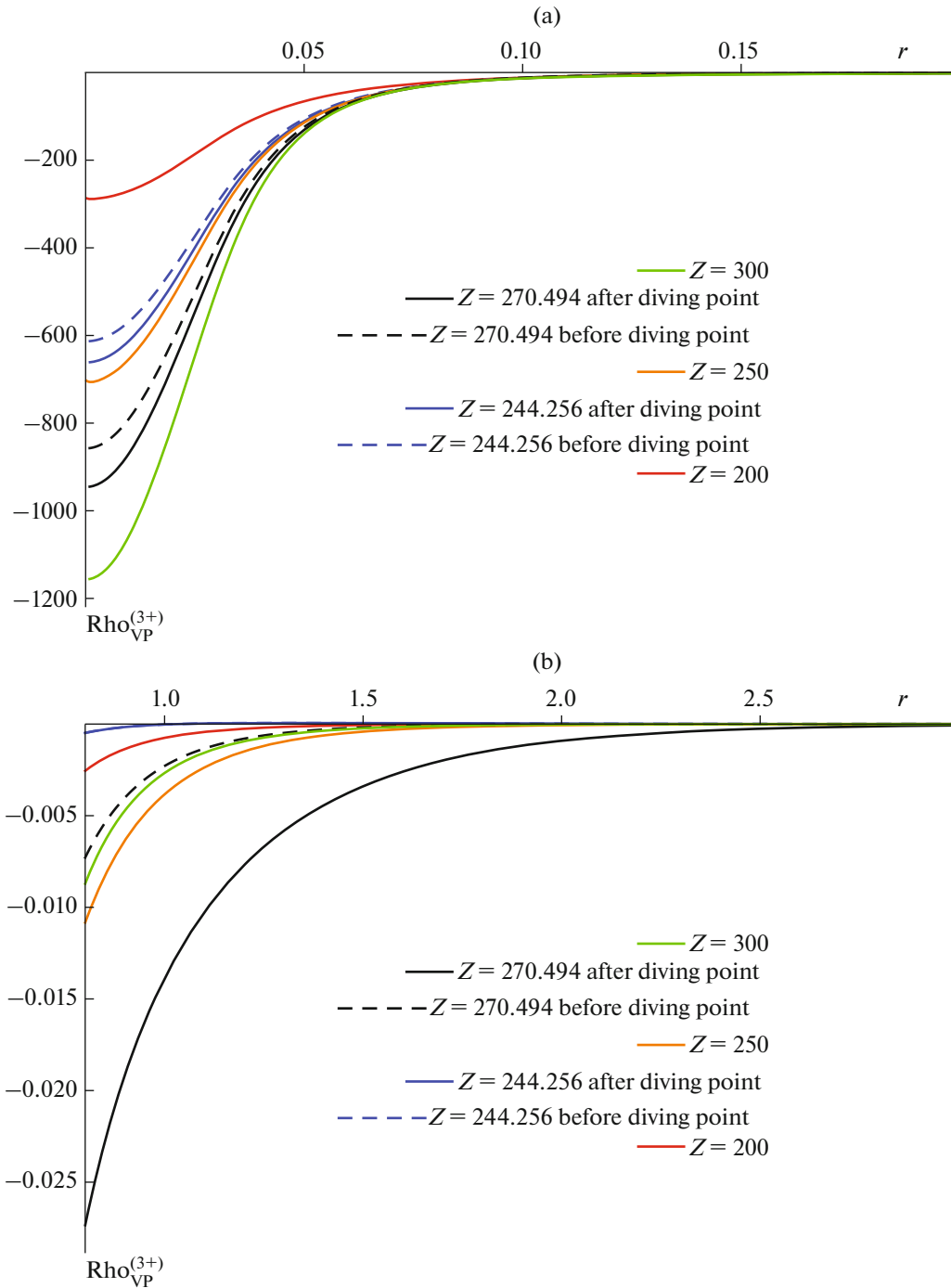


Fig. 5. Same as in Fig. 4 for (a) the domain $0 < r < 0.2$, where $\varrho_{\text{VP}}^{(3+)}(Z, r)$ changes most rapidly; (b) the domain $0.8 < r < 3.0$, where it nearly coincides with the asymptotics.

to $\mathcal{E}_{\text{D,VP},k}(Z, R)$ takes into account the contribution of levels with different parities.

This approach to calculating $\mathcal{E}_{\text{D,VP}}(Z, R)$ turns out to be very effective, since for external potentials of the form (4), (5) each partial term in the expression for $\mathcal{E}_{\text{D,VP}}(Z, R)$ immediately becomes finite. This is due

to the fact that $\delta_{\text{tot}}(k, q)$ behaves both in the IR and UV ranges with respect to the variable q much better than each of the elastic phases separately. Namely, $\delta_{\text{tot}}(k, q)$ has a finite limit for $q \rightarrow 0$ and varies as $O(1/q^3)$ for $q \rightarrow \infty$ $O(1/q^3)$. As a result, the phase integral in (67)

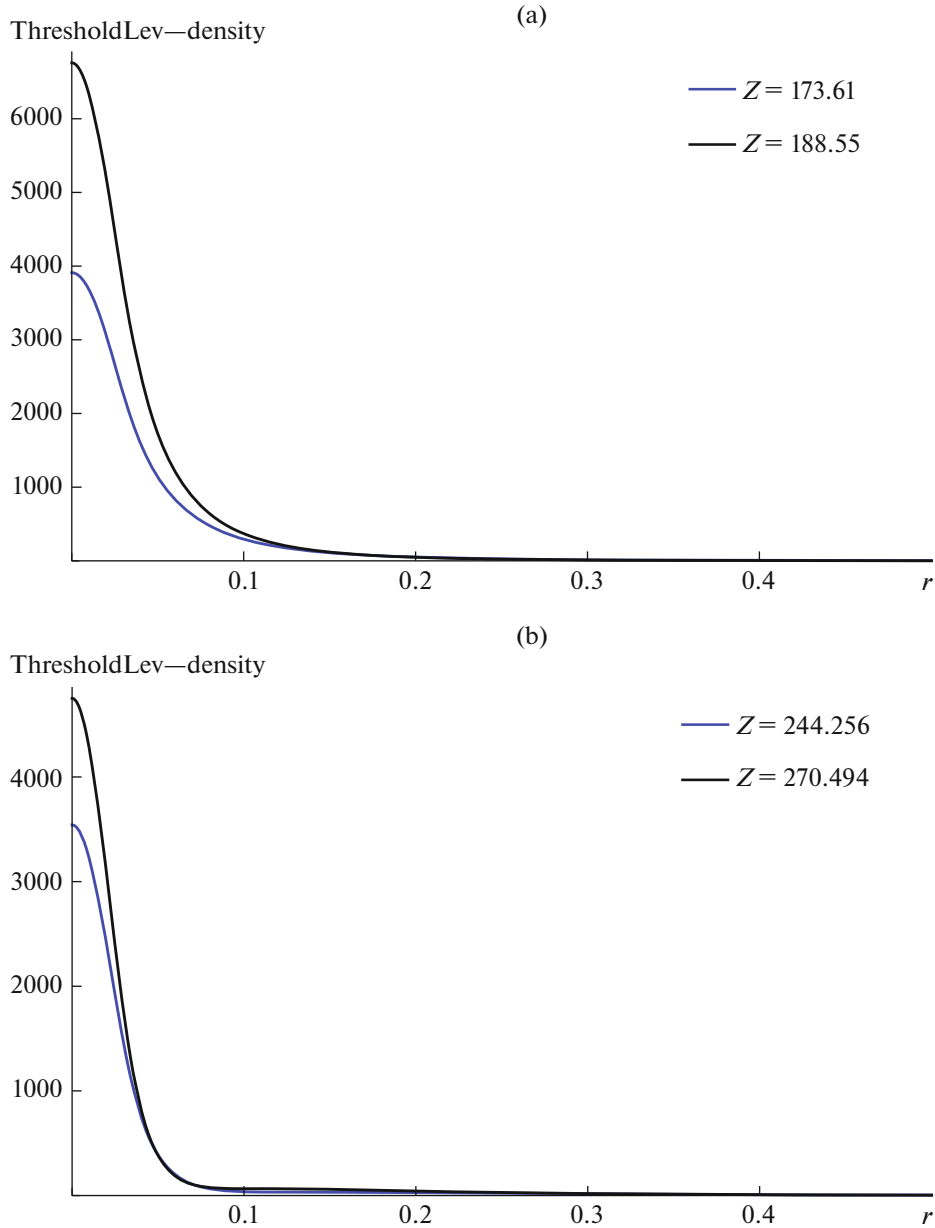


Fig. 6. Normalized functions $|\psi_v(r)|^2$ of discrete levels (for one spin projection) on the boundary of the lower continuum for (a) $v = 1s, 2p$; (b) $v = 2s, 3p$. The colors of $|\psi_{1s,2s}(r)|^2$ and $|\psi_{2p,3p}(r)|^2$ are chosen so that they coincide with the colors of the plots of the corresponding vacuum densities for $1s, 2s$ (blue) and $2p, 3p$ (black).

always converges. In addition, $\delta_{\text{tot}}(k, q)$, by construction, will automatically be an even function of the external field. In its turn, in the contribution of bound states to $\mathcal{E}_{D,VP,k}$, the focal point $\epsilon_{n,\pm k} \rightarrow 1$ is always regular, since $1 - \epsilon_{n,\pm k} \sim O(1/n^2)$ at $n \rightarrow \infty$. This circumstance makes it possible to proceed without the intermediate regularization of the Coulomb asymptotics of the external potential at $r \rightarrow \infty$, which significantly simplifies all further calculations.

In the case of $\mathcal{E}_{D,VP}(Z, R)$, the divergence of the theory manifests itself in the divergence of partial series (66) [28]. This implies the need for its regularization and subsequent renormalization, although each individual partial term $\mathcal{E}_{D,VP,k}(Z, R)$ itself is immediately finite without any additional manipulations. The need for renormalization through the fermion loop also follows from the analysis of the properties of \mathcal{Q}_{VP} carried out in Section 4, which shows that, without such renormalization, the integral vacuum

charge will not have the expected integer value in units $(-2|e|)$. In essence, the properties of ϱ_{VP} here play a role of the inspector, ensuring the fulfillment of the necessary physical conditions for the correct description of the effects of vacuum polarization beyond the framework of perturbation theory, which are not traced when calculating $\mathcal{E}_{D,VP}$ using initial expressions (1), (65).

Thus, in complete analogy with the renormalization of vacuum charge density (45), it is necessary to pass to the renormalized $\mathcal{E}_{D,VP}^{\text{ren}}(Z, R)$ using the relation

$$\mathcal{E}_{D,VP}^{\text{ren}}(Z, R) = \sum_{k=1} \mathcal{E}_{D,VP,k}^{\text{ren}}(Z, R), \quad (68)$$

where

$$\mathcal{E}_{D,VP,k}^{\text{ren}}(Z, R) = \mathcal{E}_{D,VP,k}(Z, R) + \zeta_{D,k}(R)Z^2, \quad (69)$$

and the renormalization coefficients $\zeta_{D,k}(R)$ are defined as follows:

$$\zeta_{D,k}(R) = \lim_{Z_0 \rightarrow 0} \left[\frac{\mathcal{E}_{D,VP}^{\text{PT}}(Z_0)\delta_{k,1} - \mathcal{E}_{D,VP,k}(Z_0)}{Z_0^2} \right]_R. \quad (70)$$

The main essence of procedure (68)–(70) is to allocate (at fixed R !) divergent components quadratic in Q from the nonrenormalized partial terms $\mathcal{E}_{D,VP,k}(Z, R)$ of expansion (66), replacing them with renormalized $\mathcal{E}_{D,VP}^{\text{PT}}\delta_{k,1}$, found within the framework of perturbation theory. In this case, both the convergence of entire partial series (68) and the correct value of the limit $\mathcal{E}_{D,VP}^{\text{ren}}(Z, R)$ for $Z \rightarrow 0$ at fixed R are automatically ensured.

The expression for the perturbative term $\mathcal{E}_{D,VP}^{\text{PT}}\delta_{k,1}$ follows from the general relation for the first-order perturbation theory [42, 44]

$$\mathcal{E}_{D,VP}^{\text{PT}} = \frac{1}{2} \int d\vec{r} \varrho_{VP}^{\text{PT}}(\vec{r}) A_0^{\text{ext}}(\vec{r}), \quad (71)$$

where $\varrho_{VP}^{\text{PT}}(\vec{r})$ is the perturbative vacuum density defined by expressions (46)–(49). Substituting (46), (47) into (71) yields

$$\mathcal{E}_{D,VP}^{\text{PT}} = \frac{1}{64\pi^4} \int d\vec{q} \bar{q}^2 \Pi_R(-\bar{q}^2) \left| \int d\vec{r} e^{i\vec{q}\vec{r}} A_0^{\text{ext}}(\vec{r}) \right|^2. \quad (72)$$

It should be noted that, since the function $S(x)$, determined by relations (48), (49), is strictly positive, the perturbative vacuum energy is also a strictly positive quantity.

For the case of a spherically symmetric potential $A_0^{\text{ext}}(\vec{r}) = A_0(r)$, the perturbative vacuum energy

belongs to the s -channel, which corresponds to the multiplier $\delta_{k,1}$ in (70), and has the form

$$\mathcal{E}_{D,VP}^{\text{PT}} = \frac{1}{\pi_0} \int dq q^4 \Pi_R(-q^2) \left(\int_0^\infty r^2 dr j_0(qr) A_0(r) \right)^2. \quad (73)$$

From (73), the perturbative vacuum energy in the charged sphere model is given by

$$\mathcal{E}_{D,VP,\text{sphere}}^{\text{PT}} = Q^2 / 2\pi R \int_0^\infty dq / q S(q/m) J_{1/2}^2(qR), \quad (74)$$

and, in the ball model, respectively,

$$\mathcal{E}_{D,VP,\text{ball}}^{\text{PT}} = 9Q^2 / 2\pi R^3 \int_0^\infty dq / q^3 S(q/m) J_{3/2}^2(qR). \quad (75)$$

For the source radii R sufficiently close to $R_{\text{min}}(Z)$ and satisfying the condition

$$\ln(1/2mR) \gg 1, \quad (76)$$

the integrals in (74), (75) are calculated analytically [14]:

$$\mathcal{E}_{D,VP,\text{sphere}}^{\text{PT}} = \frac{Q^2}{3\pi R} \left[\ln\left(\frac{1}{2mR}\right) - \gamma_E + \frac{1}{6} \right] \quad (77)$$

for the sphere model and

$$\mathcal{E}_{D,VP,\text{ball}}^{\text{PT}} = \frac{2Q^2}{5\pi R} \left[\ln\left(\frac{1}{2mR}\right) - \gamma_E + \frac{1}{5} \right] \quad (78)$$

for the ball model. In this case, the fraction

$$\mathcal{E}_{D,VP,\text{ball}}^{\text{PT}} / \mathcal{E}_{D,VP,\text{sphere}}^{\text{PT}} = 6/5 \quad (79)$$

exactly reproduces the ratio between the classical electrostatic energies of a uniformly charged ball ($3Z^2\alpha/5R$) and a sphere ($Z^2\alpha/2R$). In the case under consideration, however, condition (76) is too severe, so $\mathcal{E}_{D,VP}^{\text{PT}}$ is found by numerical methods directly from the original integral representations. Figure 7 shows the dependence of $\mathcal{E}_{D,VP}^{\text{ren}}(Z)$ as a function of Z at $R = R_{\text{min}}(Z)$ for the sphere model.

The renormalization of the Coulomb term is completely similar to those for the vacuum charge density and $\mathcal{E}_{D,VP}(Z, R)$. It is important to note that it in no way implies a direct substitution of $\varrho_{VP}(\vec{r}) \rightarrow \varrho_{VP}^{\text{ren}}(\vec{r})$ into expression (2). Instead, it is necessary to follow the general scheme of subtracting (for a fixed R !) the component quadratic in parameter Q from original expression (2) and then replacing it with the corresponding perturbative contribution. This means that the renormalized Coulomb term must be defined as follows

$$\mathcal{E}_{C,VP}^{\text{ren}}(Z, R) = \mathcal{E}_{C,VP}(Z, R) + \zeta_C(R)Z^2. \quad (80)$$

The renormalization coefficient $\zeta_C(R)$ is found using the relation

$$\zeta_C(R) = \lim_{Z_0 \rightarrow 0} \left[\frac{\mathcal{E}_{C,VP}^{PT}(Z_0) - \mathcal{E}_{C,VP}(Z_0)}{Z_0^2} \right]_R, \quad (81)$$

where

$$\mathcal{E}_{C,VP}^{PT} = \frac{1}{2} \int d\vec{r} d\vec{r}' \frac{\varrho_{VP}^{PT}(\vec{r}) \varrho_{VP}^{PT}(\vec{r}')}{|\vec{r} - \vec{r}'|}, \quad (82)$$

and $\varrho_{VP}^{PT}(\vec{r})$ is defined by expressions (46) and (49).

This approach provides a renormalization of the Coulomb term that is fully consistent with renormalizations of $\varrho_{VP}(\vec{r})$ and $\mathcal{E}_{D,VP}$ due to the use of the same subtraction procedure, as well as the correct value of the limit $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$ at $Z \rightarrow 0$ and fixed R . Expression (80) can be rewritten as follows:

$$\begin{aligned} & \mathcal{E}_{C,VP}^{\text{ren}}(Z, R) \\ &= \frac{1}{2} \int d\vec{r} d\vec{r}' \frac{\varrho_{VP}^{PT}(\vec{r}) \varrho_{VP}^{PT}(\vec{r}') - \varrho_{VP}^{(1)}(\vec{r}) \varrho_{VP}^{(1)}(\vec{r}')}{|\vec{r} - \vec{r}'|} \\ & \quad + \frac{1}{2} \int d\vec{r} d\vec{r}' \frac{\varrho_{VP}(\vec{r}) \varrho_{VP}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \end{aligned} \quad (83)$$

Of most interest is the fact that, in the supercritical region, the Born component $\varrho_{VP}^{(1)}$ turns out to be much larger than the perturbative term ϱ_{VP}^{PT} , as a result of which a very specific effect arises—vacuum shells, the charges of which have the same sign (negative), which are effectively attracted.

In the case under consideration, the contribution from the Coulomb term $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$ is calculated as follows. For the s -channel, general expression (83) has the form

$$\begin{aligned} \mathcal{E}_{C,VP}^{\text{ren}}(Z, R) &= (4\pi)^2 \int_0^\infty r dr \int_0^r r'^2 dr' \\ & \times [\varrho_{VP}^{PT}(r) \varrho_{VP}^{PT}(r') - \varrho_{VP}^{(1)}(r) \varrho_{VP}^{(1)}(r') + \varrho_{VP}(r) \varrho_{VP}(r')], \end{aligned} \quad (84)$$

where $\varrho_{VP}^{PT}(r)$, $\varrho_{VP}^{(1)}(r)$, $\varrho_{VP}(r)$ are the perturbative Born's (defined by expression (41)) and total vacuum charge densities calculated for the s -channel, respectively. Figures 8a–8c show the plots of the renormalized Coulomb term $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$ and its components—perturbative $\mathcal{E}_{C,VP}^{PT}(Z)$ and Born's $\mathcal{E}_{C,VP}^B(Z)$ —as functions of the charge Z of the source at $R = R_{\min}(Z)$. From Fig. 8a it is clearly seen how the renormalized Coulomb term $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$ behaves. Until the first discrete level dives into the lower continuum, the Coulomb interaction between the vacuum charge densities corresponds to their effective repulsion. When each discrete level reaches the lower continuum with the for-

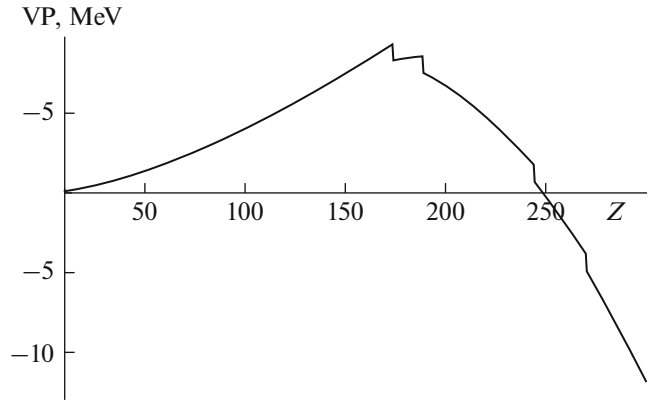


Fig. 7. Dependence $\mathcal{E}_{D,VP}^{\text{ren}}(Z)$ for potential (4) at $R = R_{\min}(Z)$ in the range $10 < Z < 300$.

mation of a charged vacuum shell, a negative jump occurs in $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$. As a result, after the first discrete level dives, the Coulomb interaction between the vacuum charge densities will correspond to the effective attraction. This nontrivial effect is a direct consequence of renormalization procedure (80)–(82). Moreover, the general shape of the plots $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$ and $\mathcal{E}_{D,VP}^{\text{ren}}(Z)$ vs. Z is the same: there are negative jumps on the plots at each $Z_{cr,i}$. The main difference is that $\mathcal{E}_{D,VP}^{\text{ren}}(Z)$ is much larger in magnitude than the Coulomb term and decreases faster and faster with increasing Z compared to $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$, which behaves much more smoothly. In addition, negative jumps in $\mathcal{E}_{D,VP}^{\text{ren}}(Z)$ are always equal to $2mc^2$, while, for $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$, the magnitude of the jumps is not a strictly fixed value, but depends on the structure of the wave functions $\psi_n(\vec{r})$ of discrete levels at the boundary of the lower continuum. The main property of jumps in $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$ is the decrease with increasing Z due to an increase in the number of nodes of the wave functions $\psi_{n,k}(\vec{r})$ at the boundary of the lower continuum.

We should also dwell on the calculation of the perturbative term $\mathcal{E}_{C,VP}^{PT}(Z, R)$ for the sphere model. In this case, the corresponding perturbative charge density $\varrho_{VP}^{PT}(r)$ has a δ -like singularity for $r = R$. Because of this, it is difficult to directly calculate the contribution from the corresponding term into integral (84). This problem can be circumvented by considering the expression for the Coulomb term written in terms of the longitudinal component of the electric field, $\mathcal{E}_{\parallel}(\vec{r})$. This is due to the fact that, for external potential (4), the Juhling potential $A_{VP,0}^{PT}(r)$ is a continuous function, and its first derivative (and, accordingly,

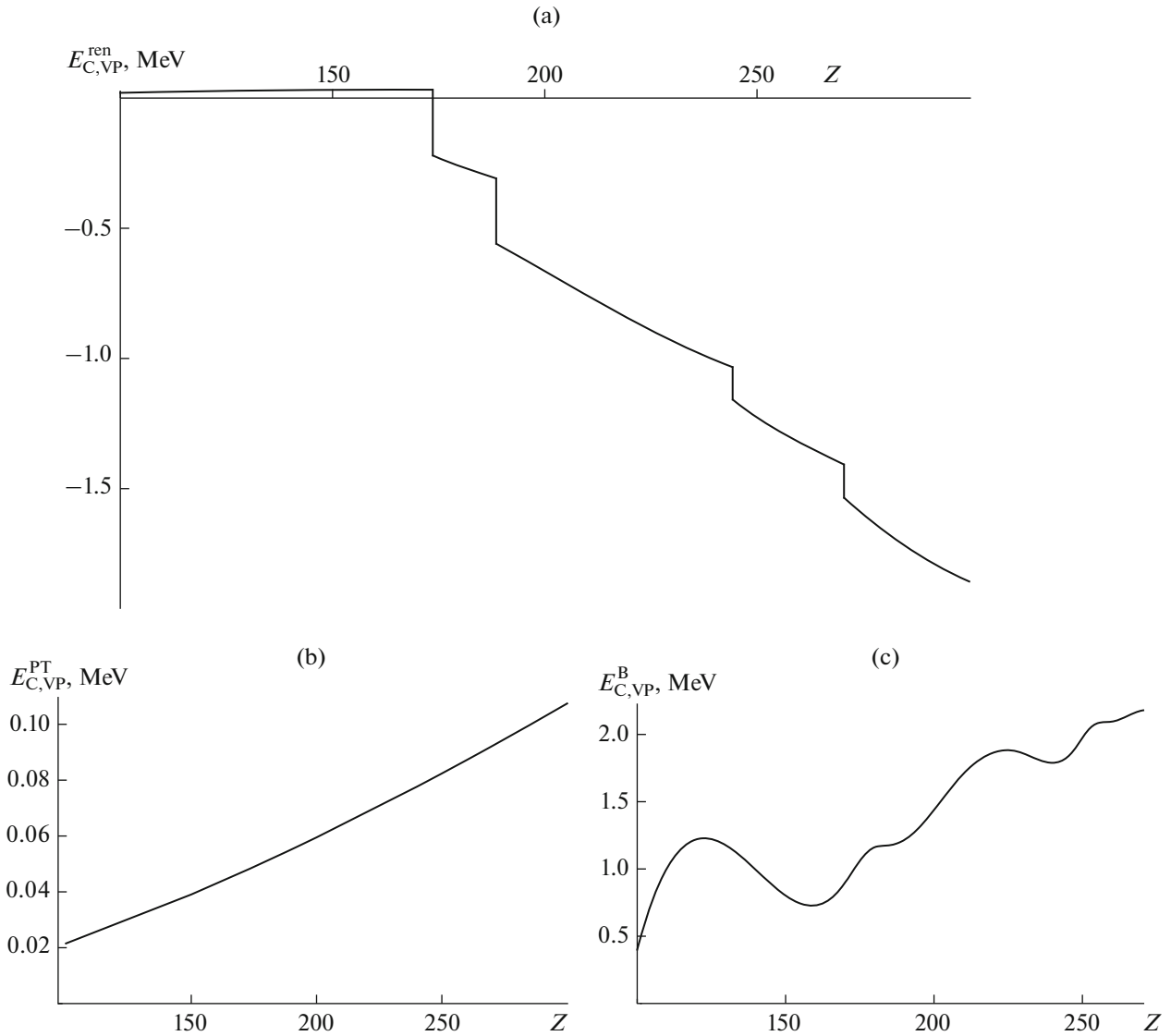


Fig. 8. Different components of the renormalized Coulomb term as a function of Z at $R = R_{\min}(Z)$ in the range $10 < Z < 300$:

(a) $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$, (b) $\mathcal{E}_{C,VP}^{\text{PT}}(Z)$, and (c) $\mathcal{E}_{C,VP}^{\text{B}}(Z)$.

$\mathcal{E}_{\parallel,VP}^{\text{PT}}(r)$ is continuous everywhere except for the finite jump at $r = R$. At the same time, for both $r < R$ and $r > R$, the radial component of the electric field $\mathcal{E}_{\parallel,VP}^{\text{PT}}(r)$ is well defined. Therefore, for the sphere model, the perturbative contribution $\mathcal{E}_{C,VP}^{\text{PT}}(Z, R)$ in the Coulomb term is found by the relation

$$\mathcal{E}_{C,VP}^{\text{PT}}(Z, R) = \frac{1}{2} \left[\int_0^R + \int_R^\infty \right] r^2 dr (\mathcal{E}_{\parallel,VP}^{\text{PT}}(r))^2. \quad (85)$$

Unlike the perturbative contribution $\mathcal{E}_{C,VP}^{\text{PT}}(Z)$, which is a monotonously increasing function, the Born term $\mathcal{E}_{C,VP}^{\text{B}}(Z)$ defined by the charge density

$\varrho_{VP}^{(1)}(r)$ behaves in a more nontrivial way. First, according to (42), (43), the asymptotics $\varrho_{VP}^{(1)}(r)$ are such that the corresponding integral defining the Born term in expression (84) converges, so that the renormalization of the Coulomb interaction is reduced to the subtraction of a finite quantity. However, the need for such a subtraction is justified, since it is part of the general renormalization scheme. Second, $\mathcal{E}_{C,VP}^{\text{B}}(Z)$ is significantly larger than other components of the Coulomb energy. Therefore, $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$ becomes negative in the supercritical region $Z > Z_{\text{cr},1}$. The most nontrivial property of $\mathcal{E}_{C,VP}^{\text{B}}(Z)$ is the noticeable response to critical charges. It does not appear as brightly as in the

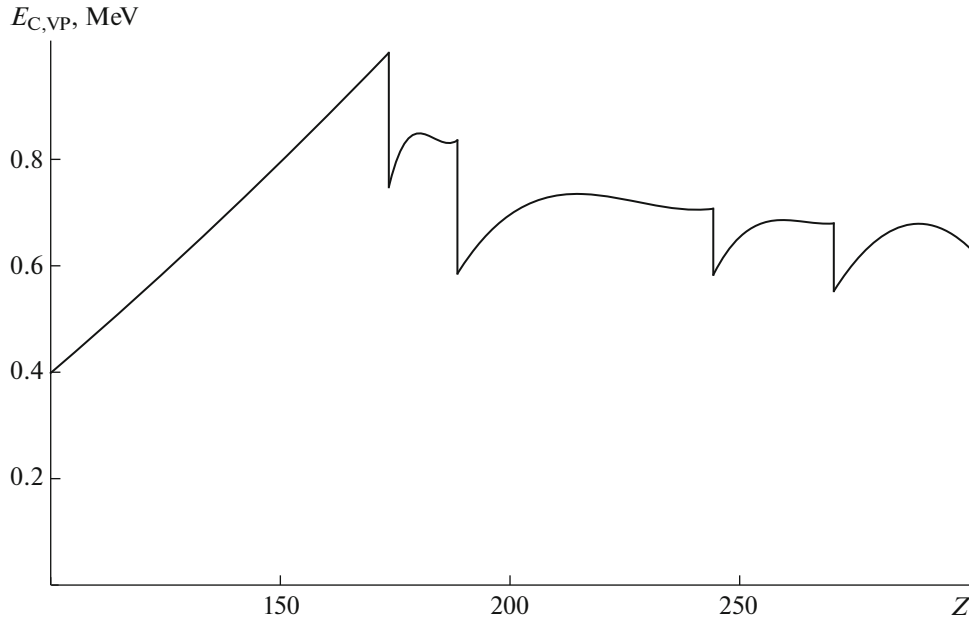


Fig. 9. Non-renormalized Coulomb term $\mathcal{E}_{C,VP}(Z)$ as a function of Z for $R = R_{\min}(Z)$ in the range $10 < Z < 300$.

renormalized Coulomb term $\mathcal{E}_{C,VP}^{\text{ren}}(Z)$, but it is expressed quite explicitly. The reason for this is that, unlike $\varrho_{VP}^{\text{PT}}(r)$, which is the result of a purely perturbative calculation, $\varrho_{VP}^{(1)}(r)$ is part of the total vacuum density and, therefore, albeit partially, reflects the information about the diving of levels into the lower continuum contained in the total vacuum density $\varrho_{VP}(r)$.

As an additional illustration of the general structure of the Coulomb term in vacuum energy, Fig. 9 shows a plot of the nonrenormalized Coulomb term $\mathcal{E}_{C,VP}(Z, R_{\min}(Z))$, which is also a finite quantity. It has negative jumps at every $Z_{cr,i}$, but is strictly positive. Thus, without renormalization replacing $\mathcal{E}_{C,VP}(Z, R)$ by $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$, the main effect of the Coulomb term would be an additional repulsion of the vacuum charge densities, which would lead to additional problems for spontaneous emission. On the other hand, $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$ acts in the right direction, but its contribution is not enough to drastically improve the overall picture.

6. CONCLUSIONS

Thus, the additional contribution from the correctly renormalized Coulomb term $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$ to the total vacuum energy turns out to be negative and can play a positive role in the energy balance of spontaneous emission in the range $170 \leq Z \leq 192$, which is currently a reference point for theoretical and experimental research on this topic within the framework of

heavy ion collisions [7, 11, 17–19]. It should also be noted that, if a level that descends into the lower continuum manifests itself as an unfilled vacancy, then the only result will be a change in the density of states $n(\vec{r})$ in accordance with the general approach [15, 45]. As a result, the resulting vacuum shell remains uncharged, and thus no negative jump in $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$ occurs. As a consequence, the main component in $\mathcal{E}_{C,VP}^{\text{ren}}(Z, R)$, in this case, will be the perturbative term $\mathcal{E}_{C,VP}^{\text{PT}}(Z, R)$, which is strictly positive and negligibly small compared to $\mathcal{E}_{D,VP}^{\text{ren}}(Z, R)$. Dependence $\mathcal{E}_{C,VP}^{\text{PT}}(Z, R)$ vs. Z for $R = R_{\min}(Z)$ is shown in Fig. 8b. It decays as $O(1/R)$ with increasing R at fixed Z . Thus, the total vacuum energy $\mathcal{E}_{VP}^{\text{ren}}(Z, R)$ in this case, up to a small positive correction that disappears at large R , coincides with its main component $\mathcal{E}_{D,VP}^{\text{ren}}(Z, R)$. For this reason, in Eq. (24), the additional term $\Delta V(r) = eA_{VP}^0(r)$ to the external potential $V(r)$, due to the induced scalar component $A_{VP}^0(r)$, which arises in the Coulomb gauge from the Poisson equation $\Delta A_{VP}^0 = -4\pi\varrho_{VP}$ [29], also turns out to be negligible and can be omitted without any loss of generality.

In addition, during the spontaneous emission of vacuum positrons, the latter carry away a lepton number equal to their amount with a minus sign. Then the corresponding positive lepton number must remain in the form of spatial density, concentrated in vacuum shells. Otherwise, in such processes either the conser-

vation law of the lepton number must be violated, or spontaneous emission itself becomes impossible. Therefore, any reliable experimental result on the problem of spontaneous emission is important for understanding the nature of the lepton number, since at present there is no evidence for the presence of any internal lepton structure. See [47] for a more detailed discussion of this issue.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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