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PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

On 10-Dimensional Exceptional Drinfel'd Algebras

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Abstract—We discuss the mathematical formalism of 10 dimensional Exceptional Drinfel'd Algebra, provide a brief explanation of developments in dualities in string theory and present our results in classification of 4 dimensional Exceptional Drinfel'd Algebras, which are the underlying Algebraic structures for the U-dualities.

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INTRODUCTION

String theory is a background dependent theory. It possesses a rich framework of dualities with which solutions on different backgrounds can be related to each other. T-duality relates the theories with big compactification radius, to the small ones—$R \sim 1/R$. Initially, T-duality was developed for spaces which had Abelian isometries [1]. In [2, 3], the abelian procedure was generalised to non-Abelian isometries. A similar duality exhibited by superstrings is the S-duality where $g \sim \frac{1}{g}$—strong-weak coupling duality. At last, M-theory provides a framework to relate all known flavors of superstrings (and their low-energy limits—SUGRAs) via U-dualities which is a broader class of duality and includes within itself T and S-dualities.

Our discussion is a generalisation of the procedure of [4] to find dual backgrounds prescribing to the U-duality transform [5, 6]. It is well known in string theory that after compactification of strings to reduce theory to a lower dimensional space, the U-dualities exhibit themselves as symmetries of the E_n groups called "the Exceptional groups".

For compactification on a 4-torus, the U-duality group E_n is—$E_4 = SL(5)$. So our task in this paper is to start with a 4-dimensional Lie algebra and expand it based on rules provided in [7] to generate the corresponding 10-dimensional Exceptional Drinfel'd Algebra (EDA) which possesses $E_4 = SL(5)$ symmetry in a manifest way by construction. More precisely, by finding these algebras, we wish to check in future work, the existence of SL(5) dualities between different EDAs which would mean the presence of U-duality between these backgrounds. This allows us to define a theory (the Exceptional Field theory) that contains the symmetries of compactified M-theory in manifest way (i.e. without compactification to a lower dimension). In the next section, we will present a brief review of the mathematical construction of these rules to define an EDA for the SL(5) theory. We will then present our results on the expansion of 4-dimensional Lie algebras to 10d EDAs describing the symmetries of M-theory (in our case, of its lower dimensional limit—1d SUGRA) in a manifest way.

We present our results based on algebras considering only those 4-dimensional Lie algebras which do not contain a U(1) cycle in them—they are not decomposible as a product of a 3-dimensional and a 1-dimensional (abelian) Lie algebra. This is because we are only interested in U-dualities between 11-dimensional supergravities (and not the Type IIB theory) [8].

EXCEPTIONAL DRINFEL'D ALGEBRA: SL(5)

This section is based on [7]. We will be focusing on the 10d case where generators of the exceptional Drinfeld algebra ED_4 are collected into the 10-dimensional representation of the SL(5) group $ED_4 = \{T_{AB}\}$, where $A, B = 1, \ldots, 5$. Multiplication table is then given by

$$T_{AB} \circ T_{CD} = \frac{i}{2} F_{AB,CD}^{GH} T_{GH}. \quad (1)$$

Table 1. Classification of 4-dimensional indecomposable real Lie algebras $\mathfrak{g}_{4,n}$ with $n = 1,\ldots,10$ [10]

$\mathfrak{g}_{4,1}$	$[T_2,T_4] = T_1$ $[T_3,T_4] = T_2$	$\mathfrak{g}_{4,5}$	$[T_1,T_4] = AT_1$ $[T_2,T_4] = BT_2$ $[T_3,T_4] = CT_3$ $ABC \neq 0$	$\mathfrak{g}_{4,9}$	$[T_2,T_3] = T_1$ $[T_1,T_4] = 2AT_1$ $[T_2,T_4] = AT_2 - T_3$ $[T_3,T_4] = T_2 + AT_3$ $A \geq 0$
$\mathfrak{g}_{4,2}$	$[T_1,T_4] = \beta T_1$ $(\beta \neq 0)$ $[T_2,T_4] = T_2$ $[T_3,T_4] = T_2 + T_3$	$\mathfrak{g}_{4,6}$	$[T_1,T_4] = AT_1$ $[T_2,T_4] = BT_2 - T_3$ $[T_3,T_4] = T_2 + BT_3$ $A > 0$	$\mathfrak{g}_{4,10}$	$[T_1,T_3] = T_1$ $[T_2,T_3] = T_2$ $[T_1,T_4] = -T_2$ $[T_2,T_4] = T_1$
$\mathfrak{g}_{4,3}$	$[T_1,T_4] = T_1$ $[T_3,T_4] = T_2$	$\mathfrak{g}_{4,7}$	$[T_2,T_3] = T_1$ $[T_1,T_4] = 2T_1$ $[T_2,T_4] = T_2$ $[T_3,T_4] = T_2 + T_3$	$2\mathfrak{g}_{2,1}$	$[T_1,T_2] = T_1$ $[T_3,T_4] = T_3$
$\mathfrak{g}_{4,4}$	$[T_1,T_4] = T_1$ $[T_2,T_4] = T_1 + T_2$ $[T_3,T_4] = T_2 + T_3$	$\mathfrak{g}_{4,8}$	$[T_2,T_3] = T_1$ $[T_1,T_4] = (1+\beta)T_1$ $[T_2,T_4] = T_2$ $[T_3,T_4] = \beta T_3$ $\beta \in [-1,1]$		

The structures constants $F_{AB,CD}^{GH}$ are defined by the following relations

$$F_{AB,CD}^{GH} = 4F_{AB,[C}^{[G}\delta_{D]}^{H]}, \tag{2}$$

$$\begin{aligned} F_{AB,C}^{D} &= \frac{1}{2}\epsilon_{ABCGH}Z^{GHD} + \frac{1}{2}\delta_{[A}^{D}S_{B]C} \\ &\quad + \frac{1}{3}\delta_{[A}^{D}\tau_{B]C} + \frac{1}{6}\delta_{C}^{D}\tau_{AB}, \end{aligned} \tag{3}$$

where τ is antisymmetric and S is a symmetric tensor, while $Z^{[ABC]} = 0$. For the algebra to be an EDA, components of the constants Z^{ABC}, S_{AB}, and τ_{AB} under decomposition $SL(5) \longleftrightarrow GL(4)$ must be defined as

$$\begin{gathered} Z^{abc} = \frac{1}{6}\epsilon^{abcd}f_{de}^{e} + \frac{1}{4}\epsilon^{abef}f_{ef}^{c}, \quad S_{5a} = f_{ab}^{b} - 3Z_a, \\ \tau_{5a} = \frac{9}{2}Z_a - \frac{1}{2}f_{ab}^{b}, \quad Z^{5[a,b]} = \frac{1}{6}\tilde{f}_c^{\ abc}, \\ S_{ab} = \frac{1}{3}\tilde{f}_{(a}^{cde}\epsilon_{b)cde}, \quad \tau_{ab} = -\frac{1}{6}\tilde{f}_{[a}^{cde}\epsilon_{b]cde}, \\ Z^{ab,5} = -Z^{5a,b} + Z^{5b,a}. \end{gathered} \tag{4}$$

The constants $F_{AB,C}^{D}$ have the same structure as the embedding tensor of [9], and in this language the above construction implies that only the geometric flux (anholonomy coefficients) and Q-flux are turned on. The former is given by the structure constants f_{ab}^{c} of the geometric subalgebra $\mathfrak{g}$ and the latter is given by $\tilde{f}_a^{bcd}$. The algebra is Leibniz with the fundamental identity given by the quadratic relations analogous to those of 7d maximal gauged SUGRA [9]:

$$2F_{AB[C}^{G}F_{GD]H}^{I} - F_{ABG}^{I}F_{CDH}^{G} + F_{ABH}^{G}F_{CDG}^{I} = 0. \tag{5}$$

In terms of structure constants f_{ab}^{c} and dual constants $f_a^{\ bcd}$, the conditions become

$$\begin{gathered} 6f_{f[a}^{[c}\tilde{f}_{b]}^{de]f} + f_{ab}^{f}\tilde{f}_f^{cde} - \frac{1}{3}\tilde{f}_{[a}^{cde}f_{b]f}^{f} = 0, \\ \tilde{f}_c^{\ abc}f_{bd}^{d} = 0, \quad f_{de}^{a}\tilde{f}_c^{bde} - \frac{1}{3}\tilde{f}_c^{\ abd}f_{de}^{e} = 0, \\ \tilde{f}_c^{\ abg}\tilde{f}_g^{def} - 3\tilde{f}_c^{\ g[de}\tilde{f}_g^{\ f]ab} = 0. \end{gathered} \tag{6}$$

RESULTS AND DISCUSSION

Here we present all the 4 dimensional Lie algebras of interest. We have listed all those algebras which do not contain a U(1) cycle.

Using the above table, and the mathematical formalism presented in the last passage, we derive all 10-dimensional EDA.

Table 2. All possible structure constant of 10d EDAs for each $\mathfrak{g}_{4,n}$ with $n = 1,\ldots,10$ and $2\mathfrak{g}_{2,1}$. The constants A, B, C, are the same as in the previous table

	EDA	Structure constants $\tilde{f}_d^{abc}$
$\mathfrak{g}_{4,1}$	1.	$\tilde{f}_2^{123} = \tilde{f}_4^{134}$, $\tilde{f}_2^{124} = \tilde{f}_3^{134}$ $\tilde{f}_4^{123} = \dfrac{\tilde{f}_3^{123}\tilde{f}_4^{134} - \tilde{f}_4^{124}\tilde{f}_4^{134}}{2\tilde{f}_3^{134}}$ $\tilde{f}_3^{124} = \dfrac{(\tilde{f}_4^{124} - \tilde{f}_3^{123})\tilde{f}_3^{134}}{2\tilde{f}_4^{134}}$
	2.	• $\tilde{f}_2^{123} = -\tilde{f}_4^{134}$, $\tilde{f}_4^{124} = \tilde{f}_3^{123}$, $\tilde{f}_4^{234} = \tilde{f}_1^{123}$
	3.	$\tilde{f}_2^{124} = \tilde{f}_3^{134}$ $\tilde{f}_4^{124} = \tilde{f}_3^{123}$
	4.	$\tilde{f}_4^{124} = \tilde{f}_3^{123}$, $\tilde{f}_4^{234} = \tilde{f}_1^{123}$
$\mathfrak{g}_{4,2}$	5.	$\tilde{f}_4^{124} = -\dfrac{1}{6}(1+2\beta)\tilde{f}_3^{123}$
	6	$\tilde{f}_4^{234} = \dfrac{1}{3\beta}(\beta - 4)\tilde{f}_1^{123}$
$\mathfrak{g}_{4,3}$	7.	$\tilde{f}_4^{234} = \dfrac{1}{3}\tilde{f}_1^{123}$
$\mathfrak{g}_{4,4}$	8.	$\tilde{f}_4^{124} = -\tilde{f}_3^{123}$
$\mathfrak{g}_{4,5}$	9.	$\tilde{f}_4^{124} = \dfrac{-2A - 2B + C}{3C}\tilde{f}_3^{123}$
	10.	$\tilde{f}_4^{134} = \dfrac{1}{3B}(2A - B + 2C)\tilde{f}_2^{123}$
	11.	$\tilde{f}_4^{234} = \dfrac{1}{3A}(A - 2B - 2C)\tilde{f}_1^{123}$
$\mathfrak{g}_{4,6}$	12.	$\tilde{f}_4^{234} = \dfrac{1}{3A}(A - 2B - 2C)\tilde{f}_1^{123}$
$\mathfrak{g}_{4,7}$	13.	$\tilde{f}_4^{124} = -\dfrac{5}{3}\tilde{f}_3^{123}$
$\mathfrak{g}_{4,8}$	14.	$\tilde{f}_4^{124} = -\dfrac{1}{3\beta}(4-\beta)\tilde{f}_3^{123}$
	15.	$\tilde{f}_4^{134} = \dfrac{1}{3}(1+4B)\tilde{f}_2^{123}$
$\mathfrak{g}_{4,9}$		$\tilde{f}_d^{abc} = 0$ or imaginary
$\mathfrak{g}_{4,10}$		$\tilde{f}_d^{abc} = 0$
$2\mathfrak{g}_{2,1}$	16.	$\tilde{f}_4^{123} = \tilde{f}_2^{123}$, $\tilde{f}_2^{134} = \tilde{f}_4^{134}$

CONCLUSIONS

Our current work is focused on finding U-dualities between 11d SUGRAs by checking the existence of an $E_4 = SL(5)$ transformation between the above derived classes of EDA. If such a transformation exists between a pair of EDAs, the corresponding algebras will become equivalent upto an $SL(5)$ transform relating them. This would hence confirm whether or not U-dualities exists between 11d SUGRAs in the framework of Exceptional Field theory.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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