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PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

On the Search for a Gravitational Chiral Anomaly Outside Curved Spacetime

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Abstract—In the last two decades it has been shown that quantum anomalies not only play an important role in particle physics, but also find novel applications in the physics of quantum fluids, leading to previously unknown nondissipative transport phenomena. In this paper we will discuss some aspects related to the search for manifestations of the gravitational chiral anomaly in a vortical and accelerated media.

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1. INTRODUCTION

The quantum world is essentially different from the classical world. For example, some conservation laws that are valid in classical physics are violated at the quantum level due to the creation and annihilation of pairs of particles and antiparticles in a vacuum. This effect is well known in modern quantum field theory and is called quantum anomalies. In particular, there is a chiral quantum anomaly due to which the axial current conservation law is violated,

$$\nabla_\mu j_A^\mu = \frac{C}{\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{N}{\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\kappa\lambda}R^{\kappa\lambda}_{\rho\sigma}, \quad (1)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor; $R_{\mu\nu\kappa\lambda}$ is the Riemann tensor; and C and N are numerical factors that depend on the type of fields under consideration, for example, for Dirac fields $C = -1/(16\pi^2)$ and $N = 1/(384\pi^2)$. Thus, there are two parts of this anomaly: gauge and gravitational.

Quantum anomaly (1) plays a huge role in fundamental physics, and its applications have entered modern textbooks. For example, the decay $\pi^0 \to 2\gamma$ occurs precisely due to this anomaly (gauge part), and the renormalizability of the theory also requires compensation for anomalies.

Manifestations (1) have been found in a completely new field: the relativistic hydrodynamics of quantum fluids [1]. It turns out that there are a number of new effects in a liquid placed in external fields associated with quantum anomalies. It is remarkable that not only a real field, such as a magnetic field $\mathbf{B}$, but also, for example, rotation can act as such a field. Rotation is characterized by either angular velocity $\boldsymbol{\Omega}$ or *local* angular velocity, vorticity $\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$, where u_μ is four-velocity. Also, the role of the external field can be played by the acceleration of the medium $a_\mu = u^\nu\partial_\nu u_\mu$. In this paper, we will consider effects in which acceleration and vorticity are classical external fields, although there are modern approaches where they become dynamic quantum quantities [2].

As an illustration, consider the most famous of the new phenomena, called the chiral magnetic effect (CME). Consider a medium in a magnetic field consisting of particles with a certain spin. Hydrodynamics is constructed as an expansion in terms of gradients. It turns out that, in a linear order along the gradients, an electric current of the form

$$\mathbf{j} = \sigma_B\mu_A\mathbf{B}, \quad (2)$$

arises, where μ_A is the axial chemical potential characterizing the imbalance between the left and right chiral particles and σ_B is the magnetic conductivity (and this current is associated with the calibration part of the anomaly

$$\sigma_B = -8C. \quad (3)$$

Conclusion (3) is rather nontrivial, but was obtained by various methods [1, 3–5].

Result (2) looks paradoxical for several reasons. First, it is known from classical electrodynamics that the current (if you do not take complex effects in plasma) flows along the electric field, not the magnetic field. Also, surprisingly, (2) is associated with an anomaly $F\tilde{F} \sim \mathbf{E}\cdot\mathbf{B}$, although it exists even when there is no electric field $\mathbf{E} = 0$, and anomaly (1), respectively, is equal to zero. Finally, current (2) is nondissipative, making it similar to superconductivity. At this point, there are indications of (2) in a solid, and a search is underway in collisions of heavy ions and on the lattice [6].

The question of the gravitational part of anomaly (1) was much more nontrivial. This work is devoted to discussing this issue.

2. CHIRAL VORTICAL EFFECT

The first candidate for the role of the anomalous current associated with the gravitational part of the anomaly was the Chiral vortical effect. According to this effect, an axial current arises in a medium with rotation, similar to (2), but with the replacement of the magnetic field by vorticity (we set $\mu_A = 0$)

$$j_A^\nu = (\sigma_T T^2 + \sigma_\mu \mu^2)\omega^\nu, \qquad (4)$$

where T is the temperature; μ is the usual chemical potential associated with the electric charge; σ_T, σ_μ and $\sigma_{\mu 5}$ are some numbers. Notice that (4) is valid for the case of massless particles (for the massive case, see [7]). As in the case (3), it can be shown that [1, 4, 5]

$$\sigma_\mu = -8C. \qquad (5)$$

Equation (5), was obtained in the general case and is confirmed by most direct calculations. At the same time, σ_T should be determined by the gravitational part of the anomaly [8–10]

$$\sigma_T = 64\pi^2 N. \qquad (6)$$

However, in this case the proof turns out to be more nontrivial than in the case of a gauge anomaly, where the effect follows directly from the hydrodynamic equations. Consider, in particular, the output [10]. We consider a spacetime with a horizon and rotation on the horizon, similar to the space of a rotating black hole. Current $T^2\omega^\nu$ can be obtained directly from anomaly $R\tilde{R}$ at (1), integrating it from the horizon to infinity, up to a constant. This constant plays a significant role and can be determined from the additional boundary condition that the current is equal to zero at the horizon. As a result, at infinity, given that the radiation of a black hole has a Hawking temperature, current $T^2\omega^\nu$ remains.

This conclusion immediately leads to (6) and is confirmed for spin 1/2, where $\sigma_T = 64\pi^2 N = 1/6$.

However, as early as in [9] it was noted that, in the case of higher spins, the described mechanism may encounter difficulties. This was clearly shown in the example of the Rarita–Schwinger–Adler theory [11], which includes fields with spins 3/2 and 1/2. A gravitational chiral quantum anomaly has been found in [12]

$$\nabla_\mu j_A^\mu = \frac{-19}{384\pi^2\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\kappa\lambda}R^{\kappa\lambda}_{\rho\sigma}, \qquad (7)$$

and the vortical current was calculated in [13]

$$j_A^\nu = \left(\frac{5}{6}T^2 + \frac{5}{2\pi^2}\mu^2\right)\omega^\nu. \qquad (8)$$

It can be seen that, for (8), (5) is performed, but (6) is not. Thus, the problem remains of finding a universal anomalous transport phenomenon corresponding to the gravitational anomaly.

3. VORTICAL FLOW OF ENERGY

Let us try to approach this problem from the other side and analyze both phenomena—gravitational anomaly and transport phenomena—from the point of view of quantum correlators.

One way to calculate the anomaly is based on considering the correlator of two energy-momentum tensors and one axial current operator. As shown in [14] for a wide class of theories, such a three-point function is equal to the universal function $f(x,y,z)$ up to a coefficient N:

$$\left\langle T\hat{T}^{\mu\nu}(x)\hat{T}^{\sigma\rho}(y)\hat{j}_A^\omega(z)\right\rangle_c = N\times f(x,y,z), \qquad (9)$$

which corresponds to the anomaly from (1). At the same time, (9) is calculated in the usual flat spacetime.

Consider now the transport coefficients. They can be found from the expansion of the density operator [7, 15]

$$\hat{\rho} = \frac{1}{Z}\exp\left\{-\beta_\mu(x)\hat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\hat{J}_x^{\mu\nu} + \frac{\mu}{T}\hat{Q} + \frac{\mu_A}{T}\hat{Q}_A\right\}, \qquad (10)$$

where $\beta_\mu = u_\mu/T$, $\hat{P}^\mu$ is the momentum operator, $\hat{J}_x^{\mu\nu}$ are the generators of the Lorentz transformations shifted by vector x_μ, and $\hat{Q}$ and $\hat{Q}_A$ are the vector and axial charges. Equation (10) is written for the case of global thermodynamic equilibrium

$$\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu = 0. \qquad (11)$$

We will be interested in the vortical energy flow

$$T_{\mu\nu} = (A_1 T^2\mu_A + A_2\mu^2\mu_A + A_3\mu_A^3)(u_\mu\omega_\nu + u_\nu\omega_\mu). \qquad (12)$$

Coefficient A_1 can be determined, for example, by expanding (10) in series with Q_5 and $\hat{J}^{\mu\nu} \sim \int (y^\mu T^{0\nu} - y^\nu T^{0\mu})d^3y$. Then A_1 it is determined by the correlator at a finite temperature (in imaginary time) of the form

$$\frac{1}{T^2}\int_0^{1/T} d\tau_x d\tau_y \times \int d^3x d^3y y^i \left\langle T_\tau \hat{j}_A^0(-i\tau_x,\mathbf{x})\hat{T}^{0j}(-i\tau_y,\mathbf{y})\hat{T}^{03}(0)\right\rangle_{T,c}, \qquad (13)$$

where the averaging is done at $\varpi = \mu = \mu_A = 0$. Compare (13) and (9). The correlator in (9) has a universal form, and, for example, for the Rarita–Schwinger–Adler theory differs by −19 times from the case of Dirac fields [12]. It would seem then that $A_1 \sim N$. However, it is not that simple. To do this, let us recall the laws of conservation in an external field

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda + F_A^{\nu\lambda} j_\lambda^A, \qquad (14)$$

where we added a term with an axial field to the usual term. Due to the terms on the right side—the Lorentz force—it turns out that, in fact, $A_1 = 2\sigma_T$ from (4) (see, for example, [5, 9]). If we now consider, for example, (8), then instead of $A_1^{RSA} = -19A_1^{Dirac}$, it turns out that $A_1^{RSA} = 5A_1^{Dirac}$.

Why did correlator $\langle \hat{T}\hat{T}\hat{j}_A \rangle$ not lead to a gravitational anomaly? The answer seems to be related to the fact that (13), unlike (9), must be calculated at the finite temperature (and then integrated). That is, generally speaking, two different correlators are computed.

However, there is another possibility where the anomaly should still be clearly manifested.

4. KINEMATIC VORTICAL EFFECT

Recall how the formulas (3) and (5) were obtained in [4]. The initial laws are conservation laws and the second law of thermodynamics,

$$\begin{gathered}\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda, \quad \partial_\mu j^\mu = 0, \\ \partial_\mu j_5^\mu = C\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \quad \partial_\mu s^\mu \geq 0.\end{gathered} \qquad (15)$$

It can be shown that, for the simultaneous fulfillment of all relations (15), it is necessary to add additional terms linear in gradients to the standard expression for currents and entropy flux in an ideal fluid, including terms of the form (2) and (4) with unknown coefficients σ_B and σ_μ. After that $\partial_\mu s^\mu \geq 0$, the condition becomes an equality and leads to a system of equations for unknown coefficients. The solution of the corresponding system of equations leads to (3) and (5).

Based on this analysis, it follows that an anomaly that has order n in terms of gradients should contribute to the current in order $n - 1$. In particular, anomaly $F\tilde{F}$ was 2nd order in terms of gradients and resulted in CVE and CME, which are 1st order.

Then it would seem that the gravitational chiral anomaly $R\tilde{R}$, being of the fourth order in terms of the metric gradients, should lead to terms of the 3rd order. Similar terms have been obtained from microscopic theory in a number of papers. Thus, for the Dirac fields it was proved that the current has the form

$$j_\mu^{A(3)} = \left(-\frac{\omega^2}{24\pi^2} - \frac{a^2}{8\pi^2}\right)\omega_\mu. \qquad (16)$$

At first glance, it seems that generalization [4] in the case of gravity and gravity anomaly is not so simple, due to the relative complexity of gravity and a higher order in gradients. Works [5, 15], where it was shown (in flat space) that, under the condition of global thermodynamic equilibrium (11), many formulas are noticeably simplified, are promising. In particular, there is no need to consider the entropy flow and it is sufficient to consider the law of conservation of current.

The corresponding generalization as a result was constructed in [16]. First, it is necessary to construct an expansion for the current in the third order in terms of gradients. Get

$$\begin{gathered}j_\mu^{A(3)} = \xi_1(T) w^2 \omega_\mu + \xi_2(T)\alpha^2 w_\mu \\ + \xi_3(T)(\alpha w)\alpha_\mu + \xi_4(T) A_{\mu\nu} w^\nu + \xi_5(T) B_{\mu\nu}\alpha^\nu,\end{gathered} \qquad (17)$$

where $\xi_n(T)$ are the unknown transport coefficients; $\alpha_\mu = u^\nu \nabla_\nu \beta_\mu$ and $w_\mu = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta} u^\nu \nabla^\alpha \beta^\beta$ are the "thermal" acceleration and vorticity, which are reduced to the usual kinematic acceleration and vorticity $\alpha_\mu = a_\mu/T$, $a_\mu = u^\nu \nabla_\nu u_\mu$, and $w_\mu = \omega_\mu/T$ in the state of global equilibrium; and $A_{\mu\nu} = u^\alpha u^\beta R_{\alpha\mu\beta\nu}$ and $B_{\mu\nu} = \frac{1}{2}\epsilon_{\alpha\mu\eta\rho} u^\alpha u^\beta R_{\beta\nu}^{\eta\rho}$ are $\epsilon_{\alpha\mu\eta\rho}$ the Levi–Civita symbol in curved space.

In (17) there are both terms c with ξ_1, ξ_2, and ξ_3, which survive in the flat spacetime limit, and terms c with ξ_4 and ξ_5, which explicitly depend on the curvature. Simplicity (17) follows from the condition of global equilibrium, as well as the additional condition $R_{\mu\nu} = 0$ imposed on the external gravitational field.

Further, it is necessary to require that current divergence (17) was equal to (1) (the gravitational part). The corresponding equation breaks down into a sum of independent terms. Equating to zero the coefficients in front of each of the terms, we obtain a system of equations for the coefficients ξ_n. In this case, from dimensional considerations, one can immediately obtain that $\xi_1 = T^3\lambda_1$, $\xi_2 = T^3\lambda_2$, $\xi_3 = T^3\lambda_3$, $\xi_4 = T\lambda_4$, $\xi_5 = T\lambda_5$. The solution of the system of equations has the form

$$\begin{gathered}(\lambda_1 - \lambda_2)/32 = \mathcal{N}, \quad \lambda_3 = 0, \\ \lambda_4 = 8\mathcal{N} + \lambda_1/2, \quad \lambda_5 = -24\mathcal{N} + \lambda_1/2.\end{gathered} \qquad (18)$$

It is remarkable that, even passing to the limit of flat space, there remains a current of the form

$$j_\mu^{A(3)} = \lambda_1(\omega_\nu \omega^\nu)\omega_\mu + \lambda_2(a_\nu a^\nu)\omega_\mu, \qquad (19)$$

which is nevertheless associated with anomaly (1) by the first of the relations in (18). Corresponding anomalous current (19) in [16] was called the kinematic vortical effect (KVE), because it explicitly depends only on the kinematic quantities: acceleration and vorticity. Note, however, that λ_1 and λ_2 become explicit functions of the parameters of the medium for massive particles.

5. CONCLUSIONS

In this paper, we talked about the search for manifestations of the gravitational chiral quantum anomaly in hydrodynamics. Compared to the case of the gauge anomaly, the derivation of the consequences of the gravitational chiral anomaly is more nontrivial and is closely related to the consideration of the properties of space that has an event horizon. Such nontriviality is connected, first and foremost, with the violation of the direct correspondence in order of the derivative between the current and the anomaly in this case [9]. However, in the case of higher spins, the situation is even more complicated. This is confirmed by the direct calculation of CVE in the Rarita–Schwinger–Adler model.

On the other hand, from the point of view of the initial hydrodynamic equations, the gravitational chiral anomaly should correspond to a third-order effect in gradients. Indeed, the current in a medium with vorticity and with acceleration turns out to be anomalous. The derivation of this connection is a generalization of the well-known calculation for the gauge anomaly to the case of curved spacetime and is based on the consideration of only basic hydrodynamic equations.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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