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PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

Sunrise Integral in Non-Relativistic QCD with Elliptics

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Abstract—The main steps of the process of obtaining the result [1] in terms of elliptic polylogarithms for a two-loop sunrise integral with two different internal masses with pseudothreshold kinematics for all orders of the dimensional regulator are shown.

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INTRODUCTION

Feynman integrals allow a Laurent expansion with respect to a dimensional regulator, and the coefficients of this expansion can often be explicitly computed in terms of well-known special functions such as multiple polylogarithms (MPL) and multiple elliptic polylogarithms (eMPL) (see the recent paper [2] and references and discussions therein). In practice, it is often possible to truncate the Laurent series, since only a few orders of expansion are required to calculate physically significant quantities. Nevertheless, it is interesting to investigate the analytical structure of these coefficients at higher orders, or, more generally, at all orders of the dimensional regulator.

This article shows the main stages of the consideration [1] of a two-loop sunrise integral topology with two different internal masses, denoted m and M, and pseudo-threshold kinematics $p^2 = -m^2$ [3] (see also [4, 5]). This integral family arises when considering the nonrelativistic limit of Quantum Chromodynamics.

The analytical structure of the sunrise topology considered in [1] was studied using differential equations in [6–10] and using the effective mass analysis in [11–13] (see the recent review in [14]). Moreover, this integral family admits a closed-form solution in terms of ${}_3F_2$-hypergeometric functions, as shown in [3] (the corresponding off-shell diagrams with equal masses are much more complicated and their explicit solution requires hypergeometric Appell functions F_2 [15]). In [1], we obtained an expression in terms of eMPL, valid for all orders of the dimensional regulator[1].

[1] Similar results in more complicated cases can be found in [16–19].

THE SUNRISE INTEGRAL

Following [3] we study the sunrise integral topology defined as,

$$J_{i_1,i_2,i_3}(m^2, M^2) = \iint \frac{d^D k_1 d^D k_2}{[k_2^2 - m^2]^{i_1}[k_1^2 - M^2]^{i_2}[(k_1 - k_2 - q)^2 - M^2]^{i_3}}\Bigg|_{q^2=-m^2}, \tag{1}$$

with $D = 4 - 2\epsilon$. As it was observed in [1],

$$J_{1,2,2} = \hat{N}_1[J^{(1)}_{1,2,2}(t) - (2t)^{\varepsilon-1} J^{(2)}_{1,2,2}(t) - (2t)^{\varepsilon} J^{(3)}_{1,2,2}(t)], \quad \left(t = \frac{m^2}{2M^2}\right), \tag{2}$$

where,

$$J^{(1)}_{1,2,2}(t) = \frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2},1+\frac{\epsilon}{2},\frac{3}{2}+\frac{\epsilon}{2}\\ 2-\epsilon,\frac{5}{4},\frac{7}{4}\end{matrix}\Bigg| -t^2\right) = -\frac{\hat{K}}{2^{2\epsilon+2}\varepsilon t^2} I^{(1)}(t),$$

$$J^{(2)}_{1,2,2}(t) = \frac{1}{(1+2\varepsilon)(1-\varepsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{1}{2}+\epsilon,1+\frac{\epsilon}{2},1+\epsilon\\ \frac{3}{2}-\frac{\epsilon}{2},\frac{3}{4}+\frac{\epsilon}{2},\frac{5}{4}+\frac{\epsilon}{2}\end{matrix}\Bigg| -t^2\right) = \frac{\hat{K}}{2^{4\epsilon+1}t^{1-\varepsilon}} I^{(2)}(t), \tag{3}$$

$$J^{(3)}_{1,2,2}(t) = \frac{1+\varepsilon}{\varepsilon(2-\varepsilon)(3+2\varepsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2}+\frac{\epsilon}{2},1+\epsilon,\frac{3}{2}+\epsilon\\ 2-\frac{\epsilon}{2},\frac{5}{4}+\frac{\epsilon}{2},\frac{7}{4}+\frac{\epsilon}{2}\end{matrix}\Bigg| -t^2\right) = -\frac{\hat{K}}{2^{4\epsilon+2}\varepsilon t^2} I^{(3)}(t)$$

and $\hat{K}$ is defined as,

$$\hat{K} = \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+\varepsilon)}, \tag{4}$$

while the factors $I^{(i)}(t)$ represent the relevant integrals,

$$I^{(1)}(t) = \int_0^1 dp\, p^{\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}} \times \left((p^2t^2+1)^{-\frac{1}{2}}(1-\varepsilon J_1(p))-1\right),$$

$$I^{(2)}(t) = \int_0^1 dp\, p^{3\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}}\left(p^2t^2+1\right)^{-\epsilon-\frac{1}{2}} J_2(p), \quad (5)$$

$$I^{(3)}(t) = \int_0^1 dp\, p^{2\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}} \times \left(\left(p^2t^2+1\right)^{-\frac{\epsilon}{2}-\frac{1}{2}}(1-\frac{\varepsilon}{2}J_3(p))-1\right),$$

with

$$J_1(p) = q(p)^{\epsilon}\int_0^{q(p)} dz((1-z)^{-\frac{1}{2}}-1)z^{-\epsilon-1},$$
$$q(p) = \frac{p^2t^2}{p^2t^2+1},$$
$$J_2(p) = \int_0^{pt} dz z^{-\epsilon}(z^2+1)^{\epsilon-\frac{1}{2}}, \quad (6)$$
$$J_3(p) = q(p)^{\frac{\epsilon}{2}}\int_0^{q(p)} dz((1-z)^{-\frac{\epsilon}{2}-\frac{1}{2}}-1)z^{-\frac{\epsilon}{2}-1},$$

We remark that integral $J_{1,2,2}$ has a finite ϵ expansion. Other sunrise integrals, $J_{1,1,2}$ and $J_{1,1,1}$, considered in [1], contain singularities but their consideration is beyond the slope of this short paper.

ALL ORDERS RESULT IN TERMS OF ELLIPTIC POLYLOGARITHMS

We are interested in the computation of iterated integrals of the form,

$$\int_0^x dx_1 R_1(x_1, y(x_1))\int_0^{x_1} dx_2 R_2(x_2, y(x_2)) \ldots \int_0^{x_{n-1}} dx_n R_n(x_n, y(x_n)), \quad (7)$$

where R_i are rational functions of their arguments and $y(x)$ is an elliptic curve,

$$y(x) = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}. \quad (8)$$

All iterated integrals of the form (7) can be expressed in terms of eMPLs:

$$E_4\left({n_1,\ldots,n_k \atop c_1,\ldots,c_k};x\right) = \int_0^x dt\,\varphi_{n_1}(c_1,t)E_4\left({n_2,\ldots,n_k \atop c_2,\ldots,c_k};t\right), \quad (9)$$
$$E_4(;x) = 1.$$

By construction, the kernels $\varphi_n(c,x)$ have at most simple poles, and they are (see [20] for a detailed discussion)

$$\varphi_0(0,x) = \frac{c_4}{y(x)}, \quad \varphi_1(c,x) = \frac{1}{x-c},$$
$$\varphi_{-1}(\infty,x) = \frac{x}{y(x)}, \quad (10)$$
$$\varphi_{-1}(c,x) = \frac{y(c)}{(x-c)y(x)} - (\delta_{c0}+\delta_{c1})\frac{1}{x-c},\ldots$$

where

$$c_4 = \frac{1}{2}\sqrt{a_{13}a_{24}} \text{ with } a_{ij} = a_i - a_j. \quad (11)$$

Moreover we define,

$$E_4\left({\vec{1} \atop \vec{0}};x\right) \equiv \frac{\log(x)^n}{n!}, \quad (12)$$

where $\vec{1}$ and $\vec{0}$ are vectors with elements equal to 1 and 0 respectively, and $n = \text{length}(\vec{1}) = \text{length}(\vec{0})$.

EMPLs are a generalisation of MPLs defined recursively as,

$$G(a_1,a_2,\ldots,a_n;x) = \int_0^x \frac{dt}{t-a_1}G(a_2,\ldots,a_n,t), \quad (13)$$
$$G(;x) \equiv 1,$$

and,

$$G(\vec{0},x) \equiv \frac{\log(x)^n}{n!}. \quad (14)$$

By definition we see that MPLs are a subset of eMPLs,

$$E_4\left({1,\ldots,1 \atop c_1,\ldots,c_n};x\right) = G(c_1,c_2,\ldots,c_n;x), \quad c_i \neq \infty. \quad (15)$$

As with all iterated integrals, eMPLs satisfy a shuffle algebra with the shuffle product defined as

$$E_4\begin{pmatrix} a_1 & \ldots & a_n \\ & & & ;x \\ a_1' & \ldots & a_n' \end{pmatrix} E_4\begin{pmatrix} b_1 & \ldots & b_m \\ & & & ;x \\ b_1' & \ldots & b_m' \end{pmatrix} = \sum_{\vec{c}=\vec{a}\vec{b}} E_4\begin{pmatrix} c_1 & \ldots & c_{n+m} \\ & & & ;x \\ c_1' & \ldots & c_{n+m}' \end{pmatrix}. \quad (16)$$

The vector $\vec{c}$ is the vector obtained by shuffling all $\vec{a}$ and $\vec{b}$ while preserving the order of the elements $\vec{a}$ and $\vec{b}$ respectively.

REGULARISATION

As we will see below, we are interested in computing definite integrals of the form,

$$\int_0^1 f(x)dx = F(1) - F(0), \; \frac{\partial F(x)}{\partial x} = f(x). \quad (17)$$

In some cases, the primitive is ill-defined when evaluated on the boundaries of integration, and two constraints must be satisfied in order to evaluate the definite integral:

$$\int_0^1 f(x)dx = \lim_{x\to 1} F(x) - \lim_{x\to 0} F(x) \equiv \mathrm{Reg}_{0,1} F(x). \quad (18)$$

COMPACT REPRESENTATION

The analysis in [1] implies that all integrals are formally evaluated as

$$\sum_{l=1}^{n'} C_l \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!\,j!} \int_0^1 dx k_{1,l}(x) L_{1,l}^i(x) \int_1^x dz k_{2,l}(z) L_{2,l}^j(z), \quad (19)$$

where C_l are some coefficients, $L_{i,j}(x)$ are combinations of eMPLs of depth one, while $k_{i,j}(x)$ are combinations of integration kernels. These integrals can be directly computed in terms of eMPL by shuffle expanding the eMPL products of the integrands and recursively using the definition of eMPL.

To make the notation more compact and to make the properties of the result in terms of eMPL clear, we use the following notation for double integrals of (19). Denoting the primitive $k_{i,j}(x)$ as $K_{i,j}(x)$ and defining the bilinear *-operator as,

$$E_4\left({\vec{n} \atop \vec{c}}; x\right) * E_4\left({\vec{m} \atop \vec{d}}; x\right) = E_4\left({\vec{n}\ \vec{m} \atop \vec{c}\ \vec{d}}; x\right), \quad (20)$$

we can write (19) in the following form

$$\mathrm{Reg}_{0,1} \sum_{l=1}^{n'} C_l \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!\,j!} K_{1,l}(x) * L_{1,l}(x)^i \times \left[K_{2,l}(z) * L_{2,l}(z)^j\right]_1^x, \quad (21)$$

where all eMPL products are shuffle expanded before the * operator is applied, and these operations are performed on the inner square brackets first. Finally, the lower and upper scripts applied to the square brackets indicate the following operation:

$$\left[F(x)\right]_1^x = F(x) - F(1). \quad (22)$$

THE INTEGRAL $I^{(2)}(t)$

We show how the solution strategy of the previous section works in practice by considering the integral $I^{(2)}(t)$ in Eq. (5). The dependence on the elliptic curve is made explicit by changing the variable $p \to (1 - x^2)$,

$$I^{(2)}(t) = \int_0^1 dx \frac{2}{t(1-x^2)y(x)} \left(\frac{(1-x^2)^3}{t^2 x^2 y(x)^2}\right)^{\epsilon} \times \int_0^{t(1-x^2)} dz \frac{1}{\sqrt{z^2+1}} \left(z + \frac{1}{z}\right)^{\epsilon}, \quad (23)$$

where the inner integral can be expressed as

$$\int_0^{t-tx^2} dz \frac{1}{\sqrt{z^2+1}} \left(z + \frac{1}{z}\right)^{\epsilon} = -\int_1^x dz \frac{2z}{y(z)} \left(\frac{ty^2(z)}{1-z^2}\right)^{\epsilon}. \quad (24)$$

All the ε-powers are expanded in ε:

$$\left(\frac{ty^2(x)}{1-x^2}\right)^{\epsilon} = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \log^i \left(\frac{ty^2(x)}{1-x^2}\right). \quad (25)$$

The resulting logarithm is Kn be expressed in terms of eMPLs as

$$\log\left(\frac{ty^2(x)}{1-x^2}\right) = \log(ty^2(0)) + \int_0^x dz \frac{2z\left(t^2(z^2-1)^2 - 1\right)}{t^2(z^2-1)y(z)^2}, \quad (26)$$

where $y^2(0) = t^{-2} + 1$. The integrand can be written in terms of the integration kernels:

$$\frac{2z\left(t^2(z^2-1)^2 - 1\right)}{t^2(z^2-1)y(z)^2} = \sum_{i=1}^{4} \varphi_1(a_i, z) - \varphi_1(-1, z) - \varphi_1(1, z), \quad (27)$$

where we denoted with a_i the four roots of the elliptic curve,

$$a_1 = -\frac{\sqrt{t-i}}{\sqrt{t}},\ a_2 = \frac{\sqrt{t-i}}{\sqrt{t}}, \quad a_3 = -\frac{\sqrt{t+i}}{\sqrt{t}},\ a_4 = \frac{\sqrt{t+i}}{\sqrt{t}}. \quad (28)$$

Upon integration we find,

$$L_4 \equiv \log\left(\frac{ty^2(x)}{1-x^2}\right) = \sum_{i=1}^{4} E_4\left({1 \atop a_i}; x\right) - E_4\left({1 \atop -1}; x\right) - E_4\left({1 \atop 1}; x\right) + \log(t^2+1) - \log(t). \quad (29)$$

Finally, all prefactors can be expressed in terms of integration kernels

$$\frac{2}{t(1-x^2)y(x)} = \varphi_{-1}(-1, x) - \varphi_{-1}(1, x) - \varphi_1(1, x). \quad (30)$$

By taking the primitive of Eq. (30) we obtain,

$$K_4 \equiv E_4\left({-1 \atop -1}; x\right) - E_4\left({-1 \atop 1}; x\right) - E_4\left({1 \atop 1}; x\right). \quad (31)$$

Applying these methods to all relevant logarithms and prefactors, we get the result in terms of integrals of the form (19), which are directly evaluated in eMPL, for example, by Eq. (21):

$$I^{(2)}(t) = \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!\,j!} K_4 * L_5^i \left[K_5 * L_4^j\right]_1^x, \quad (32)$$

where,

$$L_5 = -\sum_{i=1}^{4} E_4\left(\begin{smallmatrix}1\\a_i\end{smallmatrix};x\right) + 3E_4\left(\begin{smallmatrix}1\\-1\end{smallmatrix};x\right) - 2E_4\left(\begin{smallmatrix}1\\0\end{smallmatrix};x\right) + 3E_4\left(\begin{smallmatrix}1\\1\end{smallmatrix};x\right) - \log\left(t^2+1\right), \quad (33)$$

$$K_5 = -2E_4\left(\begin{smallmatrix}-1\\\infty\end{smallmatrix};x\right).$$

CONCLUSIONS

In this short paper, we have shown the main stages of the study [1] of the sunrise integral $J_{1,2,2}$ with two different internal masses and pseudothreshold kinematics in dimensional regularization. This integral admits a closed form solution in terms of hypergeometric functions [3], and we have used this representation as the starting point of our analysis. In particular, in [1] we shown that all relevant hypergeometric functions can be represented as iterated integrals depending on one elliptic curve (see also Eq. (5)). When these integrals are expanded in terms of a dimensional regulator, the expansion coefficients are iterated integrals in terms of rational functions on the corresponding elliptic curve with at most simple poles. Calculating $I_1^{(2)}(t)$ in Eq. (5), we have shown a way to represent the sunrise integral $J_{1,2,2}$ in terms of eMPLs, which is true for all orders of the dimensional regulator.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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