
**PHYSICS OF ELEMENTARY PARTICLES
AND ATOMIC NUCLEI. THEORY**

The Contour Gauge in Use: Telling Untold

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Abstract—In quantum field theory, we discuss the main features of the (non-local) contour gauge which extends the local axial-type gauge used in most approaches. Based on the gluon geometry, we demonstrate that the contour gauge does not suffer from the residual gauge. We discuss the useful correspondence between the contour gauge conception and the Hamiltonian (Lagrangian) formalism. Having compared the local and non-local gauges, we again advocate the advantage of the contour gauge use.

Keywords: contour gauge, path group, gauge symmetry

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1. INTRODUCTION

Since any gauge theories can be considered as the systems with dynamical constraints, it is difficult to overestimate the role of gauge conditions which are nothing but the additional constraints appeared in the Hamiltonian formalism. As well-known, the gauge theory can be consistently quantized if it is possible to uniquely resolve all constraint conditions eliminating the unphysical degrees of freedom. However, in the Hamiltonian (Lagrangian) approach, the constraint conditions with respect to the generalized momenta and coordinates have, as a rule, very nontrivial forms and, hence, finding unique solutions of these equations is not a simple task. It sometimes becomes even impossible in some cases due to the presence of the Gribov ambiguities.

Fortunately, as it is known for many years, it turns out that the infinite group orbit volume, which leads to the problem with quantization, can be factorized out to the insubstantial normalization factor thanks to the gauge invariance property of the corresponding Lagrangian (or Hamiltonian) of a theory [1]. The mentioned infinite volume that has been included in the factorized prefactor of the functional integration is now not the principle obstruction for the quantization of the gauge theory.

Nowadays, only the local type of gauges can traditionally be called as the most popular gauges used for the practical applications. At the same time, some of useful local gauges, for example the axial-type of gauges, might suffer from the residual gauge freedom which, in its turn, can ultimately give rise to the spuri-

ous singularity. In contrast to the local (axial) gauge condition, the use of contour gauge, as a class of non-local gauges, allows us indeed to fix completely the gauge freedom in the most simply form without an additional assumptions. In this case, there is no the residual gauge freedom unless the boundary conditions for gluons relate to the non-trivial topology (see also [2]).

Notice that this complete gauge fixing is provided by the construction of contour gauge from the very beginning. Indeed, within the contour gauge conception the gauge orbit representative should be first fixed and, then, a certain local gauge condition which is correlated with the given (gauge) orbit representative should be searched. In some sense, the opposite logic of the gauge fixing compared to the use of usual local gauges takes place.

Among the modern phenomenology theorists, there is such a prejudice that the study of any non-local gauge conditions in QCD is not attractive because specific contour-gauge techniques adopted for the QCD phenomenology are very complicated and the efforts to study the differential geometry details cannot be required. In the present short paper, we attempt to break this superficial and wrong impression. Namely, the differential geometry technique and the interpretation of gluons as a connection on the principle fiber bundle are merely needed (a) for the demonstration that with the help of the contour gauge the gauge freedom can be uniquely fixed upto the residual freedom; (b) for a proof of the fact that in the local gauge, given by $A^+ = 0$, two different representa-

tions of transverse gluon field through the strength tensor, related to the integrations from x to $+\infty$ and from $-\infty$ to x , are actually not equivalent ones. We stress that without the contour gauge which is more general in comparison with any axial local gauges, it is not possible to distinguish these two different representations. At the same time, the assumption of equivalence leads to many problems, for example, with the gauge invariance of Drell–Yan-like hadron tensors.

Since the interest to the details of contour gauge applications increases in the phenomenological community, it is worth to make public the important subtleties based on the mathematical technique adjusted to the physical language. Moreover, we focus here on the important explanations which are being remained untold ones in the preceding publications. On the other hand, since in the recent literature one can still find a wrong representation of the transverse gluon field through the strength tensor considered in the local axial gauge, we treat this fact as a lack of explanation of what means the contour gauge and of how to use it. All these underlie our motivation of the present paper.

In the first part of the paper, in order to clarify the main features of contour gauge as a gauge without the residual gauge freedom, we shortly remind the elements of Hamiltonian and Lagrangian formalisms of quantization. Actually, the Hamiltonian formalism might be considered as something useless from the point of view of phenomenological applications. But, on the other hand, the Hamiltonian formalism is a more convenient frame to understand the contour gauge in the context that how the contour gauge condition defines the manifold surface crossing over a group orbit of the fiber uniquely. Meanwhile, there are no doubts that the Lagrangian formalism is assumed to be better designed for the practical applications. We combine both formalisms depending on the current goals.

The attempt to present an alternative explanations of the contour gauge conception has been done in the second part of the paper. Apart, we also implement a comparative analysis of the local and non-local gauges of axial type.

2. THE LAGRANGIAN AND HAMILTONIAN SYSTEMS WITH THE DYNAMICAL CONSTRAINTS

As well-known, the local axial gauge, $A^+ = 0$, suffers from the residual gauge freedom demanding the additional requirements to fix it. In the most cases, the formulation of additional requirements is not a trivial task within the Lagrangian system. Indeed, if one demands simultaneously $A^+ = A^- = 0$, the maximal gauge fixing is got no problem in a classical theory

only. However, in a quantum theory, the simultaneous conditions $A^+ = 0$ and $A^- = 0$ as delta-function arguments in the corresponding functional integration (which lead to the effective Lagrangian with $1/\xi_1(nA)^2$ and $1/\xi_2(n^*A)^2$) result in the absence of the well-defined gauge propagator because the corresponding kinematical operator cannot be inverted. In this connection, it is necessary to establish an alternative method of gauge fixing compared to the classical approaches with the effective Lagrangian including both $(nA)^2$ and $(n^*A)^2$. The contour gauge conception gives us such an alternative and effective method.

To better understand subtleties related to the contour gauge, it is worth to remind the main stages of the Lagrangian (L-system) and Hamiltonian (H-system) approaches to the quantization of gauge fields (see, for example, [3]). In this section, we use the textbook language to show that the H-system is the most adequate approach in order to demonstrate the gauge-field quantization “philosophy” with the resolved additional constraints. However, due to the special role of a time-coordinate in H-system, the L-system is more suitable for a practical use. In other words, the H-system is needed for a help to see how the contour gauge fixes uniquely the total gauge freedom, but the main computation procedure has been formulated in terms of L-system.

2.1. H-System with Dynamical Constraints

Let us consider the H-system, defined by $H(p_i, q_i)$, where the phase space Γ has been formed by the generalized momenta p_i and coordinates q_i and, in addition, $2m$ constraints on p_i and q_i have been imposed. Traditionally, these $2m$ constraints are denoted as $\varphi_a(p_i, q_i)$ and $\chi_a(p_i, q_i)$.

We suppose that the H-system we consider has an equivalence orbit which is nothing but the gauge group orbit in the gauge theory. Since the phase space Γ is overfilled by unphysical degrees, in the most ideal case, we have to resolve all the constraints. The additional constraints $\chi_a(p_i, q_i)$ are necessary to fix uniquely the orbit representative in order to quantize the H-system. After resolving all constraints, we deal with the quantized H-system where the physical phase space Γ^* of dimension $2(n - m)$ is a subspace of the initial space Γ of dimension $2n$ and is a fundament for H-system in terms of physical configurations, $H^*(p_i^*, q_i^*)$.

Hence, in terms of the functional integration, the amplitude between the initial $|q_1^{\text{in}}, \dots, q_n^{\text{in}}; t^{\text{in}}\rangle$ and the

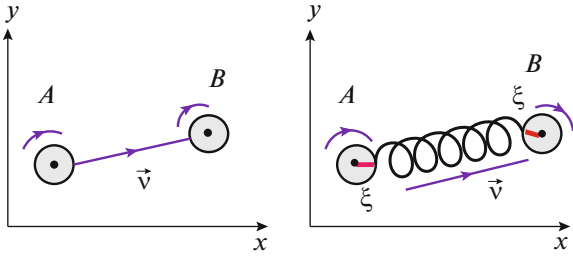


Fig. 1. The H-system with unresolved and resolved constraints. The left panel represents the system with the inner (gauge) symmetry; the right panel corresponds to the system with the fixed (preferable) site ξ on the ball surface.

final $\langle q_1^f, \dots, q_n^f; t^f |$ states takes the following form (modulo the unimportant normalization factors) [1, 3]

$$\begin{aligned} & \langle q_1^f, \dots, q_n^f; t^f | q_1^{\text{in}}, \dots, q_n^{\text{in}}; t^{\text{in}} \rangle \\ &= N \int Dp_i(t) Dq_i(t) \delta(\varphi_a) \delta(\chi_a) \det\{\varphi_a, \chi_a\} \\ & \times \exp \left\{ i \int_{t_{\text{in}}}^{t_f} dt [p_i \partial_0 q_i - H(p_i, q_i)] \right\}, \end{aligned} \quad (1)$$

where $\{\dots, \dots\}$ denotes the Poisson brackets.

As usual, the delta-function $\delta(\varphi_a)$ of Eq. (1) can be presented through the integration over the Lagrange factor λ_a as

$$\delta(\varphi_a) = \int (d\lambda_a) e^{i\lambda_a \varphi_a(p_i, q_i)} \quad (2)$$

giving the generalized Hamiltonian of system which reads

$$H'(p_i, q_i) = H(p_i, q_i) + \sum_a \lambda_a \varphi_a(p_i, q_i). \quad (3)$$

If we now suppose that the constraint conditions (see the delta-function arguments) have somehow been resolved, the amplitude is given by the following functional integration

$$\begin{aligned} & \langle q_1^f, \dots, q_n^f; t^f | q_1^{\text{in}}, \dots, q_n^{\text{in}}; t^{\text{in}} \rangle N \int \mathcal{D}p_i^*(t) \mathcal{D}q_i^*(t) \\ & \times \exp \left\{ i \int_{t_{\text{in}}}^{t_f} dt [p_i^* \partial_0 q_i^* - H^*(p_i^*, q_i^*)] \right\}, \end{aligned} \quad (4)$$

where only the physical generalized momenta and coordinates are forming the integration measure and the Hamiltonian. In other words, the generating functional of (4) corresponds to the H-system with the dynamical constraints which have been resolved and, therefore, there is no a (gauge) freedom associated with the arbitrary Lagrange factor λ_a . It would be an ideal situation which, as known, cannot be realized practically in the most of cases. However, the contour gauge as a class of non-local gauges gives a possibility to realize practically the mentioned ideal situation

because the contour gauge condition has a unique solution by construction, see below.

It is instructive to illustrate the difference between the H-system with unresolved and resolved constraint conditions with the help of a trivial mechanical example (see Fig. 1). Let us consider the homogenous ball which moves from the point A to the point B . The ball has a spherical symmetry under the rotation around the inertia center. For the simplicity, we focus on the rotation of the ball in some plane. Since the ball surface is homogenous, we do not have a chance to observe the rotation of the moving ball unless we mark some point on the surface. The invisible ball rotation around its center of inertia corresponds, roughly speaking, to the inner (gauge) transformations of the H-system and does not affect much the trajectory provided the angle velocity is constant, see the left panel of Fig. 1. In this case, the inertia center plays a role of “physical” configurations of the H-system, while the moving of different sites on the ball surface is invisible and relates to the “unphysical” configurations.

If we mark the site on the ball surface by a dash, we break down the rotation symmetry in the meaning of that we choose the preferable site of the ball surface and the ball rotation becomes visible. Therefore, in this case, we can describe the moving dash together with the inertia center as “physical” configurations of the H-system when the ball position varies from A to B , see the right panel of Fig. 1.

In the context of the contour gauge conception, the marked dash on the ball corresponds to the certain gauge function θ of the photon (gluon) gauge transforms which has been fixed by the contour gauge condition. Moreover, in the above mentioned ideal case of resolved constraints, we would merely deal with the photon (or gluon) as a massless vector field (transforming as a spinor of 2-rank under the Lorentz group) which is described by the Hamiltonian without the gauge transforms.

Notice that the Abelian and/or non-Abelian gauge theory can be treated as the H-system where the amplitudes of the state transitions take the forms which are similar to Eq. (1) provided the following correspondences between the canonical variables and the dynamical constraints (see below)

$$\begin{aligned} (p_i; q_i; t) & \Rightarrow ((E_0, E_i); (A_0, A_i); x), \\ \delta(\varphi_a) & \Rightarrow \delta(\partial_i E_i) \delta(E_0), \\ \delta(\chi_a) & \Rightarrow \delta(\partial_i A_i) \delta(A_0), \\ \det\{\varphi_a, \chi_a\} & \Rightarrow \Phi(A). \end{aligned} \quad (5)$$

2.2. L-System with Dynamical Constraints and Its Connection with H-System

In the most typical gauge theories, resolving the additional (gauge) conditions is not a simple task. The gauge conditions, leading to the equation system for

the gauge function $\theta(x)$, would have a unique solution θ_0 . In fact, it might be practically impossible.

To avoid the need for finding the unique solution θ_0 , Faddeev and Popov have proposed the approach (FP-method) where the infinite group orbit volume can be factorized out to the insubstantial normalization factor thanks to the gauge invariance property of the corresponding Lagrangian (or Hamiltonian) of theory [1].

For the sake of simplicity, we are now going to dwell on $U(1)$ gauge theory. The H-system can be readily stemmed from the Lagrangian formalism of first order where the Lagrangian is defined as

$$\mathcal{L}_{(1)} = -\frac{1}{2} \left[\partial_\mu A_\nu(x) - \partial_\nu A_\mu - \frac{1}{2} F_{\mu\nu} \right] F_{\mu\nu}, \quad (6)$$

with A_μ and $F_{\mu\nu}$ being independent field configurations. The Lagrangian of Eq. (6) can be written in 3-dimensional form as

$$\begin{aligned} \mathcal{L}_{(1)} = & E_i \partial_0 A_i + A_0 \partial_i E_i \\ & + \frac{1}{2} E^2 + \frac{1}{4} B_i \varepsilon_{ijk} \partial_k A_j + \frac{1}{2} B^2, \end{aligned} \quad (7)$$

where $E_i = F_{i0}$ and A_i imply the generalized momenta and coordinates, respectively; $B_i = 1/2 \varepsilon_{ijk} F_{jk}$ are the implicit variables in the integration measure, see below. From the Euler–Lagrange equations we can get the equations which do not include ∂_0 and give the constraint conditions written in the forms of

$$\begin{aligned} E_0 = 0 & \text{—the primary condition,} \\ \partial_i E_i = 0 & \text{—the secondary condition.} \end{aligned} \quad (8)$$

As well-known, any gauge theory can be considered as the theory with the constraints which are applied on the field configurations. Moreover, the constraint conditions of Eq. (8) have to be completed by the additional (gauge) conditions, for example

$$A_0 = 0 \quad \text{and} \quad \partial_i A_i = 0. \quad (9)$$

From the theoretical point of view, the full set of conditions defined by Eqs. (8) and (9) should eliminate all unphysical degrees of freedom for the correct quantization of H- (or L-) system.

Using the FP method, we adhere the functional integration approach to quantization of gauge fields and start from the functional integration written for the L-system as [3]

$$\int \mathcal{D}\theta(x) \int \mathcal{D}A_\mu \delta(F[A]) \Phi(A) e^{iS[A]}, \quad (10)$$

where the infinite group orbit volume given by $\mathcal{D}\theta(x)$ has been factorized out in the integration measure due to the gauge invariant action $S[A]$ and functional

$\Phi(A)$. Hence, the exact magnitude of the group volume prefactor, i.e.

$$v_g \equiv \int \mathcal{D}\theta(x) = \int \prod_x [d\theta(x)] = \{\infty, 0\}, \quad (11)$$

is irrelevant because the prefactor should be cancelled by the corresponding normalization of Green functions.

In Eq. (11), the infinite group volume corresponds to the standard case of unresolved gauge conditions, while the zero group volume appears in the case of resolved gauge conditions. The latter is a principal observation of this section. Indeed, the integration measure $[d\theta(x)]$ can be defined on the group manifold as an invariant measure,

$$\int [d\theta] f(\theta) \sim \sum_{\theta \in G} f(\theta). \quad (12)$$

$\prod_x [d\theta(x)]$ denotes the product of the invariant measures defined on the structure group G of fiber over each point of the Minkowski space. If the gauge function θ is not fixed, we have the infinite integration over the group invariant measure, otherwise (that is, if the gauge function θ is somehow fixed) the integration is equal zero for each point of the Minkowski space but the value of fixed θ_i can vary from one point to another.

Equation (10) can be identically rewritten as

$$\int \mathcal{D}A_\mu \mathcal{D}F_{\mu\nu} \delta(F[A]) \Phi(A) e^{iS[A,F]}, \quad (13)$$

where the functional of action $S[A, F]$ is given the Lagrangian of Eq. (6). Focusing on the three-dimensional forms, we obtain that the functional integration is given by

$$\int \mathcal{D}A_i \mathcal{D}A_0 \mathcal{D}B_i \mathcal{D}E_i \delta(F[A]) \Phi(A) e^{iS[A,E,B]}, \quad (14)$$

where $S[A, E, B]$ is now defined by Eq. (7). Integrations over B_i and A_0 in Eq. (14) lead to the following functional integral:

$$\begin{aligned} & \int \mathcal{D}A_i \mathcal{D}E_i \delta(F[A]) \Phi(A) \delta(\partial_i E_i) \\ & \times \exp\{i \int dz [E_i \partial_0 A_i + H(E_i, A_i)]\}, \end{aligned} \quad (15)$$

where the Hamiltonian is given by

$$H(E_i, A_i) = \frac{1}{2} E^2 - \frac{1}{2} (\varepsilon_{ijk} \partial_j A_k)^2 \quad (16)$$

and the gauge condition can be written as $F(A) = \partial_i A_i$. On the other hand, Eq. (15) can be presented in the equivalent form as

$$\begin{aligned} & \int \mathcal{D}A_i \mathcal{D}A_0 \mathcal{D}E_i \mathcal{D}E_0 \delta(\partial_i A_i) \Phi(A) \delta(\partial_i E_i) \delta(A_0) \delta(E_0) \\ & \times \exp\{i \int dz [E_i \partial_0 A_i + E_0 \partial_0 A_0 + H(E_i, A_i; E_0, A_0)]\}, \end{aligned} \quad (17)$$

where the primary and secondary constraints together with the gauge conditions of Eq. (9) have been explicitly shown in the functional integrand. This representation of H-system resembles the functional integration presented by Eq. (1).

Notice that, in H-system, it is not a problem to write all constraints through the delta functions because we have no needs to invert the kinematical-like operator as it would be necessary in L-system forming the Feynman rules. At the same time, the H-system approach is not convenient for the practical computation in QFT.

In conclusion, the basic reason for writing this section in the textbook language is the following: the section is preparing a reader for the main features of contour gauge uses. Indeed, we have reminded the differences between the H-systems with and without resolved additional dynamical constraints presenting the mechanical illustration in Fig. 1. As well known, within the FP-method of L- and H-systems there is no need to find a unique solution of the system of the additional (gauge) conditions with respect to the gauge function $\theta(x)$. Thanks for the gauge invariance of Lagrangian (Hamiltonian) and functional $\Phi(A)$, the infinite group orbit volume defined by $[d\theta]$ has been factorized out in the corresponding functional integration measure giving a possibility to quantize the gauge theory modulo the residual gauge problems. However, as mentioned, if we resolve the additional (gauge) condition with respect to θ , the group representative on each of orbit can be fixed uniquely and the factorized group integral defined through the Riemann summations should be equal to zero. Since, the group volume has been cancelled by the corresponding normalization, it does not mean that the functional integration of Eq. (10) disappears. This case takes place if we use the contour gauge conception.

3. THE CONTOUR GAUGE CONCEPTION

In this section, we concentrate on the description of the contour gauge conception which implies that the corresponding gauge condition can be formally resolved in order to find the unique gauge orbit representative. A few decades ago, the contour gauge had intensively studied due to the fact that the quantum gauge theory should not suffer from the Gribov ambiguities (see, for example, [4–6]).

It states that the gauge function can be completely fix (in the H-system, see Eq. (1)) or the unphysical gluons can be eliminated (in the L-system, see Eq. (10)) if we demand the path dependent functional (Wilson path functional) to be equal to unity, i.e.

$$\mathbf{g}(x|A)|_{P(x_0,x)} \equiv \mathbb{P} \exp \left\{ ig \int_{P(x_0,x)} d\omega_\mu A_\mu(\omega) \right\} = \mathbf{1}, \quad (18)$$

where the path $P(x_0, x)$ between the points x_0 and x is fixed. The axial (light-cone) gauge, $A^+ = 0$, is a par-

ticular case of the non-local contour gauge determined by Eq. (18) provided the fixed path is given by the straightforward line connecting $\pm\infty$ with x . By construction, the contour gauge does not possess the residual gauge freedom in the finite space, i.e. in what follows the boundary gluon configurations have assumed to be equal to zero (see [2] where it is shown that the possible residual gauge can be located at the corresponding boundary). It also gives, from the technical point of view, the simplest way to fix the gauge function completely.

The contour gauge conception is closely related to the path group formalism [7, 8] and can be traced from the Mandelstam approach [9]. The geometrical interpretation of gluon fields as a connection on the principle fiber bundle $\mathcal{P}(\mathcal{X}, \pi|G)$ is the underlying basis for the use of non-local gauges. In this connection, let us define two directions: one direction is determined in the base \mathcal{X} as the tangent vector of a curve going through the point $x \in \mathcal{X}$; the other direction is defined in the fiber and can be uniquely determined as the tangent subspace related to the parallel transport [7, 8]. These two directions allow us to introduce the horizontal vector defined as

$$H_\mu = \frac{\partial}{\partial x^\mu} - ig A_\mu^a(x) D^a, \quad D^a = \mathbf{D}^a \cdot \mathbf{g} \frac{\partial}{\partial \mathbf{g}}, \quad (19)$$

where D^a denotes the corresponding shift generator along the group fiber written in the differential form and, together with the vector coefficients (connection of the principle fiber bundle) $A_\mu^a(x)$, defines the algebraic vertical (tangent) vector field on the fiber [7, 8]. The horizontal vector H_μ is invariant under the structure group G acting on the given representation of the fiber by construction.

In $\mathcal{P}(\mathcal{X}, \pi|G)$, the functional $\mathbf{g}(x|A)$ of Eq. (18) is a solution of the parallel transport equation given by

$$\frac{dx_\mu(s)}{ds} H_\mu(A) \mathbf{g}(x(s)|A) = 0, \quad (20)$$

where the fiber point $p(s) = (x(s), \mathbf{g}(x(s)))$ with the curve $x(s) \in \mathcal{X}$ parametrized by s . Eq. (20) being an algebraic differential equation takes place even if \mathbf{g} is fixed on the group because, in this case, $D^a \mathbf{g} = 0$ while $A_\mu^a(x) \neq 0$. In this connection, the condition presented by Eq. (18) implies that the full curve-linear integration goes to zero rather than the integrand itself.

It can be shown [7, 8] that every of points belonging to the fiber bundle, $\mathcal{P}(\mathcal{X}, \pi|G)$, has one and only one horizontal vector corresponding to the given tangent vector at $x \in \mathcal{X}$. We remind that the tangent vector at the point x is uniquely determined by the given path passing through x . In the frame of H-system based on

the geometry of gluons the condition of Eq. (18) corresponds to the determining of the surface on $\mathcal{P}(\mathcal{X}, \pi|G)$ that is parallel to the base plane with the path. Moreover, it singles out the identity element, $\mathbf{g} = 1$, in every fibers of $\mathcal{P}(\mathcal{X}, \pi|G)$ (see Fig. 2). This choice can be traced to the Lagrange factor λ_a which is formally fixed in H-system (see Section 2). Roughly speaking, once the group (any) element of fiber is fixed, we deal with the Lagrange factor λ_a of H-system which is also uniquely fixed. On the other hand, if we fix the group element of fiber we fix the function theta of gauge transforms as well. In this sense, we do not have the local gauge transforms (or the gauge freedom) anymore. It resembles a bit the case of spontaneous breaking of symmetry.

Notice that the path dependent functional given by the *l.h.s.* of (18) can be also gauge transformed as

$$\mathbb{P}\exp\left\{ig \int_{P(x_0,x)} d\omega_\mu A_\mu^\theta(\omega)\right\} \times \omega(x) \mathbb{P}\exp\left\{ig \int_{P(x_0,x)} d\omega_\mu A_\mu(\omega)\right\} \omega^{-1}(x_0), \quad (21)$$

where the local gauge function defined by

$$\omega(x) = e^{i\theta(x)} \equiv e^{iT^a\theta^a(x)} \quad (22)$$

with the corresponding generator T^a . From this, one can see if the Minkowski space has been realized as a loop space, the path dependent functional becomes invariant under the local gauge transforms. In the general case of arbitrary paths, imposing the condition (18) on the *r.h.s.* of (21), we are able to get the different contour gauge $\mathbf{g}(x|A) = C$ with $C = \omega(x)\omega^{-1}(x_0)$ (see (22)). From the theoretical point of view, this new contour gauge has the same status as the gauge of (18). It would correspond to the different plane $\mathbf{g} = C$ in Fig. 2 which also transects the principle fiber bundle and, therefore, it generates the different contour gauge condition which, generally speaking, is related to the previous contour gauge by the local transform. However, from the practical point of view, the contour gauge given by $\mathbf{g}(x|A) = C$ is not convenient to use for calculations because the representation of transverse gluon field through the strength tensor has a more complicated form compared to, for example, (53). Of course, the physical quantities are independent on the contour gauge choice.

Notice that since the gauge condition given by Eq. (18) selects only the identity element of group G on each fiber, it means that the gauge transforms have been reduced to the ‘‘global’’ gauge transforms. That is, the gauge function $\theta(x)$ becomes the coordinate independent and is fixed θ_0 . This situation is identical to that one can see in Fig. 1 of Section 2. Namely, the red dash on the ball surface corresponds to one particular choice of the group element fixed by the given

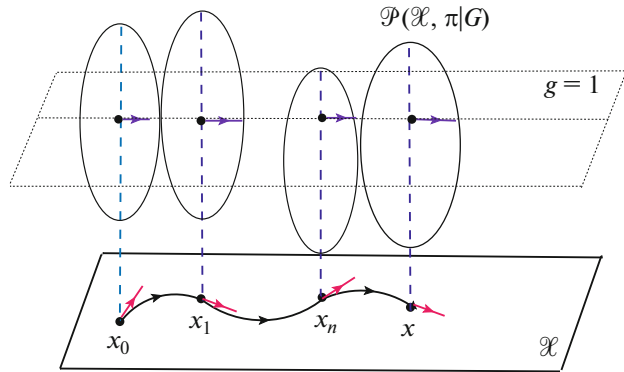


Fig. 2. The contour gauge: the plane of $\mathbf{g} = 1$ in the principle fiber bundle $\mathcal{P}(\mathcal{X}, \pi|G)$.

contour gauge. However, if we mark the line on the ball surface by the other dash, it would mean that we choose the other group element fixed by the different contour gauge. In both cases, we deal with the same description of H-system.

Since the functional $\mathbf{g}(x|A)$ depends on the whole path in \mathcal{X} , the contour gauge refers to the non-local class of gauges and generalises naturally the familiar local axial-type of gauges. It is also worth to notice that two different contour gauges can correspond to the same local (axial) gauge with the different fixing of residual gauges [10, 11]. If we would consider the local gauge without the connection with the non-local gauge, the residual gauge freedom would require the extra conditions to fix the given freedom. This statement reflects the fact that, in contrast to the local axial gauge, the contour gauge does not possess the residual gauge freedom in the finite region of a space where the boundary gluon fields are absent [2].

4. LOCAL AND NON-LOCAL GAUGE MATCHING

Since the contour gauge as a non-local kind of gauges generalizes the standard local gauge of axial type, it is worth to discuss shortly the correspondence between the local and non-local gauge transforms.

Let us begin with the axial local gauge defined by $A^+(x) = 0$. Notice that the non-local extension of this gauge is given by Eq. (18) provided the starting point is $x_0 = -\infty$. The differences between the local and non-local gauges can symbolically be demonstrated by the following trivial example. Consider two different (not arbitrary) vectors A and B , they are different by construction. We now assume that these vectors have the same projection on the certain direction given by another vector N . From the fact that the vectors have

the same projections on the vector N , it does not mean the equivalence of vectors A and B , i.e.

$$\{if(A \cdot N) = (B \cdot N)\} \not\Rightarrow \{A = B\}. \quad (23)$$

In this example, the different vectors A and B can be associated with two different contour (non-local) gauges,

$$\begin{aligned} A &\Leftrightarrow \{\mathbf{g}(x|A)|_{P(-\infty, x)} = \mathbf{1}\}, \\ B &\Leftrightarrow \{\mathbf{g}(x|A)|_{P(+\infty, x)} = \mathbf{1}\}, \end{aligned} \quad (24)$$

while the local axial gauge plays a role of the projections on N ,

$$\{(A \cdot N) = (B \cdot N)\} \Leftrightarrow \{A^+(x) = 0\}. \quad (25)$$

On the other hand, focusing on only the N vector projection, there is no additional information to see from which vectors A or B the given projection has been performed.

Coming back to the local axial gauge, we notice that the local gauge suffers from the residual gauge transformations (which can lead to the spurious pole uncertainties) till the non-local contour gauge fixes all the gauge freedom in the finite space (see for instance [2]). Indeed, consider the local axial gauge as an equation on the gauge function $\theta(x)$, i.e.

$$A^{+\theta}(x) = \omega(x)A^+\omega^{-1}(x) + \frac{i}{g}\omega(x)\partial^+\omega^{-1}(x) = 0, \quad (26)$$

where $x = (x^-, \tilde{x})$ with $\tilde{x} = (x^+, \mathbf{x}_\perp)$. We can readily find a solution of this equation which takes the form of the undetermined integration as

$$\omega_0(x) = C \mathbb{P} \exp\{-ig \int dx^- A^+(x)\}, \quad (27)$$

or the form of determined integration as

$$\begin{aligned} \omega_0(x^-, \tilde{x}) &= C(\tilde{x})\bar{\omega}(x^-, \tilde{x}), \\ \bar{\omega}(x^-, \tilde{x}) &= \mathbb{P} \exp\left\{-ig \int_{x_0^-}^{x^-} dz^- A^+(z^-, \tilde{x})\right\}. \end{aligned} \quad (28)$$

Here, x_0^- is fixed and $C(\tilde{x})$ is an arbitrary function which does not depend on x^- and it is given by

$$\omega_0(x_0^-, \tilde{x}) = C(\tilde{x}). \quad (29)$$

The arbitrariness of C -function also reflects the fact that we here deal with an arbitrary fixed starting point x_0 . We now study the residual gauge freedom requiring both $A^{+\theta}(x) = 0$ and $A^+(x) = 0$, we then have

$$\omega_0(x^-, \tilde{x})|_{\bar{\omega}=1} \equiv \omega^{\text{res}}(\tilde{x})C(\tilde{x}) \equiv e^{i\tilde{\theta}(\tilde{x})}. \quad (30)$$

One can see that the function C determines the residual gauge transforms. This situation takes place in

the local axial gauge defined by the only condition $A^+ = 0$ applied for (28).

Now, we go over to the non-local gauge which in some sense gives more information on the gauge fixing. The non-local contour gauge extends the local axial-type gauge and demands that the full (curve-linear) integral in the exponential of Eq. (18) has to go to zero. We stress that in contrast to the local gauge where the corresponding exponential disappears thanks to that the integrand A^+ goes to zero. One can demonstrate it on the example of solutions (27) or (28). Indeed, in the contour gauge the residual gauge function $\tilde{\theta}(\tilde{x})$ can be related to the path dependent functionals with A^- and A_\perp^i which also disappeared eliminating the gauge freedom and giving the physical gluon representation in the form of (56) (see [12, 13] for details). That is, if we restore the full path in the path dependent functional for a given process, we can get that

$$\begin{aligned} C(\tilde{x}) &= \tilde{C}(x_0^+, x_0^-, \mathbf{x}_0^\perp) \\ &\times \mathbb{P} \exp\left\{ig \int_{\mathbf{x}_0^\perp}^{\mathbf{x}^\perp} d\omega_\perp^i A_\perp^i(x_0^+, x_0^-, \omega_\perp)\right\} \\ &\times \mathbb{P} \exp\left\{-ig \int_{x_0^+}^{x^+} d\omega^+ A^-(\omega^+, x_0^-, \mathbf{x}_\perp)\right\}. \end{aligned} \quad (31)$$

Then, requiring the conditions as

$$A^-(\omega^+, x_0^-, \mathbf{x}_\perp) = 0, \quad (32)$$

$$\int_{\mathbf{x}_0^\perp}^{x^\perp} d\omega_\perp^i A_\perp^i(x_0^+, x_0^-, \omega_\perp) = 0, \quad (33)$$

we get that

$$C(\tilde{x})|_{\text{c.g.}} = \tilde{C}(x_0^+, x_0^-, \mathbf{x}_0^\perp). \quad (34)$$

In the contour gauge, Eq. (34) means that no the gauge freedom has left at all. We emphasize that the condition of (32) demands that the integrand is zero, while the condition of (33) is imposed on the integration which leads to the corresponding representation for the transverse gluon field (see (53), (54) and (56)). Besides, the exact value of the fixed starting point x_0 depends on the process under our consideration [12].

The other feature of the path dependent functional is that it defines the path dependent gauge transformation in the form of

$$\begin{aligned} A_\mu^{\text{p.g.}}(x) &= \mathbf{g}^{-1}(x|A)|_{P(x_0, x)} A_\mu(x) \mathbf{g}(x|A)|_{P(x_0, x)} \\ &+ \frac{i}{g} \mathbf{g}^{-1}(x|A)|_{P(x_0, x)} \partial_\mu \mathbf{g}(x|A)|_{P(x_0, x)}, \end{aligned} \quad (35)$$

where the starting point x_0 is now fixed. Hence, having calculated the derivative of \mathbf{g} in (35), we get that (here, for the simplicity, $A_\mu(-\infty) = 0$)

$$A_\mu^{\text{p.g.}}(x) \int_{-\infty}^x dz_\alpha \frac{\partial z_\beta}{\partial x_\mu} \quad (36)$$

$$\times \mathbf{g}^{-1}(z|A)|_{P(-\infty, z)} G_{\alpha\beta}(z|A) \mathbf{g}^{-1}(z|A)|_{P(-\infty, z)},$$

where $x_0 = -\infty$. If we are now focusing on the case of $x = (0^+, x^-, \mathbf{0}_\perp)$, we readily obtain that the gluon representation reads

$$A_\mu^{\text{p.g.}}(x^-) = \int_{-\infty^-}^{x^-} dz^- [z^-; -\infty^-]_{A^+}^{-1} \quad (37)$$

$$\times G^{+\mu}(z^-|A)[z^-; -\infty^-]_{A^+},$$

where the traditional notations

$$[z_2; z_1]_A = \mathbf{g}(z_2|A)|_{P(z_1, z_2)} \quad (38)$$

have been introduced. The representation given by (37) is an important result for our further considerations.

So, in this section, we have demonstrated that the contour gauge gives a possibility to illuminate the unphysical gluon components, meanwhile the local axial gauge fixes the gauge partially leaving room for the residual gauge freedom.

5. THE ADVANTAGE OF THE CONTOUR GAUGE USE

In the preceding sections, the different (mathematical) aspects of the contour gauge have been presented in the general, even formal, manner. In this section, we want to concentrate our attention on the certain examples which, first, relate to the practical use of the contour gauge and, second, demonstrate the preponderance of the non-local gauges compared to the local gauges.

As discussed in [10, 11, 13], the Drell–Yan-like processes with the polarized hadrons give the unique example where the contour gauge use shows the definite advantage from the practical point of view. In particular, the contour gauge use allows to find the new contributions to the Drell–Yan-like hadron tensor which restore and ensure the gauge invariance of the corresponding hadron tensors [10, 11, 13]. It is important, however, to stress that the mentioned new contributions are invisible if we would work within the frame in the local gauge.

In the similar manner, due to the contour gauge conception the ξ -process of DVCS-amplitude which clarifies the gauge invariance of the non-forward processes takes the closed form again [14]. From the practical point of view, it is instructive to consider the appearance of standard and non-standard diagrams

contributing to the well-known deeply virtual Compton scattering (DVCS) amplitude in the frame of the factorization procedure. The gluons radiated from the internal quark of the hard subprocess generate the standard diagrams, while the non-standard diagrams are formed by the gluon radiations from the external quark of the hard subprocess (see [14] for details).

In the most cases, it is sufficient to exponentiate only the longitudinal components of gluon field, A^- and A^+ , which are related to the unphysical degrees. Indeed, the standard diagram contributions give the gauge invariant quark string operator which reads

$$\bar{\Psi}^{(\text{st})}(0^+, 0^-, \mathbf{0}_\perp | A^+) \{\gamma\} \Psi^{(\text{st})}(0^+, z^-, \mathbf{0}_\perp | A^+), \quad (39)$$

where

$$\bar{\Psi}^{(\text{st})}(0^+, 0^-, \mathbf{0}_\perp | A^+) \quad (40)$$

$$= \bar{\Psi}(0^+, 0^-, \mathbf{0}_\perp) [0^+, 0^-, \mathbf{0}_\perp; 0^+, +\infty^-, \mathbf{0}_\perp]_{A^+},$$

$$\Psi^{(\text{st})}(0^+, z^-, \mathbf{0}_\perp | A^+) \quad (41)$$

$$= [0^+, +\infty^-, \mathbf{0}_\perp; 0^+, z^-, \mathbf{0}_\perp]_{A^+} \Psi(0^+, z^-, \mathbf{0}_\perp),$$

and $\{\gamma\}$ stands for γ -matrices the exact form of which is now irrelevant.

The non-standard diagram contributions result in the string operator defined as

$$\bar{\Psi}^{(\text{non-st})}(0^+, 0^-, \mathbf{0}_\perp | A^-) \{\gamma\} \Psi^{(\text{non-st})}(0^+, z^-, \mathbf{0}_\perp | A^-), \quad (42)$$

where

$$\bar{\Psi}^{(\text{non-st})}(0^+, 0^-, \mathbf{0}_\perp | A^-) \quad (43)$$

$$= \bar{\Psi}(0^+, 0^-, \mathbf{0}_\perp) [-\infty^+, 0^-, \mathbf{0}_\perp; 0^+, 0^-, \mathbf{0}_\perp]_{A^-},$$

$$\Psi^{(\text{non-st})}(0^+, z^-, \mathbf{0}_\perp | A^-) \quad (44)$$

$$= [0^+, z^-, \mathbf{0}_\perp; -\infty^+, z^-, \mathbf{0}_\perp]_{A^-} \Psi(0^+, z^-, \mathbf{0}_\perp).$$

Hence, using the contour gauge conception, we eliminate all the Wilson lines with the longitudinal gluon fields A^+ and A^- demanding that

$$[0^+, 0^-, \mathbf{0}_\perp; 0^+, +\infty^-, \mathbf{0}_\perp]_{A^+} = \mathbf{1}, \quad (45)$$

$$[0^+, +\infty^-, \mathbf{0}_\perp; 0^+, z^-, \mathbf{0}_\perp]_{A^+} = \mathbf{1}$$

and

$$[-\infty^+, 0^-, \mathbf{0}_\perp; 0^+, 0^-, \mathbf{0}_\perp]_{A^-} = \mathbf{1}, \quad (46)$$

$$[0^+, z^-, \mathbf{0}_\perp; -\infty^+, z^-, \mathbf{0}_\perp]_{A^-} = \mathbf{1}.$$

Equations (45) and (46) give rise to the local gauge conditions given by $A^+ = 0$ and $A^- = 0$.

With respect to the Wilson line with the transverse gluons A_\perp^i , we remind that we work here within the factorization procedure applied for DVCS-amplitude. In this case, the Wilson lines with the transverse gluon

fields are considered in the form of an expansion due to the fixed twist-order, and the transverse gluons correspond to the physical configurations of L-system.

Thus, the DVCS process gives us the example how the unphysical gluon degrees can be illuminated from the consideration with the help of contour gauge.

We are now going over to the Drell–Yan (DY) process with one transversely polarized hadron [10]

$$N^{(\uparrow\downarrow)}(p_1) + N(p_2) \Rightarrow \ell(l_1) + \bar{\ell}(l_2) + X(P_X), \quad (47)$$

where the virtual photon producing the lepton pair ($l_1 + l_2 = q$) has a large offshellness, i.e. $q^2 = Q^2 \rightarrow \infty$, while all the transverse momenta are small and integrated out in the corresponding cross-sections $d\sigma$. Here, the contour gauge use results in the gauge invariant hadron tensor and provides the new contributions to single spin asymmetries. Having considered this hadron tensor in the asymptotical regime associated with the very large Q^2 , the factorization theorem can be applied for the given hadron tensor as well as for the DVCS process. As a result, the DY hadron tensor takes a form of convolution as

$$\begin{aligned} & \text{Hadron tensor} \\ & = \{\text{Hard part } (pQCD)\} \otimes \{\text{Soft part } (npQCD)\}, \end{aligned} \quad (48)$$

where both the hard and soft parts should be independent of each other and are in agreement with the ultraviolet and infrared renormalizations. Moreover, the relevant single spin asymmetries (SSAs), which is a subject of experimental studies, can be presented as

$$d\sigma(N^{(\uparrow)}) - d\sigma(N^{(\downarrow)}) \sim \mathcal{L}_{\mu\nu} H_{\mu\nu}, \quad (49)$$

where $\mathcal{L}_{\mu\nu}$ and $H_{\mu\nu}$ are the lepton and hadron tensors, respectively. The hadron tensor $H_{\mu\nu}$ includes the the polarized hadron matrix element which takes a form of

$$\begin{aligned} & \langle p_1, S^\perp | \bar{\psi}(\lambda_1 \tilde{n}) \gamma^+ g A_\perp^\alpha(\lambda_2 \tilde{n}) \psi(0) | p_1, S^\perp \rangle \\ & \stackrel{\mathcal{F}}{=} i \varepsilon^{\alpha+S^\perp-} (p_1 p_2) B(x_1, x_2), \end{aligned} \quad (50)$$

where $\stackrel{\mathcal{F}}{=}$ stands for the Fourier transform between the coordinate space, formed by positions $\lambda_i \tilde{n}$, and the momentum space, realized by fractions x_i ; the light-cone vector \tilde{n} is a dimensionful analog of n . In Eq. (50), the parametrizing function B describes the corresponding parton distribution.

5.1. The Case of Local Gauge

In the studies, see for example [15–18], where the local light-cone gauge $A^+ = 0$ has been used, B -function of Eq. (50) is given by a purely real function. That is, we have

$$B(x_1, x_2) = \frac{\mathcal{P}}{x_1 - x_2} T(x_1, x_2) \quad (51)$$

where the function $T(x_1, x_2) \in \Re$ parametrizes the corresponding projection of $\langle \bar{\psi} G_{\alpha\beta} \psi \rangle$ and obeys $T(x, x) \neq 0$.

In Eq. (51), the pole at $x_1 = x_2$ is treated as a principle value and it obviously leads to $B(x_1, x_2) \in \Re$.

Indeed, within the local gauge $A^+ = 0$, the statement on that B is the real function stems actually from the ambiguity in the solutions of the trivial differential equation, which is equivalent to the definition of $G_{\mu\nu}$,

$$\partial^+ A_\perp^\alpha = G_\perp^{\alpha+}. \quad (52)$$

The formal resolving of Eq. (52) leads to two representations written as

$$A_{(1)}^\mu(z) = \int_{-\infty}^z d\omega^- G^{+\mu}(\omega^-) + A^\mu(-\infty), \quad (53)$$

$$A_{(2)}^\mu(z) = -\int_z^\infty d\omega^- G^{+\mu}(\omega^-) + A^\mu(\infty). \quad (54)$$

We stress that within the approaches backed on the local axial gauge use, there are no evidences to think that Eqs. (53) and (54) are not equivalent each other.

That is, the local gauge $A^+ = 0$ inevitably leads to the following logical scheme (see [11] for details)

$$\boxed{A_{(1)}^\mu \stackrel{\text{loc.g.}}{=} A_{(2)}^\mu} \rightarrow \boxed{B(x_1, x_2) = \mathcal{P} \frac{T(x_1, x_2)}{x_1 - x_2} \in \Re}. \quad (55)$$

This equation demonstrates that the equivalence of Eqs. (53) and (54) causes the representation of B -function as in Eq. (51). The discussion on the boundary configurations can be found in [11].

Regarding the DY process, the physical consequences of the use of B presented by Eqn. (51) are the problem with the photon gauge invariance of DY-like hadron tensors and the losing of significant contributions to SSAs [10, 11]. Besides, based on the local

gauge $A^+ = 0$, the representation of gluon field as a linear combination of Eqs. (53) and (54) has been used in the different studies (see [19–21]).

5.2. The Case of Non-Local Gauge

In contrast to the local gauge $A^+ = 0$, as discussed in Section 3, we can infer that the path dependent non-local gauge (see Eq. (18)) fixes unambiguously the representation of gluon field which is given by either Eqs. (53) or (54). Indeed, fixing the path $P(x_0, x)$, a solution of Eq. (18) takes the form of (see [10, 11, 22] for details)

$$A_\mu^{c.g.}(x) = \int_{P(x_0, x)} dz_\alpha \frac{\partial z_\beta}{\partial x_\mu} G_{\alpha\beta}(z|A), \quad (56)$$

where the boundary configuration $A_{\mu}^{c:g}(x_0)$ has assumed to be zero. By direct calculation, we can show that the non-local gauge $\mathbf{g}(P(-\infty, x)|A) = 1$ leads to the gluon field representation of Eq. (53), while the non-local gauge $\mathbf{g}(P(x, +\infty)|A) = 1$ corresponds to the gluon field representation of Eq. (54). Moreover, we can readily check that [10, 11]

$$\text{Eq. (53)} \rightarrow \boxed{B_+(x_1, x_2)} \neq \boxed{B_-(x_1, x_2)} \leftarrow \text{Eq. (54), (57)}$$

where

$$B_+(x_1, x_2) = \frac{T(x_1, x_2)}{x_1 - x_2 + i\epsilon} \in \mathbb{C}, \quad (58)$$

$$B_-(x_1, x_2) = \frac{T(x_1, x_2)}{x_1 - x_2 - i\epsilon} \in \mathbb{C}. \quad (59)$$

Notice that both the non-local contour gauges, i.e. $\mathbf{g}(P(-\infty, x)|A) = \mathbf{1}$ and $\mathbf{g}(P(x, +\infty)|A) = \mathbf{1}$, can be projected into the same local gauge $A^+ = 0$. As mentioned, the projection given by $A^+ = 0$ does not give a possibility to understand which of the non-local gauges generates the local gauge.

To conclude, we can state that, considering DY-like processes, the corresponding non-local gauge gives rise to the correct representation of B -functions (see Eq. (58)), which has the non-zero imaginary part. This enables us to find the new significant contributions to the hadron tensors that ensure ultimately the gauge invariance [10, 11]. In a similar manner, with the help of Eq. (58) we can fix the prescriptions for the spurious singularities in the gluon propagators [13].

6. CONCLUSIONS

In the paper, we have made public the important subtleties based on the mathematical technique adjusted to the physical language. We have presented the important explanations and analysis hidden in the preceding publications which should help to clarify the main advantages of the use of non-local contour gauges. To this goal, the combination of the Hamilton and Lagrangian approaches to the gauge theory has been exploited in our consideration. Since the contour gauge is mainly backed on the geometrical interpretation of gluon fields as a connection on the principle fiber bundle, we have provided the illustrative demonstration of geometry of the contour gauge. In this connection, the Hamiltonian formalism is supposed to be more convenient for understanding the subtleties of contour gauges. Indeed, the Lagrange factor λ_a fixation has a direct treatment in terms of the orbit group element which has been uniquely chosen by the corresponding plan transecting the principle fiber bundle (see Fig. 2). While, as shown, the Lagrangian formalism is very well designed for the practical uses to the

eliminate the unphysical gluon degree of freedom from the corresponding amplitudes.

Also, we have reminded the details of that studies where the local axial-type gauge can lead to the certain ambiguities in the gluon field representation. These ambiguities may finally produce incorrect results. Meanwhile, as demonstrated in the paper, the non-local contour gauge can fix this kind of problems and, for example, can provide not only the correct (gauge invariant) final result but also find the new contributions to the hadron tensors of DY-like processes [10, 11]. Thus, the use of contour gauge conception gives a possibility (i) to find a solution of the gauge-invariance problem, discovered in the Drell–Yan hadron tensor, by the correct description of the gluon pole that is appearing in the corresponding parton distributions; (ii) to discover the new sizeable contributions to the single-spin asymmetries which are under intensive experimental studies.

In the context of the contour gauge use, the recent progress is mainly related to the studies of the so-called gluon pole contributions to the Drell–Yan-like processes [10, 11]. However, the practical profit of the non-local contour gauge is not limited by the study of gluon poles which manifest in the different hard processes. With the help of non-local gauges, we plan to adopt the method based on the geometric quantization [23] to the investigation of different asymptotical (hard) regimes in QFT.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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