ISSN 1547-4771, Physics of Particles and Nuclei Letters, 2017, Vol. 14, No. 2, pp. 331–335. © Pleiades Publishing, Ltd., 2017.

PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

Integrability of Calogero–Coulomb Problems[1]

Tigran Hakobyan[a, b] and Armen Nersessian[a, b, *]

[a] *Yerevan State University, Yerevan, 0025 Armenia*

[b] *Tomsk Polytechnic University, Tomsk, 634050 Russia*

**e-mail: arnerses@ysu.am*

Received March 31, 2016

Abstract—In this short review we describe the integrability properties of the Calogero-type perturbations of one- and two-center Coulomb problems and of the Stark–Coulomb problem. We present the explicit expressions of their constants of motion and show that these systems admit partial separation of variables.

DOI: 10.1134/S1547477117020133

INTRODUCTION

The Coulomb problem is maximally superintegrable due to the conservation of the Runge–Lenz vector. The Hamiltonian admits separation of variables in several coordinate systems. Any such coordinate system possesses its own integrable perturbations. Among them there are the Coulomb problem in constant electric field (Coulomb–Stark problem) and the two-center Coulomb problem. They admit a separation of variables, respectively, in parabolic and elliptic coordinates. Note that both systems are not exactly solvable. In the Coulomb-Stark problem, one can get analytically only the perturbative spectrum, while in the two-center Coulomb system, the energy spectrum can be constructed only numerically, except for some special cases [1]. Nevertheless, the separation of variables is crucial in their study.

The well-known rational Calogero model [2], which describes one-dimensional particles, interacting with inverse-square potential,

$$\mathcal{H}_0 = \sum_i \frac{p_i^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2}, \qquad (1)$$

is another example of maximally superintegrable system [3]. It possesses higher-order (in momenta) integrals of motion, which had been constructed by the Lax pair [4]. The inverse-square potential in one dimension possesses various integrable generalizations [see Refs. [5, 6] for the review], which have many applications in physics and mathematics.

The mixture of the Coulomb and Calogero potentials gives rise to a more general integrable N-dimensional system [7]. Recently we have shown together with Olaf Lechtenfeld that the Calogero–Coulomb system is also superintegrable [8]. This property can be understood in the action-angle language. An explicit form of the complete set of constants of motion can be derived by taking proper deformations of the corresponding integrals of the underlying Coulomb system, then forming the symmetric polynomials on them [8, 9]. This method differs from the standard construction [4], so that the deformations of the Liouville integrals do not commute any more. Nevertheless, the functional independence of the constricted integrals of motion is preserved.

In this review based on Ref. [9, 10], we consider in this context the N-dimensional Coulomb, Coulomb–Stark and two-center Coulomb problems with the additional Calogero potential (we will refer them as Calogero–Coulomb, Calogero–Coulomb–Stark and two-center Calogero–Coulomb problems). These systems have a highlighted direction, along which the full rotational symmetry of the initial one-center Coulomb problem is broken down to the $SO(N-1)$ symmetry. It is defined, respectively, by the external filed direction and by the line connecting two Coulomb charges. We show that under the proper choice of this highlighted direction, both systems still remain integrable and admit partial separation of variables. In fact, the Schrödinger equation decouples into three parts, only one of which depends on the inverse-square interaction term. The latter can be treated as a deformation of the Schrödinger equation for the $SO(N-1)$ angular momentum, usually referred as an angular Calogero Hamiltonian [11–15].

Calogero–Coulomb problem. The Calogero–Coulomb problem is a mixture of the N-particle rational Calogero model (1) and of the N-dimensional Coulomb system [7]:

$$\mathcal{H}_\gamma = \frac{p^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r}. \qquad (2)$$

[1] The article is published in the original.

It inherits most of the properties of the original Coulomb system and possesses hidden symmetries given by an analog of Runge–Lenz vector [8, 9]. It is convenient to describe this system by means of the Dunkl operators which make transparent the analogy with the initial Coulomb problem. Let us consider instead the extended Hamiltonian in this regard,

$$\mathcal{H}^{\rm gen}_{\gamma} = \frac{\pi^2}{2} - \frac{\gamma}{r} = \frac{p^2}{2} + \sum_{i<j}\frac{g\,(g - s_{ij})}{(x_i - x_j)^2} - \frac{\gamma}{r}. \qquad (3)$$

The modified momentum is expressed in terms of the Dunkl operators by

$$\pi = -\imath\nabla, \quad \nabla_i = \partial_i - \sum_{j\neq i}\frac{g}{x_i - x_j}s_{ij}. \qquad (4)$$

The operator s_{ij} permutes the i-th and j-th coordinates. On the symmetric wavefunctions the generalized Hamiltonian $\mathcal{H}^{\rm gen}_{\gamma}$ reduces to the Calogero–Coulomb Hamiltonian (2). The Dunkl operators commute mutually like ordinary partial derivatives. However, their commutations with coordinates are nontrivial deformations of the Heisenberg algebra relations [16],

$$[\pi_i, x_j] = -\imath S_{ij}. \qquad (5)$$

The operators S_{ij} for $i \neq j$ are just rescaled permutations: $S_{ij} = -gs_{ij}$, and the S_{ii} are defined by through the relation $\sum_j S_{ij} = 1$.

Let us define the deformed angular momentum operator via the Dunkl momentum [11, 17]:

$$L_{ij} = x_i\pi_j - x_j\pi_i. \qquad (6)$$

It preserves generalized Calogero–Coulomb Hamiltonian [8] and satisfies the deformed angular momentum commutation relations [11].

The deformed Runge–Lenz vector preserving generalized Calogero–Coulomb Hamiltonian reads [9]

$$A_i = \frac{1}{2}\sum_j\{L_{ij}, \pi_j\} + \frac{\imath}{2}[\pi_i, S] - \frac{\gamma x_i}{r}. \qquad (7)$$

It contains the permutation-group invariant element which vanishes in the absence of the Calogero term

$$S = \sum_{i<j} S_{ij}. \qquad (8)$$

The Calogero–Coulomb problem can be obtained by the restriction of the extended Hamiltonian (3) to the symmetric wavefunctions. Therefore, its constants of motion can be constructed by taking the symmetric polynomials on the components of the Dunkl angular momentum and Runge-Lenz vector [8, 9]:

$$\mathcal{L}_{2k} = \sum_{i<j} L_{ij}^{2k}, \quad \mathcal{A}_k = \sum_i A_i^k. \qquad (9)$$

The expressions above demonstrate that the Calogero-Coulomb problem is a superintegrable system, like the pure Calogero [3] and Coulomb models. Note that the the square of Dunkl angular momentum is related to the the angular part $\mathcal{J}$ of the Calogero Hamiltonian [11]:

$$\mathcal{L}_2 = 2\mathcal{J} + S(S - N + 2). \qquad (10)$$

In two dimension the symmetries of the Calogero–Coulomb system, based on the dihedral group D_2, have been studied also in Ref. [18].

Coulomb–Calogero–Stark problem. Consider the N-dimensional Coulomb problem in constant electric field F in the presence of the Calogero interaction:

$$\mathcal{H}_{\gamma,F} = \frac{p^2}{2} + \sum_{i<j}^{N}\frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r} + Fx_0, \qquad (11)$$

where x_0 is the normalized center-of-mass coordinate (see Eq. (14) below). The external field is aligned in the direction $(1,1,\ldots,1)$, which ensures the permutation invariance of the Hamiltonian. In the absence of the external field, this model is reduced to the Calogero–Coulomb model, considered above.

The generalized Hamiltonian is defined in terms of the Dunkl momentum (4) as follows:

$$\mathcal{H}^{\rm gen}_{\gamma,F} = \frac{\pi^2}{2} - \frac{\gamma}{r} + Fx_0. \qquad (12)$$

The entire Dunk angular momentum tensor (6) is not an integral of motion any more. Instead, its components, which are orthogonal to the external field, are preserved,

$$L^{\perp}_{ij} = L_{ij} + \frac{1}{N}\sum_k (L_{jk} - L_{ik}). \qquad (13)$$

Alternatively, one can express them in terms of the Jacobi coordinates, which separate the center-of-mass from the relative motion. They are defined by the orthogonal map [13, 19]

$$\begin{aligned} x_0 &= \frac{1}{\sqrt{N}}(x_1 + \ldots + x_N), \\ \tilde{x}_k &= \frac{1}{\sqrt{k(k+1)}}(x_1 + \ldots + x_k - kx_{k+1}), \end{aligned} \qquad (14)$$

where $1 \le k \le N-1$. The first coordinate describes the center of mass, while the others, marked by tilde, characterize the relative motion.

Denote now by $\tilde{L}_{ij}$ the components of the deformed *relative* angular momentum, rotated by the Jacobi transformation. The algebra generated by $L^{\perp}_{ij}$, in fact, coincides with the $\tilde{L}_{ij}$, which are responsible for the relative motion ($1 \le i, j \le N-1$). In the absence of Calogero interaction, they form the $SO(N-1)$ subalgebra, which describes the rotations in the hyperspace, orthogonal to the center-of-mass direction. Apart from the deformed relative angular momen-

tums, the modified component of the Runge–Lenz vector (7) along the field direction is preserved as well. It reads

$$A = x_0\left(2\mathscr{H}^{\rm gen}_{\gamma,F} + \frac{\gamma}{r}\right) - \left(rp_r + \frac{N-1}{2t}\right)p_0 - \frac{F}{2}\left(r^2 + 3x_0^2\right). \quad (15)$$

This invariant commutes with the deformed relative angular momentum. In the $g = 0$ limit, one can extract from these symmetry generators the standard Liouville integrals of the Coulomb–Stark system. The $N-2$ integrals can be chosen to be the quadratic Casimir elements of the naturally embedded algebras $SO(2) \subset ... \subset SO(N-1)$. They are described in the relative angular coordinates and momenta. The last two integrals are given by the Hamiltonian and the modified component of the Runge–Lenz vector, which had been constructed for $N = 3$ in Ref. [20]. Out of the $g = 0$ point, we deal with the deformed quantities, and the Liouville property can not be extended straightforwardly. Nevertheless, in the presence of a constant uniform electric field, the generalized Calogero–Coulomb model (12) still remains an integrable system.

The integrals of the pure Calogero–Coulomb system (11) obtained by the restriction to the symmetric wavefunctions, must be symmetric too. Since the longitude component of the Runge–Lenz vector (15) obeys this condition, it remains as a correct integral for this system, $[A, \mathscr{H}_{\gamma,F}] = 0$. We should take symmetric expressions of the kinematical constants of motion too, as in the absence of the electric field [9]. For this purpose it is more suitable to use the angular momentum in Jacobi coordinates:

$$[\mathscr{H}_{\gamma,F}, \tilde{\mathscr{L}}_{2k}] = 0, \quad \tilde{\mathscr{L}}_{2k} = \sum_{1\le i<j\le N-1} \tilde{L}_{ij}^{2k}. \quad (16)$$

The first member of this family is the square of the relative Dunkl angular momentum. It is related to the angular part of the Calogero model with reduced center of mass $\tilde{\mathscr{J}}$, which we call the relative angular Calogero Hamiltonian, by the same formula as Eq. (10) above. So, we have proved the integrability of the Calogero–Coulomb–Stark system. It is well known that the Coulomb–Stark system admits separation of variables in parabolic coordinates. It appears that the Calogero–Coulomb–Stark system admits complete separation of variables in parabolic coordinates for $N = 2, 3$ and partial separation for $N > 3$ [10].

In the Jacobi coordinates (14), the last system acquires the following form:

$$\mathscr{H}_{\gamma,F} = \frac{p_0^2}{2} - \frac{\gamma}{\sqrt{x_0^2 + \tilde{x}^2}} + Fx_0 + \tilde{\mathscr{H}}_0, \quad (17)$$

where the last term is the Calogero Hamiltonian (1) with reduced center of mass. We pass to the parabolic coordinates $(\xi, \eta, \varphi_\iota)$, where φ_ι are the relative angular variables, and

$$\xi = r + x_0, \quad \eta = r - x_0. \quad (18)$$

In new coordinates the Hamiltonian (17) is expressed as follows:

$$\mathscr{H}_{\gamma,F} = -\frac{2}{\xi+\eta}(\gamma + B_\xi + B_\eta) + \frac{\tilde{\mathscr{J}}}{\xi\eta} + \frac{F}{2}(\xi - \eta), \quad (19)$$

where we have shorten the kinetic term using the notation

$$B_\xi = \frac{1}{\xi^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}\xi^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \quad (20)$$

Further we proceed by extending straightforwardly the steps, applied for the usual Coulomb system in external field in Ref. [21]. Employing the following ansatz to the total wavefunction

$$\Psi(\xi, \eta, \varphi_\iota) = \Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota), \quad (21)$$

we decouple Schrödinger equation $\mathscr{H}_{\gamma,F}\Psi = E\Psi$ into three parts. The two of them depend, respectively, on ξ and η,

$$\left(B_\xi + \frac{E}{2}\xi - \frac{F}{4}\xi^2 - \frac{\tilde{q}(\tilde{q}+N-3)}{4\xi} + \lambda_1\right)\Phi_1(\xi) = 0, \quad (22a)$$

$$\left(B_\eta + \frac{E}{2}\eta + \frac{F}{4}\eta^2 - \frac{\tilde{q}(\tilde{q}+N-3)}{4\eta} + \lambda_2\right) \times \Phi_2(\eta) = 0. \quad (22b)$$

where $\lambda_1 + \lambda_2 = \gamma$. The last equation describes the spectrum and eigenstates of the relative angular Calogero model [12]:

$$\tilde{\mathscr{J}}(\varphi_\iota, \partial_{\varphi_\iota})\psi_{\tilde{q}}(\varphi_\iota) = \frac{\tilde{q}(\tilde{q}+N-3)}{2}\psi_{\tilde{q}}(\varphi_\iota). \quad (22c)$$

In particular, the spectrum is determined by the numbers

$$\tilde{q} = \frac{gN(N-1)}{2} + 3l_3 + ... + Nl_N \quad \text{with } l_i = 0, 1, 2, ... \quad (23)$$

For integer values of the coupling g, the angular energy spectrum is that of a free particle with angular momentum $\tilde{q}$ on the $(N-2)$-dimensional sphere, but has a significantly lower degeneracy due to the restriction to the symmetric wavefunctions [12, 14].

The longitudinal component of the Runge–Lenz vector (15) separates the equations (22a) and (22b): $A\Psi = (\lambda_2 - \lambda_1)\Psi$. The second invariant, given by the relative angular Hamiltonian $\tilde{\mathscr{J}}$, is common in both cases and separates the relative angular degrees of freedom. As in the usual Coulomb problem [22], the electric field completely removes the degeneracy in the

orbital momentum, but preserves the degeneracy with respect to q.

Two-Center Calogero–Coulomb system. Consider now the integrable two-center Coulomb system in the presence of the inverse-square Calogero potential. In order to construct the Hamiltonian of this system, we should replace, as in previous sections, the momenta operators by the Dunkl momenta, and then restrict the Hamiltonian to the symmetric wavefunction. In order to assure the permutation symmetry, we align the axis, connecting two Coulomb charges, along the center-of-mass coordinate. In the Jacobi coordinates (14), the distances to the charges are given by

$$r_1 = \sqrt{\tilde{x}^2 + (x_0 - a)^2}, \quad r_2 = \sqrt{\tilde{x}^2 + (x_0 + a)^2}. \qquad (24)$$

The generalized two-center Calogero–Coulomb Hamiltonian is

$$\mathcal{H}^{\rm gen}_{\gamma_1,\gamma_2} = \frac{\pi^2}{2} - \frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}. \qquad (25)$$

On the symmetric wavefunctions, it produces the two-center Calogero–Coulomb system,

$$\mathcal{H}_{\gamma_1,\gamma_2} = \frac{p^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}. \qquad (26)$$

Like the Calogero–Coulomb–Stark Hamiltonian, it possesses the symmetry given by the deformed angular momentum generators perpendicular to the predefined direction (13).

The modified Runge–Lenz integral of the $g = 0$ Hamiltonian have been constructed in Refs. [23, 24]. The construction can be extended to the case of nonzero coupling values by [10]

$$A = \mathcal{L}_2 + a^2 p_0^2 - 2ax_0\left(\frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}\right), \qquad (27)$$

where $\mathcal{L}_2$ is the Dunkl angular momentum square, defined in Eqs. (9).

Now, let us show that in complete analogy with the previous case, the two center Calogero–Coulomb system (26) admits complete separation of variables in the elliptic coordinates for $N = 2, 3$ and the partial separation for $N > 3$. The map from the Jacobi variables (x_0, y) to the elliptic coordinates (ξ, η) looks as follows [25]:

$$\xi = \frac{r_1 + r_2}{2a}, \quad \eta = \frac{r_1 - r_2}{2a}, \qquad \xi \geq 1 \quad -1 \leq \eta \leq 1, \qquad (28)$$

where r_i is the distance from the i-th Coulomb charge (24).

The two-center Calogero–Coulomb Hamiltonian (26) in elliptic coordinates reads

$$\mathcal{H}_{\gamma_1,\gamma_2} = \frac{1}{2a^2(\xi^2 - \eta^2)}(B_\eta - B_\xi) + \frac{\tilde{\mathcal{J}}(\varphi_\iota, \partial_{\varphi_\iota})}{a^2(\xi^2 - 1)(1 - \eta^2)} - \frac{\gamma_1}{a(\xi + \eta)} - \frac{\gamma_2}{a(\xi - \eta)}, \qquad (29)$$

where the operator B_ξ from the kinetic energy part acquires the following form:

$$B_\xi = \frac{1}{(\xi^2 - 1)^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}(\xi^2 - 1)^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \qquad (30)$$

Then, choosing the wavefunction $\Psi(\xi, \eta, \varphi_\iota) = \Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota)$, we can separate the variables in the Schrödinger equation into the three parts. The first two equation are

$$\left(B_\xi - \frac{\tilde{q}(\tilde{q} + N - 3)}{\xi^2 - 1} + 2a(\gamma_1 + \gamma_2)\xi + 2a^2E\xi^2 - \lambda\right)\Phi_1(\xi) = 0, \qquad (31a)$$

$$\left(B_\eta - \frac{\tilde{q}(\tilde{q} + N - 3)}{\eta^2 - 1} + 2a(\gamma_1 - \gamma_2)\eta + 2a^2E\eta^2 - \lambda\right)\Phi_2(\eta) = 0. \qquad (31b)$$

The third equation is inherited from the Stark case (22c). It describes the energy eigenstates of the relative angular Calogero Hamiltonian and its spectrum, depending on the composite quantum number $\tilde{q}$ (23). In the absence of the Calogero term, it determines the spectrum and energy states of a free particle system on $(N-2)$-dimensional sphere. Obviously, the partial states $\Phi_{1,2}$ in the first two equations depend on the energy level E and the $\tilde{q}$. The parameter λ in the first two equations separates the variables ξ and η. It coincides with the eigenvalue of the slightly redefined Runge–Lenz invariant for the two center Calogero–Coulomb system (27) with the Dunkl angular momentum square replaced by the doubled angular Calogero Hamiltonian,

$$A = 2\mathcal{J} + a^2 p_0^2 - 2ax_0\left(\frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}\right). \qquad (32)$$

We follow the steps done above for the parabolic case. First, use the total wavefunction Ψ instead of the partial ones, $\Phi_{1,2}$, in Eqs. (31a) and (31b). Next, cancel out the energy E by taking appropriate combinations of both equations. This yields the eigenstate equation for the modified Runge–Lenz invariant: $A\Psi = \lambda\Psi$.

ACKNOWLEDGMENTS

This work was partially supported by the Armenian State Committee of Science Grants 15RF-039 and 15T-1C367. It was done within ICTP programs NET68 and OEA-AC-100.

REFERENCES

1. I. Komarov and S. Slavyanov, "The two Coulomb centres problem at large centre separation," J. Phys. B **1**, 1066 (1968); Yu. Demkov and I. Komarov, "Hypergeometric partial solutions in the problem of two Coulomb centers," Theor. Math. Phys. **38**, 174 (1979).
2. F. Calogero, "Solution of a three-body problem in one dimension," J. Math. Phys. **10** (1969); "Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials," J. Math. Phys. **12**, 419 (1971).
3. S. Wojciechowski, "Superintegrability of the Calogero-Moser system," Phys. Lett. A **95**, 279 (1983).
4. J. Moser, "Three integrable hamiltonian systems connected with isospectral deformations," Adv. Math. **16**, 197 (1975).
5. M. Olshanetsky and A. Perelomov, "Classical integrable finite dimensional systems related to lie algebras," Phys. Rep. **71**, 313 (1981).
6. A. P. Polychronakos, "Physics and mathematics of Calogero particles," J. Phys. A **39**, 12793 (2006).
7. A. Khare, "Exact solution of an N-body problem in one dimension," J. Phys. A **29**, L45 (1996).
8. T. Hakobyan, O. Lechtenfeld, and A. Nersessian, "Superintegrability of generalized Calogero models with oscillator or Coulomb potential," Phys. Rev. D: Part. Fields **90**, 101701(R) (2014).
9. T. Hakobyan and A. Nersessian, "Runge-Lenz vector in Calogero-Coulomb problem," Phys. Rev. A **92**, 022111 (2015).
10. T. Hakobyan and A. Nersessian, "Integrability and separation of variables in Calogero-Coulomb-Stark and two-center Calogero-Coulomb systems," Phys. Rev. D: Part. Fields **93**, 045025 (2016).
11. M. Feigin and T. Hakobyan, "On dunkl angular momenta algebra," J. High Energy Phys. **11**, 107 (2015).
12. M. Feigin, O. Lechtenfeld, and A. Polychronakos, "The quantum angular Calogero-Moser model," J. High Energy Phys. **1307**, 162 (2013).
13. T. Hakobyan, A. Nersessian, and V. Yeghikyan, "Cuboctahedric higgs oscillator from the Calogero model," J. Phys. A **42**, 205206 (2009).
14. F. Correa and O. Lechtenfeld, "The tetrahexahedric angular Calogero model," J. High Energy Phys. **10**, 191 (2015).
15. T. Hakobyan, D. Karakhanyan, and O. Lechtenfeld, "The structure of invariants in conformal mechanics," Nucl. Phys. B **886**, 399 (2014); T. Hakobyan, S. Krivonos, O. Lechtenfeld, and A. Nersessian, "Hidden symmetries of integrable conformal mechanical systems," Phys. Lett. A **374**, 801 (2010); T. Hakobyan, O. Lechtenfeld, A. Nersessian, and A. Saghatelian, "Invariants of the spherical sector in conformal mechanics," J. Phys. A **44**, 055205 (2011); O. Lechtenfeld, A. Nersessian, and V. Yeghikyan, "Action-angle variables for dihedral systems on the circle," Phys. Lett. A **374**, 4647 (2010).
16. C. F. Dunkl, "Differential-difference operators associated to reflection groups," Trans. Am. Math. Soc. **311**, 167 (1989).
17. M. Feigin, "Intertwining relations for the spherical parts of generalized Calogero operators," Theor. Math. Phys. **135**, 497 (2003).
18. V. X. Genest, A. Lapointe, and L. Vinet, "The Dunkl-Coulomb problem in the plane," Phys. Lett. A **379**, 923 (2015).
19. T. Hakobyan, O. Lechtenfeld, and A. Nersessian, "The spherical sector of the Calogero model as a reduced matrix model," Nucl. Phys. B **858**, 250 (2012).
20. P. J. Redmond, "Generalization of the Runge-Lenz vector in the presence of an electric field," Phys. Rev. B **133**, 1352 (1964).
21. K. Helfrich, "Constants of motion for separable one-particle problems with cylinder symmetry," Theor. Chim. Acta (Berl.) **24**, 271 (1972).
22. L. Landau and E. Lifshitz, *Course of Theoretical Physics,* Vol. 3: *Quantum Mechanics: Non-Relativistic Theory* (Nauka, Moscow, 1989; Pergamon, Oxford, 1977).
23. C. Coulson and A. Joseph, "A constant of the motion for the two-centre Kepler problem," Int. J. Quantum Chem. **1**, 337 (1967).
24. H. A. Erikson and E. L. Hill, "A note about one-electron states of diatomic molecules," Phys. Rev. **76**, 29 (1949).
25. S. P. Alliluev and A. V. Matveenko, "Symmetry group of the hydrogen molecular ion (A system with separable variables)," Sov. Phys. JETP **24**, 1260 (1967).