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The Tetrahexahedric Calogero Model[1]

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Abstract—We consider the spherical reduction of the rational Calogero model (of type A_{n-1}, without the center of mass) as a maximally superintegrable quantum system. It describes a particle on the $(n-2)$-sphere in a very special potential. A detailed analysis is provided of the simplest non-separable case, $n = 4$, whose potential blows up at the edges of a spherical tetrahexahedron, tesselating the two-sphere into 24 identical right isosceles spherical triangles in which the particle is trapped. We construct a complete set of independent conserved charges and of Hamiltonian intertwiners and elucidate their algebra. The key structure is the ring of polynomials in Dunkl-deformed angular momenta, in particular the subspaces invariant and antiinvariant under all Weyl reflections, respectively.

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1. SOME HISTORY

The Calogero model has a 45-year history, starting in 1971 with the original Calogero paper [1]. Ten years later Olshanetsky and Perelemov generalized the A_{n-1} model to arbitrary finite-dimensional Lie algebras and demonstrated their classical [2] and quantum [3] integrability. In 1983, the superintegrability of the Calogero-Moser system was established by Wojciechowski [4]. Starting with their seminal 1990 paper [5] on commutative rings of partial differential operators and Lie algebras, Veselov and Chalykh initiated a series of works on intertwiners (shift operators) and the exact energy spectrum for integer couplings (multiplicities). In parallel, employing the differential-difference operators associated to reflection groups and introduced by Dunkl [6], Heckman gave an elementary construction for commuting charges and intertwiners [7]. The first investigation of the spherical reduction of the rational Calogero model (here called 'angular Calogero model') goes back to M. Feigin in 2003 [8]. The A_2 and A_3 cases were analyzed classically in 2008 by Hakobyan, Nersessian and Yeghikyan [9], and five years later the quantum energy spectra and eigenstates were derived for all angular Calogero models by M. Feigin, Lechtenfeld and Polychronakos [10]. More recently, M. Feigin and Hakobyan presented a deeper analysis of the algebra of Dunkl angular momentum operators, and just now the A_2 and A_3 angular models have been reconsidered on the quantum level by the authors [12]. This talk reviews their results.

2. THE ANGULAR (RELATIVE) CALOGERO MODEL

In the first half of the talk, let us introduce the spherical reduction of rational A_{n-1} Calogero model and present some of its salient features. In an n-particle quantum phase space with particle coordinates x^μ and momenta p_μ, where $\mu = 1, 2, \dots, n$, subject to $[x^\mu, p_\nu] = i\delta^\mu_\nu$, the rational Calogero Hamiltonian (after separating the center of mass) reads

$$H = \sum_{\mu<\nu}^{n} \left\{ \frac{1}{2n}(p_\mu - p_\nu)^2 + \frac{g(g-1)}{(x^\mu - x^\nu)^2} \right\}. \tag{1}$$

The strength of the inverse-square two-body potential is parametrized by a real coupling constant g (which could be taken $\geq \frac{1}{4}$). In the 'relative' $2(n-1)$-dimensional phase space, a radial coordinate and momentum are defined via

$$\frac{1}{n}\sum_{\mu<\nu}(x^\mu - x^\nu)^2 = r^2$$
$$\text{and}\quad \frac{1}{n}\sum_{\mu<\nu}(p_\mu - p_\nu)^2 = p_r^2 + \frac{1}{r^2}L^2 + \frac{(n-2)(n-4)}{4r^2}. \tag{2}$$

It is convenient to switch to $n-1$ 'relative' coordinates y^i and momenta p_i, with $i = 1, 2, \dots, n-1$,

$$r^2 = \sum_{i=1}^{n-1}(y^i)^2, \quad p_i \equiv p_{y^i},$$
$$L_{ij} = -i(y^i p_j - y^j p_i), \quad L^2 = -\sum_{i<j} L_{ij}^2. \tag{3}$$

[1] The article is published in the original.

In terms of polar coordinates $(r,\vec{\theta})$ on $\mathbb{R}^{n-1}$, the Hamiltonian takes the form

$$H = \frac{1}{2}p_r^2 + \frac{(n-2)(n-4)}{8r^2} + \frac{1}{r^2}H_\Omega \quad \text{with } H_\Omega = \frac{1}{2}L^2 + U(\vec{\theta}), \tag{4}$$

where the angular potential is

$$U(\vec{\theta}) = r^2\sum_{\mu<\nu}\frac{g(g-1)}{(x^\mu - x^\nu)^2} = r^2\sum_{\alpha\in\mathcal{R}_+}\frac{g(g-1)}{(\alpha\cdot y)^2} = \frac{g(g-1)}{2}\sum_{\alpha\in\mathcal{R}_+}\cos^{-2}\theta_\alpha. \tag{5}$$

Here, we introduced the A_{n-1} positive root system $\mathcal{R}_+$ and the angle θ_α between the point $\vec{\theta}\in S^{n-2}$ and the root α. H_Ω is the angular (relative) Calogero Hamiltonian, our object of interest.

In the position representation, we pass to differential operators,

$$p_i \mapsto -\mathrm{i}\partial_i \Rightarrow p_r \mapsto -\mathrm{i}\left(\partial_r + \frac{n-2}{2r}\right), \tag{6}$$

so our Hamilton operators become

$$\begin{aligned} H &\mapsto -\frac{1}{2}\left(\partial_r^2 + \frac{n-2}{r}\partial_r\right) + \frac{1}{r^2}H_\Omega \\ &= S^{-1}\left[-\frac{1}{2}\left(\partial_r^2 - \frac{(n-2)(n-4)}{4r^2}\right) + \frac{1}{r^2}H_\Omega\right]S \\ H_\Omega &\mapsto -\frac{1}{2}\sum_{i<j}(y^i\partial_j - y^j\partial_i)^2 + r^2\sum_{\alpha\in\mathcal{R}_+}\frac{g(g-1)}{(\alpha\cdot y)^2} \\ &\text{with } S = r^{\frac{n-2}{2}}. \end{aligned} \tag{7}$$

The spectrum and the eigenfunctions of H are known,

$$H\Psi_{E,q} = E\Psi_{E,q} \text{ with } E\in\mathbb{R}_{\geq 0} \quad \text{and } \Psi_{E,q}(r,\vec{\theta}) = r^{-\frac{n-3}{2}}J_{q+(n-3)/2}(\sqrt{2E}r)v_q(\vec{\theta}), \tag{8}$$

where we took advantage of the conformal invariance to separate in polar coordinates. The angular wave function $v_q(\vec{\theta})$ is an eigenfunction of the angular Hamiltonian, whose spectrum is also in the literature,

$$\begin{aligned} H_\Omega v_q &= \varepsilon_q v_q \text{ with } \varepsilon_q = \frac{1}{2}q(q+n-3) \\ &\text{and } q = \frac{1}{2}n(n-1)g + \ell \\ &\text{where } \ell = 3\ell_3 + 4\ell_4 + \ldots + n\ell_n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

The degeneracy of energy level ε_q is given by

$$\deg_n(\varepsilon_q) = p_n(\ell) - p_n(\ell-1) - p_n(\ell-2) + p_n(\ell-3) \tag{10}$$

with the restricted partitions $p_n(\ell)$ given by the simple generating function

$$p_n(t) := \sum_{\ell=0}^{\infty}p_n(\ell)t^\ell = \prod_{m=1}^{n}(1-t^m)^{-1}. \tag{11}$$

Relevant for this talk are the cases of $n = 3$ and 4,

$$\begin{aligned} \deg_3(\ell) &= \begin{cases} 0 & \text{for } \ell = 1,2 \bmod 3 \\ 1 & \text{for } \ell = 0 \bmod 3, \end{cases} \\ \deg_4(\ell) &= \left\lfloor\frac{\ell}{12}\right\rfloor + \begin{cases} 0 & \text{for } \ell = 1,2,5 \bmod 12 \\ 1 & \text{for } \ell = \text{else} \bmod 12 \end{cases}. \end{aligned} \tag{12}$$

All the interesting nontrivial structure is hidden in the angular eigenfunctions:

$$v_q(\vec{\theta}) \equiv v_\ell^{(g)}(\vec{\theta}) \sim r^{n-3+q}\left(\prod_{\mu=3}^{n}\sigma_\mu(\{\mathcal{D}_i\})^{\ell_\mu}\right)\Delta^g r^{3-n-n(n-1)g}, \tag{13}$$

which employs the Vandermonde Δ and the (mutually commuting) Dunkl operators $\mathcal{D}_i$ as arguments in the μth Newton sum $\sigma_\mu(y) = \sum_i (y^i)^\mu$,

$$\Delta = \prod_{\alpha\in\mathcal{R}_+}\alpha\cdot y \quad \text{and } \mathcal{D}_i = \partial_i - g\sum_{\alpha\in\mathcal{R}_+}\frac{\alpha_i}{\alpha\cdot y}s_\alpha, \tag{14}$$

where s_α denotes the reflection on the hyperplane orthogonal to the root α. These wave functions contain a factor of Δ^g and are directly related to Dunkl-deformed Weyl-symmetric harmonic polynomials,

$$v_\ell^{(g)}(\vec{\theta}) = r^{-q}\Delta^g\tilde{h}_\ell^{(g)} \text{ with } H(\Delta^g\tilde{h}_\ell^{(g)}) = 0. \tag{15}$$

The $\mathcal{D}_i$, y^i and s_α form a rational Cherednik algebra. The restriction 'res' of its elements to Weyl-invariant functions yields important differential operators, in particular our Hamiltonians. To make this explicit, we 'Dunkl-deform' not only the linear momenta, $\partial_i \Rightarrow \mathcal{D}_i$ but also the angular momenta,

$$L_{ij} \mapsto -(y^i\partial_j - y^j\partial_i) \Rightarrow \mathcal{L}_{ij} = -(y^i\mathcal{D}_j - y^j\mathcal{D}_i), \tag{16}$$

and define the 'pre-Hamiltonians'

$$\mathcal{H} = -\frac{1}{2}\sum_i\mathcal{D}_i^2 \quad \text{and } \mathcal{H}_\Omega = -\frac{1}{2}\sum_{i<j}\mathcal{L}_{ij}^2 + \frac{1}{2}g\Sigma_\alpha s_\alpha(g\Sigma_\alpha s_\alpha + n - 3), \tag{17}$$

whose Weyl-symmetric restriction produce

$$H = \mathrm{res}(\mathcal{H}) \quad \text{and } H_\Omega = \mathrm{res}(\mathcal{H}_\Omega) = \frac{1}{2}\mathrm{res}(\mathcal{L}^2) + \varepsilon_q(\ell = 0). \tag{18}$$

The Cherednik subalgebra generated by the $\mathcal{L}_{ij}$ and the Weyl reflections is given by the relations

$$[\mathcal{L}_{ij},\mathcal{L}_{k\ell}] = \mathcal{L}_{i\ell}\mathcal{S}_{jk} - \mathcal{L}_{ik}\mathcal{S}_{j\ell} - \mathcal{L}_{j\ell}\mathcal{S}_{ik} + \mathcal{L}_{jk}\mathcal{S}_{i\ell}, \tag{19}$$

$$\mathcal{L}_{ij}\mathcal{L}_{k\ell} + \mathcal{L}_{jk}\mathcal{L}_{i\ell} + \mathcal{L}_{ki}\mathcal{L}_{j\ell} = \mathcal{L}_{ij}\mathcal{S}_{k\ell} + \mathcal{L}_{jk}\mathcal{S}_{i\ell} + \mathcal{L}_{ki}\mathcal{S}_{j\ell}, \tag{20}$$

$$[\mathcal{S}_{ij},\mathcal{L}_{k\ell}] = 0,\ \ \{\mathcal{S}_{ij},\mathcal{L}_{ij}\} = 0,\quad \mathcal{S}_{ij}\mathcal{L}_{ik} = \mathcal{L}_{jk}\mathcal{S}_{ij}, \tag{21}$$

$$\text{with } \mathcal{S}_{ij} = \begin{cases} -g s_{ij} & \text{for } i \neq j \\ 1 + g\sum_{k(\neq i)} s_{ik} & \text{for } i = j \end{cases}. \tag{22}$$

It is a 'Dunkl deformation' of $so(n-1)$, with H_Ω being the Casimir invariant

A hallmark of Calogero models is their isospectrality, which is characterized by the existence of intertwining (or shift) operators relating the energy spectra at couplings g and $g+1$. This concept is well established for the full rational model, but is also works in the angular submodel. There, angular intertwiners are differential operators M_s in $\vec{\theta}$ of some order s, constructed with the following recipe,

$$M_s = \text{res}(\mathcal{M}_s) \quad \text{with } \mathcal{M}_s = \text{Weyl antiinvariant in } \{\mathcal{L}_{ij}\} \text{ of degree } s. \tag{23}$$

Since $[\mathcal{L}_{ij},\mathcal{H}] = 0$ and M_s has no r dependence, it follows that

$$[\mathcal{M}_s,\mathcal{H}_\Omega] = 0 \Rightarrow M_s^{(g)}H_\Omega^{(g)} = H_\Omega^{(g+1)}M_s^{(g)} \quad \text{and } M_s^{(g)}v_\ell^{(g)} \sim v_{\ell-n(n-1)/2}^{(g+1)}. \tag{24}$$

The adjoint $M_s^{(g)\dagger} = M_s^{(-g)}$ intertwines in the opposite direction, i.e. $M_s^{(-g)}v_\ell^{(g+1)} \sim v_{\ell+n(n-1)/2}^{(g)}$. It follows that for integer g we can obtain the angular eigenfunctions more directly by successively applying intertwiners to the free eigenfunctions, say at $g = 1$,

$$v_\ell^{(g)} \sim M_{s_1}^{(g-1)}M_{s_2}^{(g-2)}\cdots M_{s_{g-1}}^{(1)}v_{\ell+(g-1)n(n-1)/2}^{(1)}. \tag{25}$$

An important issue is the existence of conserved charges beyond the Hamiltonian H_Ω. Obviously, $[M_s^\dagger M_s, H_\Omega] = 0 = [M_sM_s^\dagger, H_\Omega]$, but this need not provide new quantities. However, any Weyl-invariant polynomial $\mathcal{C}_t(\mathcal{L}_{ij})$ of some degree t gives rise to a conserved charge,

$$\mathcal{C}_t(\mathcal{L}_{ij}) \text{ Weyl invariant} \Rightarrow C_t = \text{res}(\mathcal{C}_t) \text{ commutes with } H_\Omega. \tag{26}$$

We already know of $C_0 = 1$ and $C_2 = -\text{res}(\mathcal{L}^2)$ but expect $2n-5$ algebraically independent constants of motion (beyond C_0) in a superintegrable theory. Other than the Liouville charges in the full Calogero model, they will generically mix under the intertwining action,

$$M_s^{(g)}C_t^{(g)} = \sum_{s',t'}\Gamma_{st}^{s't'}(g)C_{t'}^{(g+1)}M_{s'}^{(g)} \tag{27}$$

with some coefficient functions $\Gamma_{st}^{s't'}(g)$.

3. WARMUP: THE HEXAGONAL OR PÖSCHL-TELLER MODEL

Let us illustrate the structures just mentioned on the first nontrivial example, which at $n = 3$ is the A_2 model. Its spherical reduction (to the unit circle) is known as the Pöschl–Teller model, but we call it 'hexagonal' because the potential is singular at angles $\phi = (2k+1)\pi/6$. The relation between the 3 particle coordinates x^μ and the 2 Jacobi relative coordinates y^i orthogonal to the center of mass X is

$$\begin{aligned} x^1 &= X + \frac{1}{\sqrt{2}}y^1 + \frac{1}{\sqrt{6}}y^2, \quad \partial_{x^1} = \frac{1}{3}\partial_X + \frac{1}{\sqrt{2}}\partial_{y^1} + \frac{1}{\sqrt{6}}\partial_{y^2}, \\ x^2 &= X - \frac{1}{\sqrt{2}}y^1 + \frac{1}{\sqrt{6}}y^2, \quad \partial_{x^2} = \frac{1}{3}\partial_X - \frac{1}{\sqrt{2}}\partial_{y^1} + \frac{1}{\sqrt{6}}\partial_{y^2}, \\ x^3 &= X - \frac{2}{\sqrt{6}}y^2, \quad \partial_{x^3} = \frac{1}{3}\partial_X - \frac{2}{\sqrt{6}}\partial_{y^2}. \end{aligned} \tag{28}$$

Performing the polar decomposition and introducing a complex coordinate,

$$y^1 = r\cos\phi \ \text{ and } \ y^2 = r\sin\phi \Rightarrow w := y^1 + \mathrm{i}y^2 = r\mathrm{e}^{\mathrm{i}\phi}, \tag{29}$$

the angular Hamiltonian takes the form

$$H_\Omega = \frac{1}{2}(w\partial_w - \bar{w}\partial_{\bar{w}})^2 + g(g-1)\frac{18(w\bar{w})^3}{(w^3+\bar{w}^3)^2} \quad \text{since} \tag{30}$$

$$U(\phi) = \frac{g(g-1)}{2}\sum_{k=0,1,2}\cos^{-2}\left(\phi + k\frac{2\pi}{3}\right) = \frac{9}{2}g(g-1)\cos^{-2}(3\phi) = g(g-1)\frac{18(w\bar{w})^3}{(w^3+\bar{w}^3)^2}. \tag{31}$$

Its spectrum depends on a single quantum number $\ell = 3\ell_3$, with $\ell_3 \in \mathbb{N}_0$,

$$\varepsilon_q = \frac{1}{2}q^2 \quad \text{with } q = 3g + \ell = 3(g + \ell_3) \text{ and } \deg(\varepsilon_q) = 1. \tag{32}$$

Since the third Newton sum is $\sigma_3(w,\bar{w}) = w^3 - \bar{w}^3$, the angular wave functions are constructed as

$$v_q(\phi) \equiv v_\ell^{(g)}(\phi) \sim r^q(\mathcal{D}_w^3 - \mathcal{D}_{\bar{w}}^3)^{\ell_3}\Delta^g r^{-6g} = r^{-q}\Delta^g\tilde{h}_\ell^{(g)}(w^3,\bar{w}^3), \tag{33}$$

where the ingredients are

$$\Delta \sim w^3 + \bar{w}^3 \sim r^3 \cos(3\phi) \quad \text{and} \tag{34}$$

$$\mathcal{D}_w = \partial_w - g\left\{\frac{1}{w+\bar{w}} s_0 + \frac{\rho}{\rho w + \bar{\rho}\bar{w}} s_+ + \frac{\bar{\rho}}{\bar{\rho} w + \rho \bar{w}} s_-\right\} \tag{35}$$

with $\rho = e^{2\pi i/3}$.

The application of the Dunkl operators can be evaluated analytically, arriving at

$$\tilde{h}_\ell^{(g>0)}(w^3, \bar{w}^3) = \sum_{k=0}^{\ell_3} (-1)^k \frac{\Gamma(g+k)\Gamma(g+\ell_3-k)}{\Gamma(g)\Gamma(1+k)\Gamma(1+\ell_3-k)} w^{\ell-3k} \bar{w}^{3k}. \tag{36}$$

The table below lists some low-lying hexagonal wave functions, abbreviating $(m\bar{m}) := w^{3m}\bar{w}^{3\bar{m}}$.

ℓ	$\tilde{h}_\ell^{(0)}$	$\tilde{h}_\ell^{(1)}$	$\tilde{h}_\ell^{(2)}$
0	(00)	(00)	(00)
3	(10) – (01)	(10) – (01)	(10) – (01)
6	(20) + (02)	(20) – (11) + (02)	3(20) – 4(11) + 3(02)
9	(30) – (03)	(30) – (21) + (12) – (03)	4(30) – 6(21) + 6(12) – 4(03)
12	(40) + (04)	(40) – (31) + (22) – (13) + (04)	5(40) – 8(31) + 9(22) – 8(13) + 5(04)

The simplest Weyl antiinvariant build from $\mathcal{L}_{12}$ is the Dunklized angular momentum itself,

$$\mathcal{M}_1 \sim i(w\mathcal{D}_w - \bar{w}\mathcal{D}_{\bar{w}}) \sim i(w\partial_w - \bar{w}\partial_{\bar{w}}) - ig\left\{\frac{w-\bar{w}}{w+\bar{w}} s_0 + \frac{\rho w - \bar{\rho}\bar{w}}{\rho w + \bar{\rho}\bar{w}} s_+ + \frac{\bar{\rho} w - \rho\bar{w}}{\bar{\rho} w + \rho\bar{w}} s_-\right\}, \tag{37}$$

whose Weyl-symmetric restriction gives a most simple angular intertwiner,

$$M_1 \sim i(w\partial_w - \bar{w}\partial_{\bar{w}}) - 3ig\frac{w^3 - \bar{w}^3}{w^3 + \bar{w}^3} = i\Delta^g(w\partial_w - \bar{w}\partial_{\bar{w}})\Delta^{-g} = \partial_\phi + 3g\tan 3\phi, \tag{38}$$

which allows for an even simpler recursion relation for the hexagonal wave functions,

$$\tilde{h}_\ell^{(g+1)} \sim i\Delta^{-1}(w\partial_w - \bar{w}\partial_{\bar{w}})\tilde{h}_{\ell+3}^{(g)}. \tag{39}$$

Iterating this recursion is an easier way to construct these wave functions from the ground state.

Because

$$(M_1^\dagger M_1)^{(g)} = -2H_\Omega^{(g)} + 9g^2 = -\mathrm{res}(\mathcal{L}^2) = -C_2^{(g)}, \tag{40}$$

there is no further conserved charge besides the angular Hamiltonian in the hexagonal model.

4. TETRAHEXAHEDRIC MODEL: THE SPECTRUM

Now we pass to the next and more interesting case, $n = 4$. This angular model is quite new and describes a particle on the two-sphere with a non-separable potential. We call it tetrahexahedric because the singular loci of the potential are six great circles which form the edges of a spherical polyeder called tetrahexahedron. Therefore, the particle is trapped in one of 24 identical fundamental domains (the faces), which have the shape of a (spherical) right isosceles triangle. It is convenient to pass to Walsh–Hadamard relative coordinates (due to $A_4 \simeq D_3$):

$$\begin{aligned}
x^1 &= X + \tfrac{1}{2}(+x+y+z), \\
\partial_{x^1} &= \tfrac{1}{4}\partial_X + \tfrac{1}{2}(+\partial_x + \partial_y + \partial_z), \\
x^2 &= X + \tfrac{1}{2}(+x-y-z), \\
\partial_{x^2} &= \tfrac{1}{4}\partial_X + \tfrac{1}{2}(+\partial_x - \partial_y - \partial_z), \\
x^3 &= X + \tfrac{1}{2}(-x+y-z), \\
\partial_{x^3} &= \tfrac{1}{4}\partial_X + \tfrac{1}{2}(-\partial_x + \partial_y - \partial_z), \\
x^4 &= X + \tfrac{1}{2}(-x-y+z), \\
\partial_{x^4} &= \tfrac{1}{4}\partial_X + \tfrac{1}{2}(-\partial_x - \partial_y + \partial_z),
\end{aligned} \tag{41}$$

and introduce spherical coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{42}$$

The angular momenta and the spherical Laplacian take the familiar form

$$L_x = -(y\partial_z - z\partial_y),\quad L_y = -(z\partial_x - x\partial_z),\quad L_z = -(x\partial_y - y\partial_x) \tag{43}$$

$$\text{and } L^2 = -(L_x^2 + L_y^2 + L_z^2) = -\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial_\phi^2, \tag{44}$$

and the angular Hamiltonian reads $H_\Omega = \frac{1}{2}L^2 + U(\theta,\phi)$ with

$$\begin{aligned} U(\theta,\phi) &= 2g(g-1)(x^2+y^2+z^2) \times\left(\frac{x^2+y^2}{(x^2-y^2)^2} + \frac{y^2+z^2}{(y^2-z^2)^2} + \frac{z^2+x^2}{(z^2-x^2)^2}\right) \\ &= 2g(g-1)\left\{\frac{1}{\sin^2\theta\cos^2 2\phi} + \frac{\cos^2\theta + \sin^2\theta\cos^2\phi}{(\cos^2\theta - \sin^2\theta\cos^2\phi)^2} + \frac{\cos^2\theta + \sin^2\theta\sin^2\phi}{(\cos^2\theta - \sin^2\theta\sin^2\phi)^2}\right\}. \end{aligned} \tag{45}$$

The tetrahexahedric energy spectrum is given by

$$\varepsilon_q = \frac{1}{2}q(q+1) \text{ with } q = 6g + \ell = 6g + 3\ell_3 + 4\ell_4 \text{ and } \ell_3, \ell_4 \in \mathbb{N}_0. \tag{46}$$

The corresponding wave functions can be computed from

$$v_\ell^{(g)}(\theta,\phi) \sim r^{q+1}(\mathcal{D}_x\mathcal{D}_y\mathcal{D}_z)^{\ell_3} \times (\mathcal{D}_x^4 + \mathcal{D}_y^4 + \mathcal{D}_z^4)^{\ell_4}\Delta^g r^{1-12g} = r^{-q}\Delta^g \tilde{h}_\ell^{(g)}(x,y,z), \tag{47}$$

$$\text{with}\quad \Delta = (x^2-y^2)(x^2-z^2)(y^2-z^2) \tag{48}$$

and the linear Dunkl operators

$$\begin{aligned} \mathcal{D}_x &= \partial_x - \frac{g}{x+y}s_{x+y} - \frac{g}{x-y}s_{x-y} - \frac{g}{z+x}s_{x+z} - \frac{g}{x-z}s_{z-x}, \\ \mathcal{D}_y &= \partial_y - \frac{g}{y+x}s_{x+y} - \frac{g}{y-x}s_{x-y} - \frac{g}{y+z}s_{y+z} - \frac{g}{y-z}s_{y-z}, \\ \mathcal{D}_z &= \partial_z - \frac{g}{z+x}s_{z+x} - \frac{g}{z-x}s_{z-x} - \frac{g}{z+y}s_{y+z} - \frac{g}{z-y}s_{y-z} \end{aligned} \tag{49}$$

including the elementary reflections constituting the S_4 Weyl group action,

$$\begin{aligned} s_{x+y} &: (x,y,z) \mapsto (-y,-x,+z), \\ s_{x-y} &: (x,y,z) \mapsto (+y,+x,+z), \\ s_{y+z} &: (x,y,z) \mapsto (+x,-z,-y), \\ s_{y-z} &: (x,y,z) \mapsto (+x,+z,+y), \\ s_{z+x} &: (x,y,z) \mapsto (-z,+y,-x), \\ s_{z-x} &: (x,y,z) \mapsto (+z,+y,+x). \end{aligned} \tag{50}$$

The following table lists the low-lying tetrahexahedric wave functions for $g = 0$ and $g = 1$, using the notation

$$\{rst\} := x^r y^s z^t + x^r y^t z^s + x^s y^t z^r + x^s y^r z^t + x^t y^r z^s + x^t y^s z^r.$$

ℓ_3	ℓ_4	$\tilde{h}^{(0)}_{\ell_3,\ell_4}$
0	0	{000}
1	0	{111}
0	1	{400} – 3{220}
2	0	{600} – 15{420} + 30{222}
1	1	3{511} – 5{331}
0	2	{800} – 28{620} + 35{440}
3	0	9{711} – 63{531} + 70{333}
2	1	{1000} – 45{820} + 42{640} + 504{622} – 630{442}
1	2	5{911} – 60{731} + 63{551}
4	0	36{1200} – 2376{1020} + 2445{840} – 46125{822}+4893{660} – 215250{642} + 179375{444}
0	3	101{1200} – 6666{1020} + 47100{840} + 8685{822} – 42609{660} – 40530{642}+33775{444}

ℓ_3	ℓ_4	$\tilde{h}^{(1)}_{\ell_3,\ell_4}$
0	0	{000}
1	0	{111}
0	1	3{400} − 11{220}
2	0	3{600} − 39{420} + 196{222}
1	1	5{511} − 13{331}
0	2	{800} − 20{620} + 23{440} + 12{422}
3	0	3{711} − 27{531} + 56{333}
2	1	15{1000} − 425{820} + 576{640} + 7568{622} − 14454{442}
1	2	35{911} − 476{731} + 477{551} + 204{533}
4	0	12{1200} − 456{1020} + 657{840} + 13581{822} + 1137{660} − 88842{642} + 114007{444}
0	3	813{1200} − 30894{1020} + 165652{840} + 72131{822} − 147943{660} − 169702{642} + 57527{444}

We note that these are eigenfunctions of the free model, $H_\Omega = \frac{1}{2}L^2$, since the potential is absent at $g = 0$ or 1, but they are S_4 invariant, The interacting eigenfunctions are of the same form, only the coefficients depend on g.

5. TETRAHEXAHEDRIC MODEL: INTERTWINER AND INTEGRABILITY

In order to construct the intertwiners of the tetrahexahedric model, one starts with the angular Dunkl operators,

$$\begin{aligned}
\mathcal{L}_x &= L_x + g\Big\{\frac{z}{x-y}s_{x-y} - \frac{z}{x+y}s_{x+y} - \frac{y}{x-z}s_{z-x} \\
&\quad + \frac{y}{z+x}s_{z+x} - \frac{y+z}{y-z}s_{y-z} + \frac{y-z}{y+z}s_{y+z}\Big\}, \\
\mathcal{L}_y &= L_y + g\Big\{\frac{x}{y-z}s_{y-z} - \frac{x}{y+z}s_{y+z} - \frac{z}{y-x}s_{x-y} \\
&\quad + \frac{z}{y+x}s_{x+y} - \frac{z+x}{z-x}s_{z-x} + \frac{z-x}{z+x}s_{z+x}\Big\}, \\
\mathcal{L}_z &= L_z + g\Big\{\frac{y}{z-x}s_{z-x} - \frac{y}{z+x}s_{z+x} - \frac{x}{z-y}s_{y-z} \\
&\quad + \frac{x}{z+y}s_{y+z} - \frac{x+y}{x-y}s_{x-y} + \frac{x-y}{x+y}s_{x+y}\Big\}.
\end{aligned} \tag{51}$$

It turns out that the simplest Weyl antiinvariant is cubic,

$$\begin{aligned}
\mathcal{M}_3 \sim \frac{1}{6}(&\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z + \mathcal{L}_x\mathcal{L}_z\mathcal{L}_y \\
&+ \mathcal{L}_y\mathcal{L}_z\mathcal{L}_x + \mathcal{L}_y\mathcal{L}_x\mathcal{L}_z \\
&+ \mathcal{L}_z\mathcal{L}_x\mathcal{L}_y + \mathcal{L}_z\mathcal{L}_y\mathcal{L}_x),
\end{aligned} \tag{52}$$

and taking the Weyl-symmetric reduction we obtain a first angular intertwiner,

$$\begin{aligned}
M_3 &\sim y^2z\partial_{zxx} - yz^2\partial_{xxy} + \frac{1}{2}(y^2 - z^2)\partial_{xx} \\
&+ 4g\frac{yz}{y^2 - z^2}(yz\partial_{xx} + x^2\partial_{yz} - zx\partial_{xy}) \\
&+ g\Bigg[2gy^2z^2\Bigg(\frac{8g}{(x^2-y^2)(z^2-x^2)} \\
&+ \frac{16g}{(z^2-x^2)(y^2-z^2)} - \frac{2g-1}{(x^2-y^2)^2} + \frac{2g-1}{(z^2-x^2)^2}\Bigg) \\
&- \frac{2x^2y^2}{(z^2-x^2)^2} + \frac{2x^2z^2}{(x^2-y^2)^2} - \frac{2y^2}{x^2-y^2} \\
&- \frac{2z^2}{z^2-x^2} - 2\frac{y^2+z^2}{y^2-z^2}\Bigg]x\partial_x + 2g(g-1)(g+2) \\
&\times x^2\left[\frac{y^2+z^2}{(y^2-z^2)^2} + z\left(\frac{1}{(y-z)^3} - \frac{1}{(y+z)^3}\right)\right] \\
&+ g\left(2g^2 + 8g - 1\right)\frac{y^2+z^2}{y^2-z^2} \\
&+ 2g^2(8+9g)\frac{x^2y^2z^2}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} \\
&- \frac{2}{3}g^3\frac{x^6+y^6+z^6}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} \\
&+ \text{cyclic permutations.}
\end{aligned} \tag{53}$$

In the 'potential-free frame', attained by a similarity transformation, it simplifies to

$$\Delta^{-g}M_3\Delta^g \sim y^2 z\partial_{zxx} - yz^2\partial_{xxy} + \frac{1}{2}(y^2 - z^2)\partial_{xx} + 2g\frac{y^2z^2(y^2-z^2)}{(x^2-y^2)(x^2-z^2)}\partial_{xx} + 4g\frac{xy^2z}{x^2-z^2}\partial_{xz} + 2gx\left[\frac{y^2(x^2+3z^2)}{(x^2-z^2)^2} - \frac{z^2(x^2+3y^2)}{(x^2-y^2)^2}\right]\partial_x + \text{cyclic permutations.} \tag{54}$$

The next independent antiinvariant is sextic,

$$\mathcal{M}_6 \sim \{\mathcal{L}_x^4, \mathcal{L}_y^2\} - \{\mathcal{L}_y^4, \mathcal{L}_x^2\} + \{\mathcal{L}_y^4, \mathcal{L}_z^2\} - \{\mathcal{L}_z^4, \mathcal{L}_y^2\} + \{\mathcal{L}_z^4, \mathcal{L}_x^2\} - \{\mathcal{L}_x^4, \mathcal{L}_z^2\}, \tag{55}$$

and gives rise to a rather lengthy expression (not displayed) for a second intertwiner M_6. We expect that $\Delta^{-g}M_6\Delta^g$ is more compact. All higher angular intertwiners can be reduced to M_3 and M_6.

Let us finally take a look at the conserved charges in this model. It is not hard to see that they are generated by

$$J_k := \operatorname{res}(\mathcal{L}_x^k + \mathcal{L}_y^k + \mathcal{L}_z^k) \quad \text{for} \quad k = (0,)2,4,6, \tag{56}$$

$$\text{with} \quad J_0 = C_0 = 1 \quad \text{and} \quad J_2 = -C_2 = -2H_\Omega + 6g(6g+1). \tag{57}$$

Higher conserved charges are algebraically dependent, e.g.

$$\begin{aligned} 6J_8 &= 8J_6J_2 + 3J_4J_4 - 6J_4J_2J_2 + J_2J_2J_2J_2 \\ &- 12(8+5g+12g^2)J_6 + 4(34+23g+30g^2)J_4J_2 \\ &- 8(5+3g+3g^2)J_2J_2J_2 \\ &+ 24(13+15g-102g^2-72g^3)J_4 \\ &- 4(43+70g-252g^2-144g^3)J_2J_2 \\ &- 48(1+3g)(1+4g)(1-12g)J_2. \end{aligned} \tag{58}$$

Any word in $\{J_2, J_4, J_6\}$ is conserved, but there are some relations in their algebra. Namely, J_0 and J_2 span the center, and

$$[J_2, J_4] = [J_2, J_6] = 0 \quad \text{but} \quad [J_4, J_6] \neq 0, \tag{59}$$

so J_4J_6 and J_6J_4 are two independent new words. The basic intertwining relations read

$$\begin{aligned} M_3^{(g)}J_2^{(g)} &= (J_2^{(g+1)} - 6(7+12g))M_3^{(g)}, \\ M_3^{(g)}J_4^{(g)} &= (J_4^{(g+1)} - 4(11+12g)J_2^{(g+1)} \\ &+ 48(26+73g+48g^2))M_3^{(g)} + 2M_6^{(g)}, \\ M_3^{(g)}J_6^{(g)} &= (J_6^{(g+1)} - (35+36g)J_4^{(g+1)} \\ &- 3(7+4g)J_2^{(g+1)}J_2^{(g+1)} \\ &+ 2(1111+2668g+1392g^2)J_2^{(g+1)} \\ &+ 96(457+1933g+2717g^2+1368g^3+144g^4))M_3^{(g)} \\ &+ (3J_2^{(g+1)} - (115+200g+48g^2))M_6^{(g)}. \end{aligned} \tag{60}$$

Particular conserved quantities are obtained by intertwining 'back and forth', e.g.

$$\begin{aligned} M_3^\dagger M_3 &= 12J_6 - 18J_4J_2 + 6J_2J_2J_2 \\ &- 6(11+16g-48g^2)J_4 + 3(13+24g-48g^2)J_2J_2 \\ &+ 12(1+3g)(1+4g)(1-12g)J_2, \\ M_6^\dagger M_6 &= -12J_6J_6 + 12\{J_6, J_4\}J_2 \\ &- \frac{16}{3}J_6J_2J_2J_2 + 2J_4J_4J_4 - 14J_4J_4J_2J_2 \\ &+ 6J_4J_2J_2J_2J_2 - \frac{2}{3}J_2J_2J_2J_2J_2J_2 + \text{lower-order terms}, \end{aligned} \tag{61}$$

and similarly for $M_3^\dagger M_6$ and $M_6^\dagger M_3$. An additional set of 'odd' conserved charges appears due to the equality $H_\Omega^{(g)} = H_\Omega^{(1-g)}$ (here $* = 3$ or 6):

$$\begin{aligned} Q_{**\dots*}^{(g)} &:= M_*^{(g-1)}M_*^{(g-2)}\cdots M_*^{(1-g)} \\ \Rightarrow Q_{**\dots*}^{(g)}H_\Omega^{(g)} &= Q_{**\dots*}^{(g)}H_\Omega^{(1-g)} = H_\Omega^{(g)}Q_{**\dots*}^{(g)}. \end{aligned} \tag{62}$$

Combining all charges one ends up with a $\mathbb{Z}_2$ graded nonlinear algebra generated by $\{Q, J_2, J_4, J_6\}$.

6. SUMMARY AND OUTLOOK

Let us summarize. We have presented a geometrical picture of a superintegrable but not separable potential on S^{n-2}. The full set of conserved charges is characterized by the Weyl invariants built from the Dunkl-deformed angular momenta. Their algebra is largely unexplored, and it remains to be seen whether there exist bone fide Liouville charges (i.e. $n-2$ charges in involution). This angular Calogero system features a whole set of angular intertwiners (which also intertwine the full Hamiltonian), given by the Weyl antiinvariants built from the angular Dunkl operators. Their form and action on the conserved charges was elucidated in the $n = 3$ (Pöschl–Teller or hexagonal) and $n = 4$ (tetrahexahedric) cases. For integer coupling there exist additional 'odd' conserved charges which, however, have a singular action on the energy eigenstates. This can be cured by a $\mathcal{PT}$ deformation, which regularizes the potential to singular loci of codimension two and brings the (so far singular) negative-coupling states into the picture.

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