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New Cases of Integrable Ninth-Order Conservative and Dissipative Dynamical Systems

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Abstract—New cases of integrable ninth-order dynamical systems that are homogeneous in terms of some of their variables are presented, in which a system on the tangent bundle of a four-dimensional manifold can be distinguished. In this case, the force field is divided into an internal (conservative) and an external one, which has dissipation of different signs. The external field is introduced using some unimodular transformation, and it generalizes previously considered fields. Complete sets of both first integrals and invariant differential forms are given.

Keywords: invariant of dynamical system, essential singularities of invariant, system with dissipation, integrability

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1. INTRODUCTION

It is well known [1–3] that a system of ordinary differential equations can be studied more easily or can sometimes be exactly integrated if a sufficient number of its tensor invariants (not only autonomous first integrals) are found. For example, the number of required first integrals can be reduced if there is an invariant differential form of phase volume. For conservative (in particular, Hamiltonian) systems, this fact is natural when the phase flow preserves volume with a smooth (or constant) density. A more complicated situation (in the sense of smoothness of invariants) occurs for systems with attracting or repelling limit sets. For such systems, the coefficients of the sought invariants have to include, generally speaking, functions with essential singularities (see also [4–6]). According to our approach, for an autonomous system of order m to be exactly integrable, we need to know $m-1$ independent tensor invariants. Moreover, a number of additional conditions have to be imposed on these invariants to achieve exact integrability.

Important cases of integrable systems with a small number of degrees of freedom in a nonconservative force field were considered in [5, 7]. The present study extends the results of [5, 7] to a larger class of dynamical systems. Note that the emphasis in [5, 7] was on finding a sufficient number of first integrals. However, it is well known that a system may sometimes not have a complete set of first integrals, but it has a sufficient number of invariant forms.

For systems of classical mechanics, the concepts of conservativeness, force field, dissipation, etc., are quite natural. Since this paper deals with dynamical systems on the tangent bundle of a smooth manifold (position space), we need to specify these concepts for such systems in more detail.

An "overall" analysis begins with the study of reduced equations of geodesics. By applying a proper parametrization, the left-hand sides of these equations are written as the coordinates of the acceleration of a material particle, while the right-hand sides are set to zero. Accordingly, quantities that are taken in what follows to the right-hand side can be treated as generalized forces. This approach is traditional for classical mechanics, and now it is naturally extended to the more general case of the tangent bundle of a smooth manifold. As a result, we are able, in a sense, to construct "force fields." For example, introducing into the system coefficients that are linear with respect to one of the coordinates in the tangent space (with respect to one of the quasi-velocities of the system), we obtain a force field with dissipation of different signs.

Although dissipation of different signs sounds contradictory, we will nevertheless use it, taking into account that, in mathematical physics, positive dissipation means usual scattering of total energy, while negative dissipation means kind of energy pumping (in mechanics, forces ensuring energy dissipation are

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called dissipative, while forces ensuring energy pumping are called accelerating).

Conservativeness for systems on tangent bundles can be understood in the traditional sense, but the following has to be added. We say that a system is conservative if it has a complete set of smooth first integrals, which suggests that it does not have attracting or repelling limit sets. If it does, then we say that the system has dissipation of some sign. As a result, the system possesses at least one first integral (if any) with essential singularities.

In this paper, a force field is divided into an internal and an external component. The internal field does not change the conservativeness of the system, while the external field can introduce dissipation of different signs into the system. Note that the form of internal force fields is taken from the classical dynamics of a rigid body (see also [5]).

Below, we present first integrals and invariant differential forms for classes of ninth-order dynamical systems that are homogeneous with respect to some of their variables and in which a system with four degrees of freedoms can be distinguished on its eight-dimensional manifold. The force field of the system is divided into an internal (conservative) and an external component, which has dissipation of different signs. The external field is introduced using some unimodular transformation and generalizes previously considered force fields.

2. HOMOGENEOUS SYSTEMS AND THEIR SYMMETRIES

Let v, α, $\beta = (\beta_1, \beta_2, \beta_3)$, and $z = (z_1, \dots, z_4)$ be the phase variables in a smooth dynamical system whose right-hand sides are homogeneous polynomials in v and z with coefficients depending on α and β as follows:

$$(\dot{v}, \dot{z}_4, \dots, \dot{z}_1, v\dot{\alpha}, v\dot{\beta}_1, v\dot{\beta}_2, v\dot{\beta}_3)^{\mathrm{T}} = A(\alpha,\beta)P,$$
$$P^{\mathrm{T}} = (v^2, vz_4, \dots, vz_1, z_4^2, z_4z_3, z_4z_2, z_4z_1, \tag{1}$$
$$z_3^2, z_3z_2, z_3z_1, z_2^2, z_2z_1, z_1^2),$$

where $A(\alpha,\beta)$ is a 9×15 matrix. Then, choosing a new independent variable q ($dq = vdt$, $d/dq = \langle ' \rangle$, $v \neq 0$) and new phase variables Z_k, $z_k = Z_k v$, $k = 1, \dots, 4$, $Z = (Z_1, \dots, Z_4)$, we can rewrite system (1) as

$$v' = v\Psi(\alpha, Z), \quad \Psi(\alpha, Z) = A_v(\alpha,\beta)Q,$$
$$Q^{\mathrm{T}} = (1, Z_4, \dots, Z_1, Z_4^2, Z_4Z_3, Z_4Z_2, Z_4Z_1, \tag{2}$$
$$Z_3^2, Z_3Z_2, Z_3Z_1, Z_2^2, Z_2Z_1, Z_1^2),$$

$$(Z_4', \dots, Z_1', \alpha', \beta_1', \beta_2', \beta_3')^{\mathrm{T}}$$
$$= \hat{A}(\alpha,\beta)Q - (Z_4\Psi(\alpha,Z), \dots, Z_1\Psi(\alpha,Z), 0, 0, 0, 0)^{\mathrm{T}}, \tag{3}$$

where $A_v(\alpha,\beta)$ is the first row of the matrix $A(\alpha,\beta)$, while $\hat{A}(\alpha,\beta)$ is the matrix $A(\alpha,\beta)$ with the first row deleted, i.e.,

$$A(\alpha,\beta) = \begin{pmatrix} A_v(\alpha,\beta) \\ \hat{A}(\alpha,\beta) \end{pmatrix}.$$

In this case, Eq. (2) for v decouples, so eight remaining equations can be treated as system (3) on the eight-dimensional phase manifold $N^8\{Z_4, \dots, Z_1; \alpha, \beta_1, \beta_2, \beta_3\}$.

3. NINTH-ORDER SYSTEMS WITH NO EXTERNAL FORCE FIELD

Consider the following ninth-order system (from class (2), (3)):

$$v' = v\Psi(\alpha, Z),$$
$$\Psi(\alpha, Z) = -b(Z_1^2 + \dots + Z_4^2)\tilde{\Delta}(\alpha)f_4(\alpha), \tag{4}$$
$$\tilde{\Delta}(\alpha) = \frac{d\Delta(\alpha)}{d\alpha}, \quad \Delta(\alpha) = \frac{\delta(\alpha)}{f_4(\alpha)},$$

$$\alpha' = f_4(\alpha)Z_4 + b(Z_1^2 + \dots + Z_4^2)\delta(\alpha),$$
$$Z_4' = -f_4(\alpha)[\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + Df_4(\alpha)]Z_4^2 - \frac{f_1^2(\alpha)}{f_4(\alpha)}\Gamma^{\alpha}_{11}(\alpha,\beta)Z_3^2$$
$$-\frac{f_2^2(\alpha)}{f_4(\alpha)}g_1^2(\beta_1)\Gamma^{\alpha}_{22}(\alpha,\beta)Z_2^2$$
$$-\frac{f_3^2(\alpha)}{f_4(\alpha)}g_2^2(\beta_1)h^2(\beta_2)\Gamma^{\alpha}_{33}(\alpha,\beta)Z_1^2 - Z_4\Psi(\alpha,Z)$$
$$= \zeta_4(Z;\alpha,\beta),$$
$$Z_3' = -f_4(\alpha)[2\Gamma^1_{\alpha 1}(\alpha,\beta) + Df_1(\alpha)]Z_3Z_4$$
$$-\frac{f_2^2(\alpha)}{f_1(\alpha)}g_1^2(\beta_1)\Gamma^1_{22}(\alpha,\beta)Z_2^2$$
$$-\frac{f_3^2(\alpha)}{f_1(\alpha)}g_2^2(\beta_1)h^2(\beta_2)\Gamma^1_{33}(\alpha,\beta)Z_1^2 - Z_3\Psi(\alpha,Z)$$
$$= \zeta_3(Z;\alpha,\beta), \tag{5}$$
$$Z_2' = -f_4(\alpha)[2\Gamma^2_{\alpha 2}(\alpha,\beta) + Df_2(\alpha)]Z_2Z_4$$
$$-f_1(\alpha)[2\Gamma^2_{12}(\alpha,\beta) + Dg_1(\beta_1)]Z_2Z_3$$
$$-\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)}h^2(\beta_2)\Gamma^2_{33}(\alpha,\beta)Z_1^2 - Z_2\Psi(\alpha,Z)$$
$$= \zeta_2(Z;\alpha,\beta),$$

$$Z_1' = -f_4(\alpha)[2\Gamma^3_{\alpha 3}(\alpha,\beta) + Df_3(\alpha)]Z_1Z_4$$
$$- f_1(\alpha)[2\Gamma^3_{13}(\alpha,\beta) + Dg_2(\beta_1)]Z_1Z_3$$
$$- f_2(\alpha)g_1(\alpha)[2\Gamma^3_{23}(\alpha,\beta) + Dh(\beta_2)]Z_1Z_2$$
$$- Z_1\Psi(\alpha,Z) = \zeta_1(Z;\alpha,\beta),$$

$$\beta_1' = Z_3f_1(\alpha), \quad \beta_2' = Z_2f_2(\alpha)g_1(\beta_1),$$
$$\beta_3' = Z_1f_3(\alpha)g_2(\beta_1)h(\beta_2),$$

where $DQ(\pi) = d\ln|Q(\pi)|/d\pi$; $b \geq 0$; and $\Delta(\alpha)$, $f_1(\alpha)$, ..., $f_4(\alpha)$, $g_1(\beta_1)$, $g_2(\beta_1)$, $h(\beta_2)$, and $\Gamma^i_{jk}(\alpha,\beta)$ with $i, j, k = \alpha, \beta$ are smooth functions. This system can be treated as one with no external force field. Equation (4) decouples, so Eqs. (5) can be regarded as an independent system (with four degrees of freedom) on the eight-dimensional manifold $N^8\{Z_4,\ldots,Z_1;\alpha,\beta\} = TM^4\{Z_4,\ldots,Z_1;\alpha,\beta\}$ (the tangent bundle of the smooth four-dimensional manifold $M^4\{\alpha,\beta\}$, see also [7, 8]).

Let us examine the structure of system (5). For simplicity, it corresponds to the following equations of geodesics with 13 nonzero connection coefficients on the tangent bundle $TM^4\{\dot\alpha,\dot\beta;\alpha,\beta\}$ of the manifold $M^4\{\alpha,\beta\}$ (in particular, on the tangent bundle of a (four-dimensional) surface of revolution, Lobachevsky space, etc.; here, all possible $\Gamma^i_{jk}(\alpha,\beta)$ are Christoffel symbols):

$$\begin{gathered}\ddot\alpha + \Gamma^\alpha_{\alpha\alpha}(\alpha,\beta)\dot\alpha^2 + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 \\ + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 + 2\Gamma^1_{\alpha 1}(\alpha,\beta)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 + 2\Gamma^2_{\alpha 2}(\alpha,\beta)\dot\alpha\dot\beta_2 + 2\Gamma^2_{12}(\alpha,\beta)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 + 2\Gamma^3_{\alpha 3}(\alpha,\beta)\dot\alpha\dot\beta_3 + 2\Gamma^3_{13}(\alpha,\beta)\dot\beta_1\dot\beta_3 + 2\Gamma^3_{23}(\alpha,\beta)\dot\beta_2\dot\beta_3 = 0.\end{gathered} \tag{6}$$

Indeed, in the tangent space, choosing new coordinates z_1, ..., z_4 of the form

$$\begin{gathered}\dot\alpha = z_4 f_4(\alpha), \quad \dot\beta_1 = z_3 f_1(\alpha), \\ \dot\beta_2 = z_2 f_2(\alpha) g_1(\beta_1), \quad \dot\beta_3 = z_1 f_3(\alpha) g_2(\beta_1) h(\beta_2),\end{gathered} \tag{7}$$

we obtain the following relations (cf. (5)):

$$Z'_k = \zeta_k(Z;\alpha,\beta), \quad k = 1,\ldots,4. \tag{8}$$

Note that Eqs. (6) are almost everywhere equivalent to collection of (7), (8), which is present primarily in system (5) (instead of (7), it is better to use the equalities $\alpha' = Z_4 f_4(\alpha)$, $\beta_1' = Z_3 f_1(\alpha)$, $\beta_2' = Z_2 f_2(\alpha) g_1(\beta_1)$, and $\beta_3' = Z_1 f_3(\alpha) g_2(\beta_1) h(\beta_2)$).

Below are examples of problems leading to Eqs. (6).

(a) Systems on the tangent bundle of the four-dimensional sphere. Here, it is necessary to distinguish between two cases of metrics on the sphere. One case corresponds to the metric induced by the Euclidean metric of the ambient five-dimensional space. This metric is natural in studying the motion of a point over this sphere. The other case corresponds to the metric induced by symmetry groups typical for the motion of a dynamically symmetric (five-dimensional) rigid body (see also [9–11]).

(b) Systems on the tangent bundle of a more general four-dimensional surface of revolution.

(c) Systems on the tangent bundle of the Lobachevsky space in the Klein model.

System (4), (5) also involves coefficients multiplying the parameter $b \geq 0$. However, they do not violate conservativeness, since system (4), (5) has s complete set of (six) smooth first integrals.

If we consider general equations of geodesics on the tangent bundle of a four-dimensional smooth manifold, then, generally speaking, there are $n^2(n+1)/2$ functions for $n = 4$, i.e., there are 40 coefficients. It follows that the general problem of integrating the equations of geodesics is rather complicated. These connection coefficients are supplemented with functions (in our case, $f_1(\alpha)$, ..., $f_4(\alpha)$, $g_1(\beta_1)$, $g_2(\beta_1)$, $h(\beta_2)$ from (7)) determining coordinates on the tangent bundle.

For this reason, as was noted above, our consideration is restricted to “only” 13 nonzero connection coefficients ($n(n-1)+1$ ones for $n = 4$) forming the equations of geodesics (6). According to this number of connection coefficients, we choose the number of functions determining coordinates on the tangent bundle, namely, this number will be equal to 7 ($n(n-1)/2+1$ for $n = 4$). Thus, we have 20 functions characterizing only the geometry of the phase manifold and coordinates on it.

How many are algebraic and differential conditions ($B(4)$) imposed on $A(4) = 20$ functions ($A(n) = 3n(n-1)/2+2$ for $n = 4$)? Note that these conditions are sufficient for the complete integrability of the equations of geodesics. In this paper, we impose $B(4) = 16$ conditions on available $A(4) = 20$ functions.

The number $B(4)$ is composed of three terms: $B(4) = B_1(4) + B_2(4) + B_3(4)$. Here, $B_1(4)$ is the number of conditions imposed on $f_1(\alpha)$, ..., $f_4(\alpha)$, $g_1(\beta_1)$, $g_2(\beta_1)$, $h(\beta_2)$, namely,

$$\begin{gathered}f_1(\alpha) \equiv f_2(\alpha) \equiv f_3(\alpha) =: f(\alpha), \\ g_1(\beta_1) \equiv g_2(\beta_1) =: g(\beta_1),\end{gathered} \tag{9}$$

i.e., $B_1(4) = 3$ (in the general case, $B_1(n) = (n-1)(n-2)/2$). The quantity $B_2(4)$ is the number of conditions imposed on the connection coefficients, namely,

$$\begin{gathered}\Gamma^1_{\alpha 1}(\alpha,\beta) \equiv \Gamma^2_{\alpha 2}(\alpha,\beta) \equiv \Gamma^3_{\alpha 3}(\alpha,\beta) \equiv \Gamma_1(\alpha), \\ \Gamma^2_{12}(\alpha,\beta) \equiv \Gamma^3_{13}(\alpha,\beta) \equiv \Gamma_2(\beta_1), \quad \Gamma^3_{23}(\alpha,\beta) \equiv \Gamma_3(\beta_2),\end{gathered} \tag{10}$$

i.e., $B_2(4) = 6$ (in the general case, $B_2(n) = n(n-1)/2$). The quantity $B_3(4)$ is the number of algebraic and differential conditions imposed on the functions $f_1(\alpha)$, ..., $f_4(\alpha)$, $g_1(\beta_1)$, $g_2(\beta_1)$, $h(\beta_2)$ and on the connection coefficients, namely,

$$f_4^2(\alpha)[2\Gamma^1_{\alpha 1}(\alpha,\beta) + Df_1(\alpha)] + f_1^2(\alpha)\Gamma^\alpha_{11}(\alpha,\beta) \equiv 0,$$
$$f_4^2(\alpha)[2\Gamma^2_{\alpha 2}(\alpha,\beta) + Df_2(\alpha)] + f_2^2(\alpha)g_1^2(\beta_1)\Gamma^\alpha_{22}(\alpha,\beta) \equiv 0,$$
$$f_4^2(\alpha)[2\Gamma^3_{\alpha 3}(\alpha,\beta) + Df_3(\alpha)] + f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\Gamma^\alpha_{33}(\alpha,\beta) \equiv 0,$$
$$f_1^2(\alpha)[2\Gamma^2_{12}(\alpha,\beta) + Dg_1(\beta_1)] + f_2^2(\alpha)g_1^2(\beta_1)\Gamma^1_{22}(\alpha,\beta) \equiv 0, \quad (11)$$
$$f_1^2(\alpha)[2\Gamma^3_{13}(\alpha,\beta) + Dg_2(\beta_1)] + f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\Gamma^1_{33}(\alpha,\beta) \equiv 0,$$
$$f_2^2(\alpha)g_1^2(\beta_1)[2\Gamma^3_{23}(\alpha,\beta) + Dh(\beta_2)] + f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\Gamma^2_{33}(\alpha,\beta) \equiv 0,$$
$$\Gamma^\alpha_{\alpha\alpha}(\alpha,\beta) + Df_4(\alpha) \equiv 0,$$

i.e., $B_3(4) = 7$ (in the general case, $B_3(n) = n(n-1)/2+1$). Conditions (11) are based on (9), (10), due to which the number of arguments in some functions decreases.

It can be seen that, in the general case, $B(n) = B_1(n) + B_2(n) + B_3(n) = (n-1)^2 + n(n-1)/2 + 1$, moreover, $A(n) - B(n) = n$, which suggests that the number of "arbitrary" functions increases by exactly n as compared with the number of conditions imposed on them (here, n is the dimension of the considered Riemannian manifold). In our case, $A(4) - B(4) = 4$.

As will be shown later, for the complete integrability of system (4), (5), it suffices to know six independent tensor invariants: six first integrals, six independent differential forms, or a combination of integrals and forms amounting to a total of six. Of course, invariants (in particular, for the case of no external force field) can be sought in a more general form than the one considered below (cf. [5–7]). It will be shown below that the complete set consists of six, rather than eight, tensor invariants (in addition to the trivial one being the vector field of the system itself [3]).

It is well known that a first integral of the geodesics equations (6) rewritten as $\ddot{x}^i + \sum_{j,k=1}^4 \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0$, $i = 1,\ldots,4$, is the smooth function $\Phi(\dot{x};x) = \sum_{j,k=1}^4 g_{jk}(x)\dot{x}^j\dot{x}^k$, but we represent it in a simpler form by introducing suitable coordinates on the tangent bundle, thus "flattening" the quadratic form on the phase manifold.

It should be emphasized that Theorem 1 below (which also holds under more general conditions) imposes 16 algebraic and differential relations (9)–(11) on 20 11 functions, namely, on 7 functions $f_1(\alpha)$, ..., $f_4(\alpha)$, $g_1(\beta_1)$, $g_2(\beta_1)$, $h(\beta_2)$ and on 13, generally speaking, nonzero connection coefficients $\Gamma^i_{jk}(\alpha,\beta)$.

Theorem 1. *If conditions* (9)–(11) *are satisfied, then system* (4), (5) *considered on the product* $\mathbf{R}^1_+\{v\} \times TM^4\{Z_4,\ldots,Z_1;\alpha,\beta_1,\beta_2,\beta_3\}$ *has a complete set of six smooth first integrals of the form*

$$\Phi_0(v;Z_4;\alpha) = v^2(1 + 2bZ_4\Delta(\alpha)) = C_0 = \text{const};$$
$$\Phi_1(v;Z_4,\ldots,Z_1) = v^2(Z_1^2 + \ldots + Z_4^2) = C_1^2 = \text{const}; \quad (12)$$

$$\Phi_2(v;Z_3,Z_2,Z_1;\alpha) = v^2\sqrt{Z_1^2 + Z_2^2 + Z_3^2}\,\Delta(\alpha) = C_2 = \text{const}, \quad (13)$$
$$\Delta(\alpha) = A_1 f(\alpha)\exp\left\{2\int_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}, \quad A_1 = \text{const};$$

$$\Phi_3(v;Z_2,Z_1;\alpha,\beta_1) = v^2\sqrt{Z_1^2 + Z_2^2}\,\Delta(\alpha)\Psi_1(\beta_1) = C_3 = \text{const}, \quad (14)$$
$$\Psi_1(\beta_1) = g(\beta_1)\exp\left\{2\int_{\beta_{1,0}}^{\beta_1}\Gamma_2(b)db\right\};$$

$$\Phi_4(v;Z_1;\alpha,\beta_1,\beta_2) = v^2 Z_1\Delta(\alpha)\Psi_1(\beta_1)\Psi_2(\beta_2) = C_4 = \text{const}, \quad (15)$$
$$\Psi_2(\beta_2) = h(\beta_2)\exp\left\{2\int_{\beta_{2,0}}^{\beta_2}\Gamma_3(b)db\right\};$$

$$\Phi_5(\beta_2,\beta_3) = \beta_3 - \int_{\beta_{2,0}}^{\beta_2}\frac{C_4 h(b)}{\sqrt{C_3^2\Psi_2^2(b) - C_4^2}}db = C_5 = \text{const}. \quad (16)$$

Moreover, after its reduction by making the substitutions of the independent variable $d/dt = f_4(\alpha)d/d\tau$ *and of the phase variables*

$$w_4 = Z_4, \quad w_3^* = \ln|w_3|, \quad w_3 = \sqrt{Z_1^2 + Z_2^2 + Z_3^2},$$
$$w_s^* = \ln|w_s + \sqrt{1 + w_s^2}|, \quad s = 1,2, \quad (17)$$
$$w_2 = \frac{Z_2}{Z_1}, \quad w_1 = \frac{Z_3}{\sqrt{Z_1^2 + Z_2^2}}$$

the phase flow of system (4), (5) *preserves its phase volume with the density* $\rho(v) = v^3$ *on the product* $\mathbf{R}^1_+\{v\} \times TM^4\{w_4,w_3^*,w_2^*,w_1^*;\alpha,\beta_1,\beta_2,\beta_3\}$, *i.e., the corresponding differential form* $v^3 dv \wedge dw_4 \wedge dw_3^* \wedge dw_2^* \wedge dw_1^* \wedge d\alpha \wedge d\beta_1 \wedge d\beta_2 \wedge d\beta_3$ *is conserved.*

Note also that the system of equalities (11) can be treated as the possibility of transforming the quadratic form of the manifold's metric to canonical form with energy conservation law (12). The history and the state of the art in this more general problem have been covered rather extensively (we note only [9, 10]). The search for first integrals relies on the fact that the system has additional symmetry groups.

4. INTRODUCING OF AN EXTERNAL FORCE FIELD WITH DISSIPATION VIA UNIMODULAR TRANSFORMATIONS

We modify system (4), (5) with two key parameters $b \geq 0$, $b_1 \neq 0$ by introducing an external force field. If such a field is introduced by adding the coefficient $F(\alpha)f_4(\alpha)$ to the equation for Z_4' in system (18), (19), even though we set $b_1 = 0$, the resulting system is, generally speaking, not conservative. Conservativeness holds under the additional condition $b = 0$. However, we extend the force field, assuming that $b > 0$, $b_1 \neq 0$. Additionally, (as before) the independent variable t is changed to τ according to the formula $d/dt = f_4(\alpha)d/d\tau$ and the derivative with respect to τ is denoted by a prime. Then the considered system on the direct product of the numerical ray and the tangent bundle $TM^4\{Z_4,\ldots,Z_1;\alpha,\beta_1,\beta_2,\beta_3\}$ takes the form

$$\begin{gathered} v' = v\Psi(\alpha, Z),\\ \Psi(\alpha, Z) = -b(Z_1^2 + \ldots + Z_4^2)\tilde{\Delta}(\alpha) + b_1F(\alpha)\Delta(\alpha), \end{gathered} \tag{18}$$

$$\begin{gathered} \alpha' = Z_4 + b(Z_1^2 + \ldots + Z_4^2)\Delta(\alpha) + b_1F(\alpha)\overline{f}(\alpha),\\ \overline{f}(\alpha) = \frac{\mu - \Delta^2(\alpha)}{\tilde{\Delta}(\alpha)},\\ Z_4' = F(\alpha) - [\Gamma^\alpha_{\alpha\alpha}(\alpha,\beta) + Df_4(\alpha)]Z_4^2\\ -\frac{f_1^2(\alpha)}{f_4^2(\alpha)}\Gamma^\alpha_{11}(\alpha,\beta)Z_3^2 - \frac{f_2^2(\alpha)}{f_4^2(\alpha)}g_1^2(\beta_1)\Gamma^\alpha_{22}(\alpha,\beta)Z_2^2\\ -\frac{f_3^2(\alpha)}{f_4^2(\alpha)}g_2^2(\beta_1)h^2(\beta_2)\Gamma^\alpha_{33}(\alpha,\beta)Z_1^2 - Z_4\Psi(\alpha, Z),\\ Z_3' = -[2\Gamma^1_{\alpha 1}(\alpha,\beta) + Df_1(\alpha)]Z_3Z_4\\ -\frac{f_2^2(\alpha)}{f_1(\alpha)f_4(\alpha)}g_1^2(\beta_1)\Gamma^1_{22}(\alpha,\beta)Z_2^2\\ -\frac{f_3^2(\alpha)}{f_1(\alpha)f_4(\alpha)}g_2^2(\beta_1)h^2(\beta_2)\Gamma^1_{33}(\alpha,\beta)Z_1^2 - Z_3\Psi(\alpha, Z),\\ Z_2' = -[2\Gamma^2_{\alpha 2}(\alpha,\beta) + Df_2(\alpha)]Z_2Z_4\\ -\frac{f_1(\alpha)}{f_4(\alpha)}[2\Gamma^2_{12}(\alpha,\beta) + Dg_1(\beta_1)]Z_2Z_3\\ -\frac{f_3^2(\alpha)}{f_2(\alpha)f_4(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)}h^2(\beta_2)\Gamma^2_{33}(\alpha,\beta)Z_1^2 - Z_2\Psi(\alpha, Z),\\ Z_1' = -[2\Gamma^3_{\alpha 3}(\alpha,\beta) + Df_3(\alpha)]Z_1Z_4\\ -\frac{f_1(\alpha)}{f_4(\alpha)}[2\Gamma^3_{13}(\alpha,\beta) + Dg_2(\beta_1)]Z_1Z_3\\ -\frac{f_2(\alpha)}{f_4(\alpha)}g_1(\alpha)[2\Gamma^3_{23}(\alpha,\beta) + Dh(\beta_2)]Z_1Z_2 - Z_1\Psi(\alpha, Z),\\ \beta_1' = Z_3\frac{f_1(\alpha)}{f_4(\alpha)}, \quad \beta_2' = Z_2\frac{f_2(\alpha)}{f_4(\alpha)}g_1(\beta_1),\\ \beta_3' = Z_1\frac{f_3(\alpha)}{f_4(\alpha)}g_2(\beta_1)h(\beta_2), \end{gathered} \tag{19}$$

where $\mu > 0$ is a parameter. Here, the coefficients of the conservative component of the internal force field contain the parameter b, while the coefficients of the nonconservative component of the external field contain the parameter b_1.

The force field in the equations for v' and Z' is defined by the function $\Psi(\alpha, Z)$. The force field is introduced in the form of a two-dimensional column with the first row containing the coefficients of the equation for α' and with the second row containing the coefficients from the function $\Psi(\alpha, Z)$. Thus, the total force field (involving three parameters $b \geq 0$, $b_1 \neq 0$, $\mu > 0$) has the form

$$U\begin{pmatrix} b(Z_1^2 + \ldots + Z_4^2)\\ b_1F(\alpha)\end{pmatrix}, \quad U = \begin{pmatrix} \Delta(\alpha) & \overline{f}(\alpha)\\ -\tilde{\Delta}(\alpha) & \Delta(\alpha)\end{pmatrix},$$

where U is a transformation with determinant μ that is unimodular for $\mu = 1$. Specifically, if $\mu = 1$ and $\Delta(\alpha) = \cos\alpha$ or $\Delta(\alpha) = \sin\alpha$, then this transformation defines rotation by an angle of α. Moreover, this transformation introduces dissipation (of both signs, see also [5–7]) into the system.

5. INVARIANTS OF NINTH-ORDER SYSTEMS WITH DISSIPATION

Now we will integrate the ninth-order system (18), (19) under conditions (9)–(11), which ensure decoupling an independent a seventh-order subsystem.

As will be shown below, for the complete integrability of system (18), (19), it suffices to know six independent tensor invariants: six first integrals, six independent differential forms, or a combination of integrals and forms amounting to a total of six. Of course, invariants can be sought in a more general form than the one considered below.

Additionally, it should be emphasized that Theorem 2 below (which also holds under more general conditions) imposes 16 algebraic and differential relations (9)–(11) on 20 functions, namely, on 7 functions $f_1(\alpha)$, ..., $f_4(\alpha)$, $g_1(\beta_1)$, $g_2(\beta_1)$, $h(\beta_2)$ and on 13, generally speaking, nonzero connection coefficients

$\Gamma^i_{jk}(\alpha,\beta)$. In particular, a consequence of imposing 16 relations is the property $\Gamma^\alpha_{11}(\alpha,\beta) \equiv \Gamma^\alpha_{11}(\alpha) =: \Gamma_4(\alpha)$.

Then, after making the substitutions given by (17) for the phase variables, system (18), (19) splits as follows:

$$v' = v\Psi_0(\alpha, w),\quad \Psi_0(\alpha, w) = -b(w_3^2 + w_4^2)\tilde{\Delta}(\alpha) + b_1F(\alpha)\Delta(\alpha), \tag{20}$$

$$\alpha' = w_4 + b(w_3^2 + w_4^2)\Delta(\alpha) + b_1F(\alpha)\overline{f}(\alpha),$$
$$w_4' = F(\alpha) - \Gamma_4(\alpha)\frac{f^2(\alpha)}{f_4^2(\alpha)}w_3^2 - w_4\Psi_0(\alpha, w), \tag{21}$$
$$w_3' = \Gamma_4(\alpha)\frac{f^2(\alpha)}{f_4^2(\alpha)}w_3w_4 - w_3\Psi_0(\alpha, w),$$

$$w_2' = \pm w_3\sqrt{\frac{1+w_2^2}{1+w_1^2}}\frac{f(\alpha)}{f_4(\alpha)}g(\beta_1)[2\Gamma_3(\beta_2) + Dh(\beta_2)],\quad \beta_2' = \pm\frac{w_2w_3}{\sqrt{(1+w_1^2)(1+w_2^2)}}\frac{f(\alpha)}{f_4(\alpha)}g(\beta_1), \tag{22}$$

$$w_1' = \pm w_3\sqrt{1+w_1^2}\frac{f(\alpha)}{f_4(\alpha)}[2\Gamma_2(\beta_1) + Dg(\beta_1)],\quad \beta_1' = \pm\frac{w_1w_3}{\sqrt{1+w_1^2}}\frac{f(\alpha)}{f_4(\alpha)}, \tag{23}$$

$$\beta_3' = \pm\frac{w_3}{\sqrt{(1+w_1^2)(1+w_2^2)}}\frac{f(\alpha)}{f_4(\alpha)}g(\beta_1)h(\beta_2). \tag{24}$$

It can be seen that, for system (20)–(24) to be completely integrable, it suffices to indicate two independent tensor invariants of system (21), one for each of the systems (22) and (23) (after making suitable substitutions for their independent variables) and two additional tensor invariants "linking" Eqs. (20) and (24) (i.e., altogether there are six of them).

We impose some constraint on the force field. Assume that, for some $\kappa \in \mathbf{R}$,

$$\frac{f^2(\alpha)}{f_4^2(\alpha)}\Gamma_4(\alpha) = \kappa\frac{d}{d\alpha}\ln|\Delta(\alpha)| = \kappa\frac{\tilde{\Delta}(\alpha)}{\Delta(\alpha)}, \tag{25}$$

and, for some $\lambda \in \mathbf{R}$,

$$F(\alpha) = \lambda\frac{d}{d\alpha}\frac{\Delta^2(\alpha)}{2} = \lambda\tilde{\Delta}(\alpha)\Delta(\alpha). \tag{26}$$

Condition (25) is called "geometric," while condition (26) is called an "energy" one. Condition (25) is called geometric, because, among other things, it imposes a constraint on the key connection coefficient $\Gamma_4(\alpha)$, so that the corresponding coefficients of the system are reduced to homogeneous form with respect to $\Delta(\alpha)$ with the help of the functions $f(\alpha)$ and $f_4(\alpha)$ involved in the kinematic relations. Condition (26) is called an energy one because, among other things, the (external) forces become kind of "potential" with respect to the "force" function $\Delta^2(\alpha)/2$, so that the corresponding coefficients of the system are reduced to homogeneous form (again with respect to $\Delta(\alpha)$). It is the function $\Delta(\alpha)$ that, in a sense, introduces dissipation of different signs, or variable dissipation, into the system (see also [12–14]).

Theorem 2. *Assume that conditions* (25) *and* (26) *are satisfied for some* $\kappa, \lambda \in \mathbf{R}$. *Then system* (20)–(24) *has a complete set of six independent first integrals* (*of which one is smooth*, *while five, generally speaking, have essential singularities*). *Additionally*, *this system has six invariant differential forms that are independent of each other*, *but are coupled to the first integrals.*

Indeed, using the homogeneous variables u_1, u_2, $w_3 = u_1\Delta(\alpha)$, and $w_4 = u_2\Delta(\alpha)$, from system (21) we can derive the differential relations

$$\Delta\frac{du_2}{d\Delta} = \frac{\lambda - \kappa u_1^2 - u_2^2 - b_1\lambda\mu u_2}{u_2 + b(u_1^2 + u_2^2)\Delta^2 + b_1\lambda(\mu - \Delta^2)},\quad \Delta\frac{du_1}{d\Delta} = \frac{(\kappa - 1)u_1u_2 - b_1\lambda\mu u_1}{u_2 + b(u_1^2 + u_2^2)\Delta^2 + b_1\lambda(\mu - \Delta^2)}, \tag{27}$$

which easily imply the first-order equation

$$\frac{du_2}{du_1} = \frac{\lambda - b_1\lambda\mu u_2 - u_2^2 - \kappa u_1^2}{(\kappa - 1)u_1u_2 - b_1\lambda\mu u_1}. \tag{28}$$

Equation (28) has the form of an Abel equation [15–17]. Specifically, for $\kappa = -1$, it has the first integral

$$\frac{u_2^2 + u_1^2 + b_1\lambda\mu u_2 - \lambda}{u_1} = C_1 = \text{const}, \tag{29}$$

which, in the previous variables, is given by

$$\Theta_1(w_4, w_3; \alpha) = G_1\left(\frac{w_4}{\Delta(\alpha)}, \frac{w_3}{\Delta(\alpha)}\right) = \frac{w_4^2 + w_3^2 + b_1\lambda\mu w_4\Delta(\alpha) - \lambda\Delta^2(\alpha)}{w_3\Delta(\alpha)} = C_1 = \text{const}. \tag{30}$$

To calculate the divergence of the vector field $W(v; w_4, w_3^*, w_2^*, w_1^*; \alpha, \beta)$, $w_3^* = \ln|w_3|$, $w_s^* = \ln|w_s + \sqrt{1 + w_s^2}|$, $s = 1, 2$, of system (20)–(24) with dissipation, we use the function $\rho(v) = v^3$ (obtained for system (4), (5)). Then the compound system of equations of characteristics for the equation

$$\text{div}[\rho(v; w_4, w_3^*, w_2^*, w_1^*; \alpha, \beta) \times W(v; w_4, w_3^*, w_2^*, w_1^*; \alpha, \beta)] = 0 \tag{31}$$

consists of system (20)–(24) (whose right-hand side is multiplied by the function $\rho(v) = v^3$) and the additional equation

$$\rho' = -v^3b_1\lambda\mu\tilde{\Delta}(\alpha)\rho. \tag{32}$$

System (20)–(24), (32) of equations of characteristics can be assigned the following relations: two from (27) and

$$\Delta\frac{d\rho}{d\Delta} = \frac{-\rho[b_1\lambda\mu]}{u_2 + b(u_1^2 + u_2^2)\Delta^2 + b_1\lambda(\mu - \Delta^2)}. \quad (33)$$

In the general case, the desired first integrals have cumbersome expressions (in particular, if $\kappa = -1$, then equality (29) is used). With the help of Eqs. (27), we obtain an additional first integral of system (21) having the structural form

$$\Theta_2(w_4, w_3; \alpha) = G_2\left(\Delta(\alpha), \frac{w_4}{\Delta(\alpha)}, \frac{w_3}{\Delta(\alpha)}\right) = C_2 = \text{const}. \quad (34)$$

Moreover, (for $\kappa = -1$) the first integral (34) is found from the Bernoulli equation

$$\frac{d\Delta}{du_2} = \frac{(b_1\lambda\mu + u_2)\Delta + \{b[U^2(C_1,u_2) + u_2^2] - b_1\lambda\}\Delta^3}{u(u_2) + U^2(C_1,u_2)},$$

$$U(C_1,u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 + 4u(u_2)}\right\},$$

$$u(u_2) = \lambda - b_1\lambda\mu u_2 - u_2^2.$$

The expression for the first integral (34) in terms of a finite combination of elementary functions depends primarily on the closed-form expression for $\Delta(\alpha)$.

Additionally, system (20)–(24) has a smooth first integral, which, for example, for $b = -b_1$, is given by

$$\Theta_0(v; w_4, w_3; \alpha) = v^2(1 + 2bw_4\Delta(\alpha) - b^2\mu(w_3^2 + w_4^2)) = C_0 = \text{const}. \quad (35)$$

First integrals for the independent (after making substitutions for their independent variables) subsystems (22), (23) have the form

$$\Theta_3(w_2;\beta_2) = \frac{\sqrt{1 + w_2^2}}{\Psi_2(\beta_2)} = C_3 = \text{const},$$
$$\Theta_4(w_1;\beta_1) = \frac{\sqrt{1 + w_1^2}}{\Psi_1(\beta_1)} = C_4 = \text{const}, \quad (36)$$

where the functions $\Psi_s(\beta_s)$, $s = 1, 2$, are given in (14), (15). An additional first integral linking Eq. (24) is found by analogy with (16):

$$\Theta_5(\beta_2,\beta_3) = \beta_3 \mp \int_{\beta_{2,0}}^{\beta_2} \frac{h(b)}{\sqrt{C_3^2\Psi_2^2(b) - 1}}db = C_5 = \text{const}. \quad (37)$$

In turn, Eq. (33) allows us to obtain a function $\rho(v; w_4, w_3^*, w_2^*, w_1^*; \alpha, \beta_1, \beta_2, \beta_3)$ defining an invariant differential volume form. Indeed, we have the invariant relation

$$\rho\cdot\exp\left\{b_1\lambda\mu\int\frac{du_2}{U_2(C_1,u_2)}\right\} = C_\rho = \text{const},$$
$$U_2(C_1,u_2) = 2u(u_2) + C_1U(C_1,u_2).$$

It follows that a possible invariant differential volume form is given by

$$R(v; w_4; \alpha)dv \wedge d\alpha \wedge dw_4 \wedge dw_3^* \wedge dw_2^*$$
$$\wedge dw_1^* \wedge d\beta_1 \wedge d\beta_2 \wedge d\beta_3,$$
$$R(v; w_4; \alpha) = v^3\exp\left\{-b_1\lambda\mu\int\frac{du_2}{U_2(C_1,u_2)}\right\},$$
$$u_2 = \frac{w_4}{\Delta(\alpha)}.$$

Thus, the general solution of the linear partial differential equation (31) becomes

$$\rho = R(v; w_4; \alpha)\cdot\mathcal{F}[\Theta_0, \Theta_1, \ldots, \Theta_5],$$

here, $\mathcal{F}[\Theta_0, \Theta_1, \ldots, \Theta_5]$ is an arbitrary smooth function of six arguments, where $\Theta_0, \Theta_1, \ldots, \Theta_5$ are six independent first integrals (35), (30), (34), (36), and (37), respectively.

In particular, as six functionally independent solutions of Eq. (31), we can use the functions

$$\rho_0(v; w_4, w_3; \alpha) = R(v; w_4; \alpha)\cdot\Theta_0(v; w_4, w_3; \alpha),$$
$$\rho_1(v; w_4, w_3; \alpha) = R(v; w_4; \alpha)\cdot\Theta_1(w_4, w_3; \alpha),$$
$$\rho_2(v; w_4, w_3; \alpha) = R(v; w_4; \alpha)\cdot\Theta_2(w_4, w_3; \alpha),$$
$$\rho_3(v; w_4, w_2; \alpha, \beta_2) = R(v; w_4; \alpha)\cdot\Theta_3(w_2; \beta_2),$$
$$\rho_4(v; w_4, w_1; \alpha, \beta_1) = R(v; w_4; \alpha)\cdot\Theta_4(w_1; \beta_1),$$
$$\rho_5(v; w_4; \alpha, \beta_2, \beta_3) = R(v; w_4; \alpha)\cdot\Theta_5(\beta_2, \beta_3).$$

6. STRUCTURE OF INVARIANTS FOR DISSIPATIVE SYSTEMS AND APPLICATIONS

System (20)–(24) is a dynamical system with variable dissipation [12–14]. For $F(\alpha) \equiv 0$ it turns into a conservative system equivalent to (4), (5). Under certain natural conditions, the latter system has two smooth first integrals of the form (12), (13) in the coordinates w. Moreover, if $F(\alpha)$ is not identically zero, but $b_1 = 0$, then system (20)–(24) under condition (26) has a first integral of the form

$$\Theta|_{B=0}(B; v; w_4, w_3; \alpha) = v^2(w_3^2 + w_4^2 - \lambda\Delta^2(\alpha)) = \text{const}, \quad (38)$$

where $\Theta(B; v; w_4, w_3; \alpha) = v^2(w_3^2 + w_4^2 + B\lambda\mu w_4\Delta(\alpha) - \lambda\Delta^2(\alpha))$ is a family of functions depending on the parameter $B \geq 0$.

Obviously, the ratio of two first integrals (38) and (13) (in the coordinates w) is also a first integral of system (20)–(24) if $F(\alpha)$ is not identically zero, but $b_1 = 0$. However, for $b_1 > 0$, each of the functions

$$\Theta|_{B=b_1}(B; v; w_4, w_3; \alpha) = v^2(w_3^2 + w_4^2 + b_1\lambda\mu w_4\Delta(\alpha) - \lambda\Delta^2(\alpha)) = \text{const} \tag{39}$$

and (13) (in the coordinates w) taken separately is not a first integral of system (20)–(24). Nevertheless, the ratio of functions (39) and (13) (in the coordinates w) is the first integral (30) of system (20)–(24) (for simplicity, $\kappa = -1$) for any $b_1 > 0$.

Overall, as was noted above, for systems with dissipation, the property of functions as first integrals to be transcendental (in the sense that they have essential singularities) is inherited from the existence of attracting or repelling limit sets in the system [4, 13, 14].

Now we consider important cases for the functions $f(\alpha)$, $f_4(\alpha)$ determining the metric on the four-dimensional sphere and for the function $\Delta(\alpha)$:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha\sqrt{1 + \nu\sin^2\alpha}}, \quad \nu \in \mathbf{R}, \quad f_4(\alpha) \equiv -1, \quad \Delta(\alpha) = \sin\alpha, \tag{40}$$

$$f(\alpha) = \frac{1}{\sin\alpha\sqrt{1 + \nu\sin^2\alpha}}, \quad \nu \in \mathbf{R}, \quad f_4(\alpha) \equiv -1, \tag{41}$$

additionally, the following case is of interest in itself:

$$f(\alpha) = \frac{\nu_1\alpha^2}{\sqrt{\alpha^2 + \nu_2}}, \quad \nu_1, \nu_2 \in \mathbf{R}, \quad f_4(\alpha) = \nu_1\alpha. \tag{42}$$

Case (40) forms a class of systems (18), (19) with $\mu = 1$ that correspond to the motion of a dynamically symmetric five-dimensional rigid body at zero levels of cyclic integrals in a nonconservative force field. Specifically, for $\Delta(\alpha) \equiv F(\alpha) \equiv 0$, the considered system describes a geodesic flow on the four-dimensional sphere. In the case of (40), if $\Delta(\alpha) = F(\alpha)/\cos\alpha$, then the system describes the motion of a five-dimensional rigid body in the force field $F(\alpha)$ under the action of a follower force [12]. In particular, if $F(\alpha) = \sin\alpha\cos\alpha$, $\Delta(\alpha) = \sin\alpha$, then the system is equivalent to a generalized spherical pendulum in a nonconservative force field ("placed in an incoming material flow") and has a complete set of first integrals with essential singularities expressed in terms of a finite combination of elementary functions.

Case (41) forms a class of systems (18), (19) corresponding to the motion of a point over a four-dimensional sphere with a metric induced by the Euclidean metric of the ambient five-dimensional space.

Case (42) forms a class of systems (18), (19) corresponding to the motion of a point over the four-dimensional Lobachevsky space in the Klein model.

In the last two cases, the function $\Delta(\alpha)$ runs over some functional set.

To conclude, we make a remark on integrability. It is well known that the concept of integrability is rather diverse. In this paper, we have presented complete sets of not only first integrals, but also invariant differential forms for homogeneous systems of the ninth order. These sets contain almost everywhere smooth functions with essential singularities. In the case of conservative systems, if invariants are determined by smooth functions of their phase variables, then a sufficiently general dissipative force field added to the system leads to invariants whose smoothness is destroyed be the existence of essential singularities in the system. Such points characterize energy dissipation near themselves if they are attracting and energy pumping if they are repelling. The result is of additional interest, because all this happens in different parts of the phase space, but for the same dynamical system.

The above-given examples from applications are also new nontrivial cases of closed-form integrability of systems of geodesics and systems with dissipation (see also [18–20]).

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The author of this work declares that he has no conflicts of interest.

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