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MATHEMATICS

On Topological Classification of Regular Denjoy Type Homeomorphisms

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Abstract—We consider regular Denjoy type homeomorphisms of the two-dimensional torus which are the most natural generalization of Denjoy homeomorphisms of the circle. In particular, they arise as Poincaré maps induced on global cross sections by leaves of one-dimensional orientable unstable foliations of some partially hyperbolic diffeomorphisms of closed three-dimensional manifolds. The nonwandering set of each regular Denjoy type homeomorphism is a Sierpiński set, and each such homeomorphism is, by definition, semiconjugate to the minimal translation on the two-dimensional torus. For regular Denjoy type homeomorphisms, we introduce a complete invariant of topological conjugacy characterized by the minimal translation, which is semiconjugate to the given regular Denjoy type homeomorphism, with a distinguished at most countable set of orbits.

Keywords: topological classification, Denjoy type homeomorphism, Sierpiński set
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Let X be a topological space and $f : X \to X$ be a homeomorphism. A nonempty subset M of X is called a minimal set of f if M is closed and invariant under f (i.e., $f(M) = M$) and does not have nonempty closed invariant subsets other than M. If the entire space X is a minimal set of f, then f is called minimal. Well-known examples of minimal homeomorphisms are minimal rotations of the circle and minimal translations on the two-dimensional torus.[1]

Homeomorphisms of the circle with no periodic points that are topologically nonconjugate to a minimal rotation were first considered in [13]. Later, such diffeomorphisms (homeomorphisms) of the circle were called Denjoy diffeomorphisms (homeomorphisms). The nonwandering set of a Denjoy homeomorphism is minimal and homeomorphic to a Cantor set. Moreover, each such homeomorphism is semiconjugate to a minimal rotation and the complete preimage of each point of the circle under a semiconjugate map is either a point or homeomorphic to a closed interval. A topological classification of Denjoy homeomorphisms was obtained in [9].

Maps of the 2-torus that have properties characteristic of Denjoy homeomorphisms of the circle were considered in [10–12]. According to [11], we introduce following definition.

Definition 1. A homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is called a Denjoy type homeomorphism if it satisfies the following conditions:

1. f is semiconjugate to a minimal translation $g : \mathbb{T}^2 \to \mathbb{T}^2$ under a continuous map $p : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to identity (i.e., $p \circ f = g \circ p$).

2. The set $B = \{x \in \mathbb{T}^2 : p^{-1}(x)$ contains more than one point$\}$[2] is nonempty and countable.

The set B will be called the characteristic set of f. Note that, if a point x belongs to B, then all points of its orbit under the map g also belong to B.

It should be noted that the direct product of two Denjoy homeomorphisms of the circle is not a Denjoy type homeomorphism of the 2-torus.

[1] The rotation of the circle is defined as a map $R(x) = x + \alpha \pmod 1$. A rotation R is minimal if and only if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. The translation on the 2-torus is defined as a map $g(x, y) = (x + \alpha, y + \beta) \pmod 1$. A translation g is minimal if and only if the numbers α, β, and 1 are independent over the rational number field, i.e., if and only if $k_1\alpha + k_2\beta$ is not an integer for any pair of integers k_1, k_2, except for $k_1 = k_2 = 0$.

[2] By $p^{-1}(x)$ we mean the complete preimage of the point x.

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Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a Denjoy type homeomorphism. Then, by virtue of [11], the complete preimage of each point $x \in \mathbb{T}^2$ under a semiconjugate map p is connected and the nonwandering set of f is minimal. However, in contrast to Denjoy homeomorphisms of the circle, nonwandering sets of Denjoy type homeomorphisms may not be homeomorphic (in induced topologies). In this paper, we consider the subclass of Denjoy type homeomorphisms of the 2-torus (see Definition 2 below) with homeomorphic nonwandering sets. Such homeomorphisms are the most natural generalization of Denjoy homeomorphisms of the circle and admit a complete topological classification, which is obtained in Theorem 1 below.

Definition 2. A Denjoy type homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is called regular if the complete preimage of each point of its characteristic set under a semiconjugate map p is a closed embedded disk[3] and the diameters of these disks form a sequence converging to zero.

A partially hyperbolic diffeomorphism h of the 3-torus $\mathbb{T}^3$ that has a two-dimensional attractor was constructed in [10]. Specifically, it was obtained from an algebraic Anosov automorphism via a bifurcation of birth of an invariant curve (similar constructions were considered in [3, 5, 7]). According to [10], a one-dimensional orientable unstable foliation of h has a global cross section (2-torus) and its leaves induce on it a Poincaré map that is a regular Denjoy type homeomorphism.

Below, we describe the topological properties of nonwandering sets of regular Denjoy type homeomorphisms that underlie the proof of the present results.

Definition 3. A Sierpiński set on the 2-torus $\mathbb{T}^2$ is a set $S = \mathbb{T}^2 \setminus \bigcup_{k\in\mathbb{Z}} \operatorname{int} D_k$, where $\{D_k\}_{k\in\mathbb{Z}}$ is the family of sets with the following properties:

(1) for each $k \in \mathbb{Z}$, the set D_k is a closed embedded disk;

(2) $\bigcup_{k\in\mathbb{Z}} \operatorname{int} D_k$ is dense in $\mathbb{T}^2$;

(3) $D_k \cap D_{k'} = \varnothing$ for $k \neq k'$;

(4) $\operatorname{diam}(D_k)$, $k \in \mathbb{Z}$, forms a sequence converging to zero.

Let $S = \mathbb{T}^2 \setminus \bigcup_{k\in\mathbb{Z}} \operatorname{int} D_k$ be a Sierpiński set on the 2-torus $\mathbb{T}^2$. For every $k \in \mathbb{Z}$, let L_k denote the boundary of the set D_k. Since D_k is a closed embedded disk, L_k is a simple closed curve. Let $I = S \setminus \bigcup_{k\in\mathbb{Z}} L_k$. Each point $x \in I$ is called an interior point of S, and I is called the set of interior points of S.

Let Q be a standard Sierpiński carpet on the square $V = [0;1]\times[0;1]$ (for the construction, see, e.g., [2, pp. 275–276]). A set C on $\mathbb{T}^2$ is defined as $\pi|_V(Q)$, where $\pi|_V$ is the restriction of the natural projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ to the square V. Then C is a Sierpiński set. The set of interior points of C is denoted by I_C. By virtue of [4, 14], for any Sierpiński set S, there exists a homeomorphism $\theta : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\theta(S) = C$ and $\theta(I) = I_C$.

The following result is given without proof.

Lemma 1. *The nonwandering set of a regular Denjoy type homeomorphism of the* 2*-torus is a Sierpiński set.*

According to [1], a map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is called linear if it can be represented as the composition of an algebraic automorphism and a translation on the 2-torus.

Let f_1 and f_2 be regular Denjoy type homeomorphisms of the 2-torus such that f_j ($j \in \{1,2\}$) is semiconjugate to a minimal translation $g_j : \mathbb{T}^2 \to \mathbb{T}^2$ under the map $p_j : \mathbb{T}^2 \to \mathbb{T}^2$. Let B_j be the characteristic set of the homeomorphism f_j.

Theorem 1. *Let f_1 and f_2 be regular Denjoy type homeomorphisms of the* 2*-torus. Then f_1 and f_2 are topologically conjugate if and only if there exists a linear map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\varphi \circ g_1 = g_2 \circ \varphi$ and $\varphi(B_1) = B_2$.*

The key element in the proof of Theorem 1 is the proof of sufficiency, i.e., the construction of a homeomorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ that is a modification of φ under which f_1 and f_2 are conjugate. Schematically, this construction consists of the following five steps.

1. Let S_1 and S_2 denote the nonwandering sets of f_1 and f_2, respectively, and I_1 and I_2 denote the sets of interior points of S_1 and S_2, respectively. It can be directly verified that the restriction of the map p_j, $j \in \{1,2\}$, to I_j is a homeomorphism of I_j to the image $p_j(I_j)$.

2. According to [4, 14], there exist homeomorphisms $\theta_1 : \mathbb{T}^2 \to \mathbb{T}^2$ and $\theta_2 : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\theta_1(S_1) = C$, $\theta_1(I_1) = I_C$ and $\theta_2(S_2) = C$, $\theta_2(I_2) = I_C$. Then the map $h : I_C \to I_C$ defined as $h(x) = \theta_2(p_2^{-1}(\varphi(p_1(\theta_1^{-1}(x)))))$, where $x \in I_C$, is a homeomorphism.

3. There exists a homeomorphism $\zeta : C \to C$ such that $\zeta(x) = h(x)$ for all $x \in I_C$. This follows from the

[3] By the closed embedded disk, we mean the image of the closed disk $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ under the embedding $\tau : D \to \mathbb{T}^2$.

uniform continuity of the map h, which is proved below.

The map $\kappa : \mathbb{T}^2 \to \mathbb{T}^2$ is defined as $\kappa(x) = \theta_1(p_1^{-1}(\varphi^{-1}(p_2(\theta_2^{-1}(x)))))$, where $x \in \mathbb{T}^2$ and $p_1^{-1}(\varphi^{-1}(p_2(\theta_2^{-1}(x))))$ is the complete preimage of the set $\varphi^{-1}(p_2(\theta_2^{-1}(x)))$. Note that $\kappa(x) = h^{-1}(x)$ for all $x \in I_C$.

For each positive integer n, let $Q_1^n, \ldots, Q_{8^n}^n$ denote equal squares obtained at the nth step of constructing the Sierpiński carpet Q (see [2, pp. 275–276]), and let $K_1^n, \ldots, K_{8^n}^n$ denote the images of $Q_1^n, \ldots, Q_{8^n}^n$ under the map $\pi|_V$, respectively. Since $Q \subset \bigcup_{i=1}^{8^n} Q_i^n$ (for each $n \in \mathbb{N}$), it is true that $C \subset \bigcup_{i=1}^{8^n} K_i^n$ (for each $n \in \mathbb{N}$).

For a fixed $\varepsilon > 0$, we choose $m \in \mathbb{N}$ $(m \geq 2)$ such that $\mathrm{diam}(K_i^m) < \frac{\varepsilon}{2}$. For each i, the set $\tilde{K}_i^m$ is defined as $\tilde{K}_i^m = \kappa(K_i^m)$.

Let A and B be sets on the 2-torus $\mathbb{T}^2$. The distance between A and B is defined as $\mathrm{dist}(A, B) = \inf\{\rho(x, y) \mid x \in A, y \in B\}$, where ρ denotes the distance between points on $\mathbb{T}^2$ induced by the Riemannian metric.

For each set $\tilde{K}_i^m$, the quantity d_i is defined as $d_i = \min_j(\mathrm{dist}(\tilde{K}_i^m, \tilde{K}_j^m))$, where $\tilde{K}_j^m$ is a set that does not share points with $\tilde{K}_i^m$. Choose $\delta > 0$ such that $\delta < \min_{i \in \{1, \ldots, 8^m\}} d_i$. Any two points $x_1, x_2 \in I_C$ such that $\rho(x_1, x_2) < \delta$ are contained in either a single set $\tilde{K}_i^m$ or two different sets $\tilde{K}_i^m$ and $\tilde{K}_j^m$ sharing at least one point. Then $h(x_1)$ and $h(x_2)$ lie either in a single set K_i^m or two different sets K_i^m and K_j^m sharing at least one point. From this result and the relation $\mathrm{diam}(K_i^m) < \frac{\varepsilon}{2}$, it follows that $\rho(h(x_1), h(x_2)) < \varepsilon$. Thus, the map h is uniformly continuous on the set I_C.

4. The map $\chi : S_1 \to S_2$ is defined as $\chi(x) = \theta_2^{-1}(\zeta(\theta_1(x)))$, where $x \in S_1$. Then $p_2 \circ f_2|_{I_2} = g_2 \circ p_2|_{I_2} = \varphi \circ g_1 \circ \varphi^{-1} \circ p_2|_{I_2} = \varphi \circ g_1 \circ p_1 \circ \chi^{-1}|_{I_2} = \varphi \circ p_1 \circ f_1 \circ \chi^{-1}|_{I_2} = p_2 \circ \chi \circ f_1 \circ \chi^{-1}|_{I_2}$. Thus, $\chi \circ f_1|_{I_1} = f_2 \circ \chi|_{I_1}$. Since the maps χ, f_1, f_2 are continuous and I_1 is dense in S_1, it follows that $\chi \circ f_1|_{S_1} = f_2 \circ \chi|_{S_1}$.

5. Lemma 1 and the equality $\varphi(B_1) = B_2$ imply that the homeomorphism χ extends to a homeomorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\psi \circ f_1 = f_2 \circ \psi$.

Thus, the proof of the sufficiency of the conditions in Theorem 1 is complete.

Theorem 1 implies the following result.

Corollary 1. *Let f_1 and f_2 be regular Denjoy type homeomorphisms of the 2-torus such that the characteristic set of each of them consists of a single orbit. Then f_1 and f_2 are topologically conjugate if and only if there exists an algebraic automorphism $\eta : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\eta \circ g_1 = g_2 \circ \eta$.*

The necessity follows immediately from Theorem 1. Let us prove the sufficiency.

Let O_1 and O_2 be orbits that are the characteristic sets of f_1 and f_2, respectively. Since $\eta \circ g_1 = g_2 \circ \eta$, the automorphism η maps the orbits of g_1 to the orbits of g_2. Then we can choose a translation $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ such that the map $\varphi = \gamma \circ \eta$ takes the orbit O_1 to O_2. By construction, φ is a linear map. Moreover, the translations g_1 and g_2 are conjugate under φ. Then the sufficiency follows from Theorem 1.

Following [8], for any minimal translation g and any set B consisting of n orbits g $(n \geq 1)$, there exists a regular Denjoy type homeomorphism that is semiconjugate to g and its characteristic set coincides with B. From Theorem 1 and the results of [8], it follows that there exists a standard representative in each class of topological conjugacy of regular Denjoy type homeomorphisms with characteristic sets consisting of finitely many orbits. We do not know whether an example of a regular Denjoy type homeomorphism with a characteristic set consisting of a countable number of orbits has been constructed.[4]

Theorem 2. *For any minimal translation $g : \mathbb{T}^2 \to \mathbb{T}^2$ and any positive integer $n \geq 2$, there exists a continual set of pairwise topologically nonconjugate regular Denjoy type homeomorphisms of the 2-torus, each being semiconjugate to g and having a characteristic set consisting of n orbits.*

The idea of the proof of Theorem 2 can be described as follows.

Let $g : \mathbb{T}^2 \to \mathbb{T}^2$ be a minimal translation and $n \geq 2$ be a positive integer. By Theorem 1 and the results of [8], it suffices to show that there exists a continuum of sets B_ν having the following properties:

(1) Each set B_ν is the union of n orbits of g.

[4] In the one-dimensional case, according to [6], it is possible to construct a Denjoy homeomorphism of the circle with a characteristic set consisting of a countable number of orbits.

(2) For any two sets B_{ν_1} and B_{ν_2}, there does not exist a linear map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\varphi(B_{\nu_1}) = B_{\nu_2}$.

Let $O_1, \ldots, O_{n-1}$ be $n-1$ arbitrary different orbits of g. Define the set $B_\nu = \bigcup_{i=1}^{n-1} O_i \cup O_\nu$, where O_ν is an orbit of g other than $O_1, \ldots, O_{n-1}$. By direct verification, we can show that it is possible to choose a continuum of different orbits O_ν and, hence, a continuum of different sets B_ν such that the sets B_ν satisfy the required conditions.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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