ISSN 1064-5624, Doklady Mathematics, 2022, Vol. 105, No. 2, pp. 109–111. © Pleiades Publishing, Ltd., 2022.
Russian Text © The Author(s), 2022, published in Doklady Rossiiskoi Akademii Nauk. Matematika, Informatika, Protsessy Upravleniya, 2022, Vol. 503, pp. 67–69.

MATHEMATICS

Classical Solutions of the First Boundary Value Problem for Parabolic Systems on the Plane

A. N. Konenkov[a,*]

Presented by Academician of the RAS E.I. Moiseev January 11, 2022

Received January 19, 2022; revised January 19, 2022; accepted January 22, 2022

Abstract—The first boundary value problem for a second-order parabolic system with one spatial variable in a domain with nonsmooth lateral boundaries is considered. The domain can be bounded or semi-bounded. The coefficients of the system depend only on the spatial variable and satisfy the Hölder condition. The initial and boundary functions are assumed to be continuous and bounded. The existence and uniqueness of a classical solution of this problem is established.

Keywords: parabolic system, first boundary value problem, nonsmooth lateral boundary, classical solution

DOI: 10.1134/S1064562422020132

For second-order parabolic equations, the uniqueness of a classical solution to the first boundary value problem follows from the maximum principle. Here and below, by a classical solution in a domain Ω, we mean a bounded function from the class $C^{2,1}_{x,t}(\Omega)\cap C(\bar{\Omega})$ that satisfies the equation in Ω and initial and boundary conditions on the parabolic boundary of Ω. In the case when the coefficients of the equation satisfy only the Hölder condition, the existence of a classical solution to the first boundary value problem with a continuous boundary function is established in [1, Chap. 3, Sect. 4]. The proof is based on the barrier method, which also uses the maximum principle. However, for parabolic systems, the maximum principle does not hold in general [2]. Note that, if the leading coefficients of the equation have spatial derivatives satisfying the Hölder condition, then the existence of a solution can be established with the help of a double-layer potential.

For a wide class of parabolic systems, the unique solvability of boundary value problems in anisotropic Hölder spaces was established in [3]. The solutions were assumed to be sufficiently smooth, namely, all the derivatives of the solution involved in the system were assumed to be continuous in the closure of the domain.

The existence and uniqueness of solutions to the first and second boundary value problems for parabolic systems in a bounded plane domain in the class $C^{1,0}_{x,t}(\bar{\Omega})$ were proved in [4–6]. The boundary function was assumed to have a continuous derivative of order 1/2 vanishing at $t = 0$. For systems with differentiable coefficients, the uniqueness of the first boundary value problem in a semi-bounded domain in $C^{1,0}_{x,t}(\bar{\Omega})$ was proved in [7]. The existence of a solution to this problem follows from [8]. Domains with curved and nonsmooth lateral boundaries were considered in all these works.

In this paper, for a parabolic system with one space variable, we study the existence and uniqueness of a classical solution to the first boundary value problem with continuous functions in the initial and boundary conditions. The domain can be bounded or semi-bounded, and its lateral boundary can be nonsmooth.

In a strip $D = \mathbb{R}\times(0,T)$, $0 < T < \infty$, we consider the parabolic operator

$$Lu = \partial_t u - A(x)\partial_x^2 u - B(x)\partial_x u - C(x)u, \qquad (1)$$

where $u = (u_1, \dots, u_m)^{\mathrm{T}}$ and $A(x)$, $B(x)$, and $C(x)$ are $m\times m$ matrices with elements $a_{ij}(x)$, $b_{ij}(x)$, and $c_{ij}(x)$, respectively. The operator L is assumed to satisfy the uniform parabolicity condition, i.e., the eigenvalues $\lambda_k(x)$ of the matrix $A(x)$ satisfy the inequality

$$\mathrm{Re}\lambda_k(x) \geq \mu > 0, \quad \forall x\in\mathbb{R}, \quad k = 1, \dots, m; \qquad (2)$$

and the coefficients are real, bounded, and satisfy the Hölder condition:

$$a_{ij}, b_{ij}, c_{ij} \in C^{\alpha}(\mathbb{R}), \quad 0 < \alpha < 1. \qquad (3)$$

[a] Esenin Ryazan State University, Ryazan, Russia
*e-mail: a.konenkov@365.rsu.edu.ru

In the strip D, we consider a semi-bounded domain

$$\Omega = \{(x,t) \in D \mid x > g(t),\ 0 < t < T\}$$

with the base $\Omega_0 = \bar{\Omega} \cap \{t = 0\}$ and the lateral boundary

$$\Sigma = \{(x,t) \in D \mid x = g(t),\ 0 < t < T\},$$

where the function g satisfies the condition

$$g \in C^{(1+\alpha)/2}([0,T]), \quad 0 < \alpha < 1. \tag{4}$$

In Ω we consider the first boundary value problem

$$Lu = 0 \ \text{ in } \ \Omega, \quad u|_\Sigma = \psi, \quad u|_{t=0} = h. \tag{5}$$

For a set $E \subset \bar{D}$, let $C(E)$ denote the space of continuous and bounded vector functions $f : E \to \mathbb{R}^m$ with the norm

$$|f|_{0,E} = \sup_{(x,t)\in E} |f(x,t)|,$$

and let $\underset{\circ}{C}(E) = \{f \in C(E) \mid f|_{t=0} = 0\}$.

Theorem 1. *Suppose that the operator L satisfies conditions* (2), (3) *and the lateral boundary of the domain* Ω *satisfies condition* (4). *If* $\psi \in C(\Sigma)$, $h \in C(\Omega_0)$, *and the compatibility condition* $\psi(g(0)) = h(g(0))$ *holds, then problem* (5) *has a unique classical solution* $u \in C^{2,1}_{x,t}(\Omega) \cap C(\bar{\Omega})$. *It satisfies the estimate*

$$|u|_{0,\Omega} \le C(|\psi|_{0,\Sigma} + |h|_{0,\Omega_0}).$$

Under conditions (2) and (3) imposed on the coefficients of L, there exists a fundamental solution matrix $\Gamma(x,\xi,t-\tau)$ [9, Chap. 1, Sect. 3].

To establish the existence of a solution, we introduce the potential

$$W\varphi(x,t) = \int_0^t K(x, g(\tau), t-\tau)\varphi(\tau)d\tau \tag{6}$$

with the kernel

$$K(x,\xi,t) = \partial_\xi(\Gamma(x,\xi,t)A(\xi)).$$

Theorem 2. *The function* $\Gamma(x,\xi,t)A(\xi)$ *is differentiable with respect to* ξ *in the half-space* $t > 0$. *The function K satisfies the equation* $L_{x,t}K(x,\xi,t) = 0$ *and the estimates*

$$|\partial_t^l \partial_x^m K(x,\xi,t)| < C_{l,m} t^{-(m+2l+2)/2} e^{-c_{l,m}(x-\xi)^2/t}, \quad m \le 2, \quad l \ge 0, \tag{7}$$

for $t > 0$ *and* $x, \xi \in \mathbb{R}$.

Theorem 3. *The function* $W : \varphi \to W\varphi$ *maps the space* $\underset{\circ}{C}(\Sigma)$ *to* $\underset{\circ}{C}(\bar{\Omega})$; *moreover*,

$$|W\varphi|_{0,\bar{\Omega}} \le C|\varphi|_{0,\Sigma}.$$

For density $\varphi \in C(\Sigma)$ *there holds the jump relation*

$$W^{\pm}\varphi(g(t),t) = \pm\frac{\varphi(t)}{2} + W^0\varphi(g(t),t), \quad 0 < t \le T, \tag{8}$$

where $W^{\pm}\varphi$ *are the limiting values as the point* $(g(t),t)$ *is approached on the right and left, respectively, and* $W^0\varphi$ *is the direct value of the potential* $W\varphi$ *on the curve* $x = g(t)$.

Thus, $W\varphi$ has many properties of a double-layer potential and can be used, as the latter, for solving the first boundary value problem.

With the help of a Poisson-type potential [9, Chap. 1, Sect. 4], problem (5) can be reduced to a problem with a zero initial function:

$$Lu = 0 \quad \text{in} \quad \Omega, \quad u|_\Sigma = \psi, \quad u|_{t=0} = 0. \tag{9}$$

A solution u of this problem is sought in the form of potential (6) with a continuous density φ. By virtue of the jump formula (8), the problem of finding u is reduced to solving a Volterra integral equation of the second kind with a kernel having a weak singularity.

Theorem 4. *Suppose that the operator L satisfies conditions* (2), (3) *and the lateral boundary of the domain* Ω *satisfies condition* (4). *If* $\psi \in \underset{\circ}{C}(\Sigma)$, *then problem* (9) *has a unique classical solution* $u \in C^{2,1}_{x,t}(\Omega) \cap \underset{\circ}{C}(\bar{\Omega})$. *It satisfies the estimate*

$$|u|_{0,\Omega} \le C|\psi|_{0,\Sigma}.$$

There exists a function $\varphi \in \underset{\circ}{C}(\Sigma)$ *such that* $u = W\varphi$ *in* Ω.

Also we consider the first boundary value problem in a bounded domain

$$\Omega = \{(x,t) \in D \mid g_1(t) < x < g_2(t), 0 < t < T\}$$

with lateral boundaries

$$\Sigma_i = \{(x,t) \in D \mid x = g_i(t), 0 < t < T\}, \quad i = 1,2,$$

where the functions g_i satisfy the conditions

$$g_i \in C^{(1+\alpha)/2}([0,T]), \quad g_1(t) < g_2(t), \quad t \in [0,T], \quad 0 < \alpha < 1. \tag{10}$$

Theorem 5. *Suppose that the operator L satisfies conditions* (2), (3) *and the lateral boundaries of the domain* Ω *satisfy condition* (10). *If* $\psi_i \in C(\Sigma_i)$ *for* $i = 1, 2$, $h \in C(\Omega_0)$, *and the compatibility conditions* $\psi_1(g_1(0)) = h(g_1(0))$ *and* $\psi_2(g_2(0)) = h(g_2(0))$ *hold, then there exists a classical solution of the first boundary value problem*

$$Lu = 0 \quad in \quad \Omega, \quad u|_{\Sigma_1} = \psi_1, \quad u|_{\Sigma_2} = \psi_2, \quad u|_{t=0} = h.$$

This solution satisfies the estimate

$$|u|_{0,\Omega} \le C(|\psi_1|_{0,\Sigma_1} + |\psi_2|_{0,\Sigma_2} + |h|_{0,\Omega_0}).$$

For $h \equiv 0$, there exist functions $\varphi_i \in \overset{\circ}{C}(\Sigma_i)$, $i = 1, 2$, such that the solution can be represented as a sum of potentials: $u = W_1\varphi_1 + W_2\varphi_2$ in Ω, where

$$W_i\varphi_i(x,t) = \int_0^t K(x, g_i(\tau), t-\tau)\varphi_i(\tau)d\tau, \quad i = 1, 2.$$

ACKNOWLEDGMENTS

The author is grateful to Academician of the RAS E.I. Moiseev and Professor I.S. Lomov for helpful discussions.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

REFERENCES

1. A. Friedman, *Partial Differential Equations of Parabolic Type* (Prentice Hall, Englewood Cliffs, N.J., 1964).
2. V. G. Maz'ya and G. I. Kresin, "On the maximum principle for strongly elliptic and parabolic second order systems with constant coefficients," Math. USSR Sb. **53** (2), 457–479 (1986).
3. V. A. Solonnikov, "On boundary value problems for linear parabolic systems of differential equations of general form," Proc. Steklov Inst. Math. **83**, 1–184 (1965).
4. E. A. Baderko and M. F. Cherepova, "Simple layer potential and the first boundary value problem for a parabolic system on the plane," Differ. Equations **52** (2), 197–209 (2016).
5. E. A. Baderko and M. F. Cherepova, "Uniqueness of solutions to initial boundary value problems for parabolic systems in plane bounded domains with nonsmooth lateral boundaries," Dokl. Math. **102** (2), 357–359 (2020).
6. E. A. Baderko and M. F. Cherepova, "Uniqueness of solutions of the first and second initial–boundary value problems for parabolic systems in bounded domains on the plane," Differ. Equations **57** (8), 1010–1019 (2021).
7. E. A. Baderko and S. I. Sakharov, "Uniqueness of the solution of the first initial–boundary value problem for a parabolic system with differentiable coefficients in a half-strip with nonsmooth lateral boundary," Differ. Equations **57** (5), 605–614 (2021).
8. E. A. Baderko and M. F. Cherepova, "Dirichlet problem for parabolic systems with Dini continuous coefficients," Appl. Anal. **100** (13), 2900–2910 (2021).
9. S. D. Eidel'man, *Parabolic Systems* (Nauka, Moscow, 1964) [in Russian].

Translated by I. Ruzanova