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MATHEMATICS

On the Bellman Function Method for Operators on Martingales

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Abstract—It is shown how to apply the Bellman function method to general operators on martingales, i.e., to operators that are not necessarily martingale transforms. As examples of such operators, we consider the Haar transforms and an operator whose L^p-boundedness implies the Rubio de Francia inequality for the Walsh system. For the corresponding Bellman function, the Bellman induction is carried out and a Bellman candidate is constructed.

Keywords: Burkholder method, Gundy theorem, Walsh system, Rubio de Francia inequality, Haar transform

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We consider functions acting on the unit interval and, for brevity, write L^p instead of $L^p([0,1])$ and $L^p(l^2)$ instead of $L^p([0,1],l^2)$ (in the latter case, we mean-valued functions defined on the unit interval).

1. MOTIVATING EXAMPLES

In [1] Burkholder used the Bellman function method (from optimal control theory) to obtain sharp L^p-estimates of martingale transforms $(1 < p \le 2)$. First, we give two examples of operators on martingales that are not martingale transforms.

The symbol "$\sqsubseteq$" is used to denote the relation of "being a dyadic subinterval," and $J^\pm$ denotes the left and right halves of an interval J. Consider the Haar system

$$h_0 \stackrel{\text{def}}{=} 1_{[0,1]} \quad \text{and} \quad h_J \stackrel{\text{def}}{=} |J|^{-1/2}(1_{J^+} - 1_{J^-}), \quad J \sqsubseteq [0,1].$$

It is easy to see that, for any $m \in \mathbb{Z}_+$ we can define a unitary operator

$$H_m : L^2 \to L^2,$$

that establishes a one-to-one correspondence between

$$\{h_0, h_J\}_{\substack{J \sqsubseteq [0,1] \\ |J| \ge 2^{-m}}} \quad \text{and} \quad \{1_e\}_{\substack{e \sqsubseteq [0,1] \\ |e| = 2^{-m-1}}}$$

maps the other Haar basis functions into themselves, and has the property

$$\operatorname{supp} H_m h_J \subseteq J \quad \text{for all} \quad J \sqsubseteq [0,1].$$

Note that the matrix in the Haar basis that defines the action of the operator H_m on the first 2^{m+1} basis vectors coincides with the matrix of the Haar transform of order 2^{m+1} with suitably rearranged columns. It will be shown below that, for $1 < p \le 2$, the L^p-boundedness (uniform in m) of the operators H_m can be established within the classical operator theory on martingales ("discrete" version of the theory of Calderon–Zygmund operators). However, the Bellman function constructed by Burkholder in [1] does not suffice to obtain such boundedness.

Another example is as follows. Let $\mathcal{W} = \{w_n\}_{n \in \mathbb{Z}_+}$ be a standardly ordered Walsh system. Consisting of all possible products of Rademacher functions, this system resembles in properties the Fourier basis of exponentials and, in some sense, can be viewed as its discrete analogue (for more details, see, e.g., [2, Sect. 4.5]). An example supporting this analogy is the following result of [3], which states that the Rubio de Francia inequality [4] can be extended from the Fourier basis to the Walsh system.

Theorem. *Let $\{f_m\}$ be at most a countable set of functions whose Walsh spectra lie in pairwise disjoint intervals $I_m \subseteq \mathbb{Z}_+$:*

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$$f_m = \sum_{n \in I_m} (f_m, w_n) w_n.$$

If $1 < p \le 2$, *then*

$$\left\| \sum_m f_m \right\|_{L^p} \le C_p \|\{f_m\}\|_{L^p(l^2)},$$

where C_p *is a constant independent of the collections* $\{f_m\}$ *and* $\{I_m\}$.

With the help of combinatorial arguments, the proof of this theorem is reduced in [3] to checking the L^p-boundedness of an operator that is our second example. Before describing this operator, we present two well-known simple properties of Walsh functions.

1. For a function $g \in L^1$, its martingale differences $\Delta_k g$ in the standard dyadic filtration coincide with the Walsh multipliers for the intervals $\delta_0 \overset{\text{def}}{=} \{0\}$ and $\delta_k \overset{\text{def}}{=} \{2^{k-1}, \ldots, 2^k - 1\}$, $k > 0$:

$$\Delta_0 g = (g, h_0) h_0 = (g, w_0) w_0;$$

$$\Delta_k g = \sum_{\substack{J \sqsubseteq [0,1] \\ |J| = 2^{-k+1}}} (g, h_J) h_J = \sum_{n \in \delta_k} (g, w_n) w_n.$$

2. For $a, b \in \mathbb{Z}_+$ the "exponential" property $w_a(x) w_b(x) = w_{a \dotplus b}(x)$ holds, where $a \dotplus b$ is bitwise XOR (the corresponding bits in the binary decompositions of a and b are summed modulo 2). In other words, there is an isomorphism between two groups: $(\mathbb{Z}_+, \dotplus) \cong (\mathcal{W}, \times)$.

Let (j, k) be multi-indices running over a subset $\mathcal{A} \subseteq \mathbb{Z}_+^2$ and $a_{j,k} \in \mathbb{Z}_+$ be numbers such that the sets $a_{j,k} \dotplus \delta_k$ are pairwise disjoint and completely cover $\mathbb{Z}_+$. Consider an operator G that puts parts of the Walsh spectra of functions from the sequence $f = \{f_{j,k}\}_{(j,k) \in \mathcal{A}} \in L^2(l^2)$ in these sets and then combines the results into a single function:

$$Gf \overset{\text{def}}{=} \sum_{(j,k) \in \mathcal{A}} w_{a_{j,k}} \Delta_k f_{j,k}.$$

The above-presented theorem from [3] is reduced to the estimate

$$\|Gf\|_{L^p} \le C_p \|f\|_{L^p(l^2)}, \quad 1 < p \le 2,$$

with a constant C_p depending only on p. As in the case of the operators H_m, this estimate follows directly from classical operator theory on martingales, but does not follow from Burkholder's results [1].

2. FORMULATION OF THE PROBLEM

Let $h_J^i \overset{\text{def}}{=} (0, \ldots, 0, 1, 0, \ldots) h_J$, where 1 is in the ith position (the functions h_0^i are defined in a similar manner). Then the system $\{h_0^i, h_J^i\}_{\substack{i \in N \\ J \sqsubseteq [0,1]}}$ is an orthonormal basis for the functions $f = \{f_i\}_{i \in \mathbb{N}} \in L^2(l^2)$.

Definition 1. A linear bounded operator $T : L^2(l^2) \to L^2$ is said to *belong to the class* $\mathcal{G}(l^2)$ if the following conditions are satisfied for T:

1. Parseval's identity holds for the system $\{Th_0^i, Th_J^i\}_{\substack{i \in N \\ J \sqsubseteq [0,1]}}$ for any function $g \in L^2$,

$$\|g\|_{L^2}^2 = \sum_{i \in \mathbb{N}} (g, Th_0^i)^2 + \sum_{\substack{i \in \mathbb{N} \\ J \sqsubseteq [0,1]}} (g, Th_J^i)^2.$$

2. The operator T does not increase the supports of the basis functions: $\operatorname{supp} Th_J^i \subseteq J$ for $J \sqsubseteq [0,1]$.

The class $\mathcal{G}$ of linear operators $T : L^2 \to L^2$ is defined in a similar (and simpler) manner; namely, the Haar basis is considered without participating the index i and summation is taken only over J.

Condition 1 is stronger than the L^2-boundedness of the operator, but weaker than its unitarity in L^2. In turn, condition 2 coincides with the main condition of the Gundy theorem for the Haar filtration (in which the intervals are bisected sequentially from left to right). Here, we refer to the version of the Gundy theorem for vector-valued martingales stated and proved in [5, Theorem 1] (the original scalar version of the theorem can be found in [6]). If the boundedness of martingale transforms is treated as a discrete analogue of the boundedness of the Hilbert transform, then the Gundy theorem can be regarded as an analogue of the boundedness of general Calderón–Zygmund operators, and the above-mentioned main condition from it, as an analogue of the smoothness condition on the kernel of an operator. The Gundy theorem implies that operators from the classes $\mathcal{G}$ and $\mathcal{G}(l^2)$ are uniformly L^p-bounded ($1 < p \le 2$). On the other hand, the generality of conditions 1 and 2 is similar, to a large degree, to the generality of the conditions of the Gundy theorem.

Now we note that $H_m \in \mathcal{G}$ and $G \in \mathcal{G}(l^2)$. Indeed, the operators H_m are unitary and, by construction, satisfy the scalar variant of condition 2. Concerning the operator G, it is easy to see that the system $\{Gh_0^i, Gh_J^i\}_{\substack{i \in N \\ J \sqsubseteq [0,1]}}$ includes an orthonormal basis in L^2, while the other elements are zero. Condition 2 for the operator G is also satisfied.

Our goal is to extend the Burkholder method to the classes $\mathcal{G}$ and $\mathcal{G}(l^2)$.

3. RESULTS

Suppose that $1 < p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. By renormalizing the Haar functions, the classes $\mathcal{G}$ and $\mathcal{G}(l^2)$ can be extended to any interval $I \sqsubset \mathbb{R}$. Let $\langle h_I\rangle \stackrel{\text{def}}{=} \frac{1}{|I|}\int_I h$. The Bellman function, which allows us to obtain an estimate for operators from $\mathcal{G}$, is defined as

$$\mathbf{B}(x) = \mathbf{B}(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} \sup\left\{ \langle gTf\rangle_I - \langle f\rangle_I \langle gT\mathbb{1}_I\rangle_I \,\middle|\, \begin{array}{l} f, g \in L^2(I), \quad T \in \mathcal{G}, \\ \langle f\rangle_I = x_1, \quad |\langle gT\mathbb{1}_I\rangle_I|^2 = x_2, \\ \langle |f|^p\rangle_I = x_3, \quad \langle |g|^q\rangle_I = x_4 \end{array} \right\}.$$

It is easy to see that the function $\mathbf{B}$ does not depend on the choice of the interval I and that the domain $\Omega_{\boldsymbol{B}}$ consisting of points x for which the supremum is taken over a nonempty subset satisfies the inclusion

$$\Omega_{\boldsymbol{B}} \subseteq \Omega \stackrel{\text{def}}{=} \{x \in \mathbb{R}\times\mathbb{R}_+^3 \,|\, |x_1|^p \le x_3, x_2 \le x_4^{2/q}\}.$$

Definition 2. A function $B \in C(\Omega)$ is said *to belong to the class* $\mathcal{K}$ if B satisfies a boundary condition and a concavity-type geometric condition that are as follows:

1. If $|x_1|^p = x_3$, then $B(x) \ge 0$.

2. Given $x, x^{\pm} \in \Omega$ and $\Delta \in \mathbb{R}$, if

$$\frac{x^+ + x^-}{2} - x = (0, \Delta^2, 0, 0),$$

then

$$B(x) \ge \frac{|x_1^+ - x_1^-|}{2}|\Delta| + \frac{B(x^+) + B(x^-)}{2}.$$

By applying the Bellman induction, it can be proved that any such function is a majorant for $\mathbf{B}$.

Theorem 1. *If* $B \in \mathcal{K}$, *then* $\mathbf{B}(x) \le B(x)$ *for all* $x \in \Omega_{\boldsymbol{B}}$.

With the help of the Taylor formula, the concavity-type condition from the definition of the class $\mathcal{K}$ can be rewritten in the differential form

$$d^2B \le \frac{|dx_1|^2}{2B_{x_2}} \le 0.$$

Here, we mean that the Hessian at an arbitrary point from Ω is computed on the left and it acts as a quadratic form on an arbitrary vector (dx_1, dx_2, dx_3, dx_4). Relying on the differential form of the basic condition and using an argument similar to that used in [7], we can find a particular representative of $\mathcal{K}$. Namely, for $y \in \mathbb{R}_+^4$, we set

$$B_0(y) \stackrel{\text{def}}{=} 2(y_3 + y_4) - y_1^p - y_2^{q/2} - \delta \begin{cases} y_1^{2-p} y_2 + y_1^{2-p-2t(p-1)} y_2^{t+1}, & y_1^p \ge y_2^{q/2} \\ \frac{2}{q}(2+t) y_2^{q/2} + \frac{2}{p}(2 - p - t(p-1)) y_1^p, & y_1^p \le y_2^{q/2}. \end{cases}$$

Then, for each $1 < p \le 2$ it is possible to choose nonnegative constants t, δ, and C_p such that the function $B(x) = C_p B_0(|x_1|, x_2, x_3, x_4)$, $x \in \Omega$, is in $\mathcal{K}$. By using this function, Theorem 1, and the homogeneity of the function $\mathbf{B}$, it is easy to obtain the following estimate for operators $T \in \mathcal{G}$:

$$\|Tf\|_{L^p} \le C_p' \|f\|_{L^p},$$

where

$$C_p' = 2p^{1/p} q^{1/q} C_p + 1.$$

This method can potentially be used to compute the exact constants in L^p-estimates. For this purpose, it is necessary to find the function $\mathbf{B}$. We have established that $\Omega_{\boldsymbol{B}} = \Omega$ and that the function $\mathbf{B}$ satisfies properties 1 and 2 from the definition of the class $\mathcal{K}$. Therefore, $\mathbf{B}$ should be sought as the pointwise minimum of all functions from this class.

Now suppose that $f = \{f_i\}_{i\in\mathbb{N}} \in L^2(I, l^2)$ and $T \in \mathcal{G}(l^2)$. Define

$$\mathbb{1}_I^i \stackrel{\text{def}}{=} (0, \ldots, 0, 1, 0, \ldots)\mathbb{1}_I,$$

where 1 is in the ith position. By $\langle f\rangle_I$, we mean the sequence $\{\langle f_i\rangle_I\}_{i\in\mathbb{N}}$, and, by $\langle gT\mathbb{1}_I\rangle_I$, the sequence $\{\langle gT\mathbb{1}_I^i\rangle_I\}_{i\in\mathbb{N}}$. The modulus of a vector from l^2 is understood as its l^2-norm, and the product of such vectors is understood as their scalar product. With these conventions, all what was said above remains true up to an L^p-estimate of an operator $T \in \mathcal{G}(l^2)$. It should be emphasized that x_1 and dx_1 are now vectors from l^2, while the other variables remain scalars (including y_1 in the definition of the function B_0).

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