ISSN 1064-5624, Doklady Mathematics, 2020, Vol. 102, No. 3, pp. 518–523. © Pleiades Publishing, Ltd., 2020.
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MATHEMATICS

New Cases of Homogeneous Integrable Systems with Dissipation on Tangent Bundles of Three-Dimensional Manifolds

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Presented by Academician of the RAS V.V. Kozlov October 20, 2020

Received October 20, 2020; revised October 20, 2020; accepted November 12, 2020

Abstract—The integrability of certain classes of homogeneous dynamical systems on the tangent bundles of three-dimensional manifolds is shown. The force fields involved in the systems lead to dissipation of variable sign and generalize previously considered fields.

Keywords: dynamical system, integrability, dissipation, transcendental first integral

DOI: 10.1134/S1064562420060265

Research on the integrability of autonomous dynamical systems on a three-dimensional configuration manifold M^3 leads to the study of sixth-order systems on the tangent bundle TM^3. A key point, along with the geometry of M^3, is the structure of the force field present in the system. For example, the well-known problem concerning the motion of a four-dimensional pendulum on a generalized spherical hinge in a nonconservative force field leads to a dynamical system on the tangent bundle of the three-dimensional sphere with a special metric on it induced by an additional symmetry group [1, 2]. The dynamical systems describing the motion of such a pendulum have dissipation of variable sign (referred to as alternating dissipation), and the complete list of first integrals consists of transcendental functions that can be expressed in terms of a finite combination of elementary functions [2, 3].

There are also problems concerning the motion of a point on three-dimensional surfaces of revolution, the Lobachevsky plane, etc. Sometimes, in systems with dissipation, it is possible to find a complete list of first integrals consisting of transcendental functions (in the sense of complex analysis), since a complete list of even continuous autonomous first integrals has to be forgotten. The results obtained below are especially important in the context of a nonconservative force field present in the system.

In this work, we show the integrability of certain classes of homogeneous dynamical systems on tangent bundles of smooth three-dimensional manifolds. The force fields involved in the systems lead to dissipation of different sign and generalize previously considered fields [2, 3].

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1. INTEGRATION OF EQUATIONS OF GEODESICS

It is well known that, in the case of a three-dimensional Riemannian manifold M^3 with coordinates (α, β), $\beta = (\beta_1, \beta_2)$, and affine connection $\Gamma^i_{jk}(\alpha, \beta)$, the equations of geodesic lines on the tangent bundle $TM^3\{\alpha^\bullet, \beta_1^\bullet, \beta_2^\bullet; \alpha, \beta_1, \beta_2\}$ $\alpha = x^1$, $\beta_1 = x^2$, $\beta_2 = x^3$, $x = (x^1, x^2, x^3)$, have the following form (the derivatives are taken with respect to the natural parameter):

$$x^{i\bullet\bullet} + \sum_{j,k=1}^{3} \Gamma^i_{jk}(x) x^{j\bullet} x^{k\bullet} = 0, \quad i = 1, 2, 3. \tag{1}$$

Let us study the structure of Eqs. (1) under a change of coordinates on the tangent bundle TM^3. Consider a change of coordinates of the tangent space:

$$x^{i\bullet} = \sum_{j=1}^{3} R^{ij} z_j, \tag{2}$$

which can be inverted as $z_j = \sum_{i=1}^{3} T_{ji} x^{i\bullet}$, here, R^{ij} and T_{ji}, $i, j = 1, 2, 3$, are functions of x, and $RT = E$, where $R = (R^{ij})$ and $T = (T_{ji})$. Equations (2) will be referred to as new kinematic relations, i.e., linear relations on the tangent bundle TM^3. We have

$$z_i^{\bullet} = \sum_{j,k=1}^{3} T_{ij,k} x^{j\bullet} x^{k\bullet} - \sum_{j,p,q=1}^{3} T_{ij} \Gamma_{pq}^{j} x^{p\bullet} x^{q\bullet}, \qquad (3)$$

where $T_{ji,k} = \dfrac{\partial T_{ji}}{\partial x^k}$, j, i, $k = 1, 2, 3$; here, formulas (2) are substituted for $x^{i\bullet}$, $i = 1, 2, 3$, in (3), and the right-hand sides of compound system (2), (3) are homogeneous forms of suitable degrees in the quasi-velocities z_1, z_2, and z_3.

Proposition 1. *System* (1) *is equivalent to compound system* (2), (3) *in the domain where* $\det R(\alpha, \beta) \neq 0$.

Thus, the result of passing from the geodesic equations (1) to equivalent system (2), (3) depends on both substitution (2) (i.e., on the introduced kinematic relations) and on the affine connection $\Gamma^i_{jk}(\alpha, \beta)$.

Consider a rather general case of kinematic relations specified as

$$\alpha^{\bullet} = z_3 f_3(\alpha), \quad \beta_1^{\bullet} = z_2 f_1(\alpha), \quad \beta_2^{\bullet} = z_1 f_2(\alpha) g(\beta_1), \qquad (4)$$

where $f_1(\alpha)$, $f_2(\alpha)$, $f_3(\alpha)$, and $g(\beta_1)$ are smooth functions that do not vanish identically. Such coordinates z_1, z_2, and z_3 in the tangent space are introduced when we consider geodesic equations [4, 5], for example, with seven nonzero connection coefficients (in particular, on three-dimensional surfaces of revolution, in the Lobachevsky space, etc.):

$$\begin{gathered}
\alpha^{\bullet\bullet} + \Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta)\alpha^{\bullet 2} + \Gamma^{\alpha}_{11}(\alpha,\beta)\beta_1^{\bullet 2} + \Gamma^{\alpha}_{22}(\alpha,\beta)\beta_2^{\bullet 2} = 0, \\
\beta_1^{\bullet\bullet} + 2\Gamma^{1}_{\alpha 1}(\alpha,\beta)\alpha^{\bullet}\beta_1^{\bullet} + \Gamma^{1}_{22}(\alpha,\beta)\beta_2^{\bullet 2} = 0, \qquad (5) \\
\beta_2^{\bullet\bullet} + 2\Gamma^{2}_{\alpha 2}(\alpha,\beta)\alpha^{\bullet}\beta_2^{\bullet} + 2\Gamma^{2}_{12}(\alpha,\beta)\beta_1^{\bullet}\beta_2^{\bullet} = 0,
\end{gathered}$$

i.e., the other connection coefficients vanish. In the case of (4), Eqs. (3) become

$$\begin{aligned}
z_1^{\bullet} &= -f_3(\alpha)\left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_1 z_3 \\
&\quad - f_1(\alpha)\left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right] z_1 z_2, \\
z_2^{\bullet} &= -f_3(\alpha)\left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_2 z_3 \\
&\quad - \frac{f_2^2(\alpha)}{f_1(\alpha)} g^2(\beta_1)\Gamma^1_{22}(\alpha,\beta) z_1^2, \qquad (6) \\
z_3^{\bullet} &= -f_3(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_3^2 \\
&\quad - \frac{f_1^2(\alpha)}{f_3(\alpha)}\Gamma^{\alpha}_{11}(\alpha,\beta) z_2^2 - \frac{f_2^2(\alpha)}{f_3(\alpha)} g^2(\beta_1)\Gamma^{\alpha}_{22}(\alpha,\beta) z_1^2,
\end{aligned}$$

and geodesic equations (5) are equivalent to compound system (4), (6) almost everywhere on the manifold $TM^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$.

To integrate system (4), (6) completely, we need to know, generally speaking, five independent first integrals. Moreover, first integrals (specifically, for equations of geodesics) can be sought in a more general form than that considered below.

Proposition 2. *If the system of differential equalities*

$$\begin{gathered}
\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha} \equiv 0, \\
f_3^2(\alpha)\left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] \\
+ f_1^2(\alpha)\Gamma^{\alpha}_{11}(\alpha,\beta) \equiv 0, \\
f_3^2(\alpha)\left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] \qquad (7) \\
+ f_2^2(\alpha) g^2(\beta_1)\Gamma^{\alpha}_{22}(\alpha,\beta) \equiv 0, \\
f_1^2(\alpha)\left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right] \\
+ f_2^2(\alpha) g^2(\beta_1)\Gamma^1_{22}(\alpha,\beta) \equiv 0
\end{gathered}$$

holds everywhere, then system (4), (6) *has an analytic first integral of the form*

$$\Phi_1(z_3, z_2, z_1) = z_1^2 + z_2^2 + z_3^2 = C_1^2 = \text{const}. \qquad (8)$$

Example. In the three-dimensional Lobachevsky space in the Klein model with coordinates ($x = \beta_1$, $y = \beta_2, \alpha = z$), the geodesic equations (5) become

$$\begin{gathered}
\alpha^{\bullet\bullet} - \frac{1}{\alpha}(\alpha^{\bullet 2} - \beta_1^{\bullet 2} - \beta_2^{\bullet 2}) = 0, \\
\beta_1^{\bullet\bullet} - \frac{2}{\alpha}\alpha^{\bullet}\beta_1^{\bullet} = 0, \quad \beta_2^{\bullet\bullet} - \frac{2}{\alpha}\alpha^{\bullet}\beta_2^{\bullet} = 0.
\end{gathered} \qquad (9)$$

A four-parameter system that is equivalent for $\mu_1, \mu_3 \neq 0, \alpha \neq 0$ to Eqs. (9) and has a first integral of form (8) is given by

$$\begin{gathered}
\alpha^{\bullet} = z_3\mu_1\alpha, \quad z_3^{\bullet} = -z_2^2\frac{\mu_1\alpha^2}{\alpha^2 + \mu_2} - z_1^2\frac{\mu_1\mu_3^2\alpha^2}{\mu_3^2\alpha^2 + \mu_4}, \\
z_2^{\bullet} = z_2 z_3\frac{\mu_1\alpha^2}{\alpha^2 + \mu_2}, \quad z_1^{\bullet} = z_1 z_3\frac{\mu_1\mu_3^2\alpha^2}{\mu_3^2\alpha^2 + \mu_4}, \\
\beta_1^{\bullet} = z_2\frac{\mu_1\alpha^2}{\sqrt{\alpha^2 + \mu_2}}, \quad \beta_2^{\bullet} = z_1\frac{\mu_1\mu_3\alpha^2}{\sqrt{\mu_3^2\alpha^2 + \mu_4}},
\end{gathered}$$

if the first, fifth, and sixth equations of this system are treated as new kinematic relations.

The system of equalities (7) can be treated as the possibility of transforming the quadratic form of the metric into a canonical form with energy conservation law (8) (or see (18) below) depending on the problem under consideration. The history and the state of the art in this more general problem have been covered rather extensively (we note only [5, 6]).

The search for both integral (8) and (13), (15) (see below) relies on additional symmetry groups present in the system [5, 6].

It is possible to prove a separate theorem on the existence of a solution $f_1(\alpha)$, $f_2(\alpha)$, $f_3(\alpha)$, and $g(\beta_1)$ of system (7) for system (4), (6) to have analytical integral (8). However, some of the equalities (7) are not required in what follows in the study of dynamical systems with dissipation. Nevertheless, we assume hereinafter that the condition

$$f_1(\alpha) = f_2(\alpha) = f(\alpha) \tag{10}$$

holds for Eqs. (4), where the function $g(\beta_1)$ satisfies the transformed third equality in (7):

$$2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1} + \Gamma^1_{22}(\alpha,\beta)g^2(\beta_1) \equiv 0. \tag{11}$$

Thus, the function $g(\beta_1)$ depends on the connection coefficients, while constraints on the functions $f(\alpha)$ and $f_3(\alpha)$ will be given below.

Proposition 3. *If properties* (10) *and* (11) *hold and, additionally,*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma_1(\alpha), \tag{12}$$

then system (4), (6) *has a smooth first integral of the form*

$$\Phi_2(z_2, z_1;\alpha) = \sqrt{z_1^2 + z_2^2}\,\Phi_0(\alpha) = C_2 = \text{const},$$
$$\Phi_0(\alpha) = f(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}. \tag{13}$$

Proposition 4. *If the property*

$$\Gamma^2_{12}(\alpha,\beta) = \Gamma_2(\beta_1) \tag{14}$$

holds and the second equality in (12) *is satisfied* (*i.e.,* $\Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma_1(\alpha)$), *then system* (4), (6) *has a smooth first integral of the form*

$$\Phi_3(z_1;\alpha,\beta_1) = z_1\Phi_0(\alpha)\Phi(\beta_1) = C_3 = \text{const},$$
$$\Phi(\beta_1) = g(\beta_1)\exp\left\{2\int\limits_{\beta_{10}}^{\beta_1}\Gamma_2(b)db\right\}. \tag{15}$$

Proposition 5. *If conditions* (10)–(12), (14) *are satisfied, then system* (4), (6) *has a first integral of the form*

$$\Phi_4(z_2, z_1;\beta) = \beta_2 \pm \int\limits_{\beta_{10}}^{\beta_1}\frac{C_3 g(b)}{\sqrt{C_2^2\Phi^2(b) - C_3^2}}db = C_4 = \text{const}, \tag{16}$$

where, after the integral in (16) *has been evaluated, the constants* C_2 *and* C_3 *can be replaced by the left-hand sides of* (13) *and* (15), *respectively.*

Theorem 1. *If conditions* (10)–(12) *and* (14) *are satisfied, then system* (4), (6) *has a complete set of four first integrals of the form* (8), (13), (15), *and* (16).

The fact that the complete set consists of four, rather than five first integrals will be shown below.

2. INTEGRATION OF EQUATIONS OF MOTION IN A POTENTIAL FORCE FIELD

Modifying system (4), (6) yields a conservative system. Namely, we introduce a smooth force field in the projections onto the $z_k^\bullet$ axes, respectively, $k = 1, 2, 3$: $F_1(\beta_2)f_2(\alpha)$, $F_2(\beta_1)f_1(\alpha)$, and $F_3(\alpha)f_3(\alpha)$. The considered system on the tangent bundle $TM^3\{z_3, z_2, z_1;\alpha, \beta_1, \beta_2\}$ becomes

$$\begin{aligned}
\alpha^\bullet &= z_3 f_3(\alpha),\\
z_3^\bullet &= F_3(\alpha)f_3(\alpha) - f_3(\alpha)\left[\Gamma^\alpha_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right]z_3^2\\
&\quad - \frac{f_1^2(\alpha)}{f_3(\alpha)}\Gamma^\alpha_{11}(\alpha,\beta)z_2^2 - \frac{f_2^2(\alpha)}{f_3(\alpha)}g^2(\beta_1)\Gamma^\alpha_{22}(\alpha,\beta)z_1^2,\\
z_2^\bullet &= F_2(\beta_1)f_1(\alpha) - f_3(\alpha)\left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right]z_2 z_3\\
&\quad - \frac{f_2^2(\alpha)}{f_1(\alpha)}g^2(\beta_1)\Gamma^1_{22}(\alpha,\beta)z_1^2,\\
z_1^\bullet &= F_1(\beta_2)f_2(\alpha)g(\beta_1) - f_3(\alpha)\left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right]z_1 z_3\\
&\quad - f_1(\alpha)\left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right]z_1 z_2,\\
\beta_1^\bullet &= z_2 f_1(\alpha), \quad \beta_2^\bullet = z_1 f_2(\alpha)g(\beta_1),
\end{aligned} \tag{17}$$

and it is almost everywhere equivalent to the system

$$\alpha^{\bullet\bullet} - F_3(\alpha)f_3^2(\alpha) + \Gamma^\alpha_{\alpha\alpha}(\alpha,\beta)\alpha^{\bullet 2} + \Gamma^\alpha_{11}(\alpha,\beta)\beta_1^{\bullet 2} + \Gamma^\alpha_{22}(\alpha,\beta)\beta_2^{\bullet 2} = 0,$$

$$\beta_1^{\bullet\bullet} - F_2(\beta_1)f_1^2(\alpha) + 2\Gamma^1_{\alpha 1}(\alpha,\beta)\alpha^\bullet\beta_1^\bullet + \Gamma^1_{22}(\alpha,\beta)\beta_2^{\bullet 2} = 0,$$

$$\beta_2^{\bullet\bullet} - F_1(\beta_2)f_2^2(\alpha)g^2(\beta_1) + 2\Gamma^2_{\alpha 2}(\alpha,\beta)\alpha^\bullet\beta_2^\bullet + 2\Gamma^2_{12}(\alpha,\beta)\beta_1^\bullet\beta_2^\bullet = 0.$$

Proposition 6. *If equalities* (7) *are everywhere valid, then system* (17) *has a smooth first integral of the form*

$$\begin{aligned}
\Phi_1(z_3, z_2, z_1;\alpha,\beta) &= z_1^2 + z_2^2 + z_3^2 + V(\alpha,\beta) = C_1 = \text{const},\\
V(\alpha,\beta) &= V_3(\alpha) + V_2(\beta_1) + V_1(\beta_2)\\
&= -2\int\limits_{\alpha_0}^{\alpha}F_3(a)da - 2\int\limits_{\beta_{10}}^{\beta_1}F_2(b)db - 2\int\limits_{\beta_{20}}^{\beta_2}F_1(b)db.
\end{aligned} \tag{18}$$

Proposition 7. *Suppose that* $F_2(\beta_1) \equiv F_1(\beta_2) \equiv 0$. *If the conditions of Propositions* 3 *and* 4 *are satisfied, then system* (17) *has two smooth first integrals of the form* (13), (15).

Proposition 8. *If the conditions of Proposition* 5 *are satisfied, then system* (17) *has a first integral of the form* (16).

Theorem 2. *Suppose that* $F_2(\beta_1) \equiv F_1(\beta_2) \equiv 0$. *If conditions* (10)–(12) *and* (14) *are satisfied, then system* (17) *has a complete set of four first integrals of the form* (18), (13), (15), (16).

The fact that the complete set consists of four, rather than five, first integrals, will be shown below.

3. INTEGRATION OF EQUATIONS OF MOTION IN A FORCE FIELD WITH DISSIPATION

Slightly modifying system (17) with conditions (10)–(12), (14), and $F_2(\beta_1) \equiv F_1(\beta_2) \equiv 0$, we obtain a system with dissipation. Namely, dissipation (generally, alternating) is characterized not only by the coefficient $b\delta(\alpha)$, $b > 0$, in the first equation of system (19) (in contrast to (17)), but also by the following dependence of the (external) force field in the projections onto the $z_k^{\bullet}$ axes (k = 1, 2, 3), respectively: $z_1F^1(\alpha)$, $z_2F^1(\alpha)$, and $F_3(\alpha)f_3(\alpha) + z_3F_3^1(\alpha)$. The considered system on the tangent bundle $TM^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$ becomes

$$\begin{gathered}
\alpha^{\bullet} = z_3f_3(\alpha) + b\delta(\alpha),\\
z_3^{\bullet} = F_3(\alpha)f_3(\alpha)\\
- f_3(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right]z_3^2\\
- \frac{f^2(\alpha)}{f_3(\alpha)}\Gamma^{\alpha}_{11}(\alpha,\beta)z_2^2\\
- \frac{f^2(\alpha)}{f_3(\alpha)}g^2(\beta_1)\Gamma^{\alpha}_{22}(\alpha,\beta)z_1^2 + z_3F_3^1(\alpha),\\
z_2^{\bullet} = -f_3(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]z_2z_3\\
- f(\alpha)g^2(\beta_1)\Gamma^1_{22}(\alpha,\beta)z_1^2 + z_2F^1(\alpha),\\
z_1^{\bullet} = -f_3(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]z_1z_3\\
- f(\alpha)\left[2\Gamma_2(\beta_1) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right]z_1z_2 + z_1F^1(\alpha),\\
\beta_1^{\bullet} = z_2f(\alpha), \quad \beta_2^{\bullet} = z_1f(\alpha)g(\beta_1),
\end{gathered} \tag{19}$$

and it is almost everywhere equivalent to the system

$$\begin{gathered}
\alpha^{\bullet\bullet} - \left\{b\tilde{\delta}(\alpha) + F_3^1(\alpha) + b\delta(\alpha)\left[2\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right]\right\}\alpha^{\bullet}\\
- F_3(\alpha)f_3^2(\alpha) + b\delta(\alpha)F_3^1(\alpha)\\
+ b^2\delta^2(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right]\\
+ \Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta)\alpha^{\bullet 2} + \Gamma^{\alpha}_{11}(\alpha,\beta)\beta_1^{\bullet 2} + \Gamma^{\alpha}_{22}(\alpha,\beta)\beta_2^{\bullet 2} = 0,\\
\beta_1^{\bullet\bullet} - \left\{F_2^1(\alpha) + b\delta(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]\right\}\beta_1^{\bullet}\\
+2\Gamma_1(\alpha)\alpha^{\bullet}\beta_1^{\bullet} + \Gamma^1_{22}(\alpha,\beta)\beta_2^{\bullet 2} = 0,\\
\beta_2^{\bullet\bullet} - \left\{F_1^1(\alpha) + b\delta(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]\right\}\beta_2^{\bullet}\\
+ 2\Gamma_1(\alpha)\alpha^{\bullet}\beta_2^{\bullet} + 2\Gamma_2(\beta_1)\beta_1^{\bullet}\beta_2^{\bullet} = 0,\\
\tilde{\delta}(\alpha) = d\delta(a)/d\alpha.
\end{gathered}$$

Now we integrate the sixth-order system (19) with conditions (11) and

$$\Gamma^{\alpha}_{11}(\alpha,\beta) = \Gamma^{\alpha}_{22}(\alpha,\beta)g^2(\beta_1) = \Gamma_3(\alpha).$$

Assume also (by analogy with (11)) that the function $f(\alpha)$ satisfies the transformed equality in (7):

$$f_3^2(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] + \Gamma_3(\alpha)f^2(\alpha) \equiv 0.$$

In this case, a fifth-order independent subsystem decouples, namely,

$$\begin{gathered}
\alpha^{\bullet} = z_3f_3(\alpha) + b\delta(\alpha),\\
z_3^{\bullet} = F_3(\alpha)f_3(\alpha) - \frac{f^2(\alpha)}{f_3(\alpha)}\Gamma_3(\alpha)(z_2^2 + z_1^2) + z_3F_3^1(\alpha),\\
z_2^{\bullet} = -f_3(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]z_2z_3\\
- f(\alpha)g^2(\beta_1)\Gamma^1_{22}(\alpha,\beta_1)z_1^2 + z_2F^1(\alpha),\\
z_1^{\bullet} = -f_3(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]z_1z_3\\
- f(\alpha)\left[2\Gamma_2(\beta_1) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right]z_1z_2 + z_1F^1(\alpha),\\
\beta_1^{\bullet} = z_2f(\alpha), \quad \beta_2^{\bullet} = z_1f(\alpha)g(\beta_1).
\end{gathered}$$

To integrate the system completely, we need to know, generally speaking, five independent first integrals. However, after making the change of variables $z_1, z_2 \to z, z_*$, $z = \sqrt{z_1^2 + z_2^2}$, $z_* = z_2/z_1$, system (19) splits into

$$\begin{gathered}
\alpha^{\bullet} = z_3f_3(\alpha) + b\delta(\alpha),\\
z_3^{\bullet} = F_3(\alpha)f_3(\alpha) - \frac{f^2(\alpha)}{f_3(\alpha)}\Gamma_3(\alpha)z^2 + z_3F_3^1(\alpha),\\
z^{\bullet} = \frac{f^2(\alpha)}{f_3(\alpha)}\Gamma_3(\alpha)zz_3 + zF^1(\alpha),
\end{gathered} \tag{20}$$

$$z_*^\bullet = (\pm)z\sqrt{1+z_*^2}f(\alpha)\left[2\Gamma_2(\beta_1) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right],$$
$$\beta_1^\bullet = (\pm)\frac{zz_*}{\sqrt{1+z_*^2}}f(\alpha), \tag{21}$$

$$\beta_2^\bullet = z_1 f(\alpha)g(\beta_1). \tag{22}$$

It can be seen that, for the complete integrability of system (20)–(22), it suffices to indicate two independent first integrals of system (20), one first integral of independent system (21) (after changing the independent variable), and an additional first integral for "coupling" Eq. (22) (altogether four first integrals).

Assume also that, for some $\kappa \in \mathbf{R}$, it is true that

$$\frac{f^2(\alpha)}{f_3^2(\alpha)}\Gamma_3(\alpha) = \kappa\frac{d}{d\alpha}\ln|\Delta(\alpha)|,$$
$$\Delta(\alpha) = \frac{\delta(\alpha)}{f_3(\alpha)} \tag{23}$$

and, for some $\lambda_3^0, \lambda_k^1 \in \mathbf{R}$,

$$F_3(\alpha) = \lambda_3^0\frac{d}{d\alpha}\frac{\Delta^2(\alpha)}{2},$$
$$F_k^1(\alpha) = \lambda_k^1 f_3(\alpha)\frac{d}{d\alpha}\Delta(\alpha), \quad \kappa = 1,2,3. \tag{24}$$

Here, $F_1^1(\alpha) = F_2^1(\alpha) = F^1(\alpha)$, i.e., $\lambda_1^1 = \lambda_2^1 = \lambda^1$. Condition (23) will be referred to as geometric, and conditions of group (24), as energy ones.

Condition (23) is called geometric, because, among other things, it imposes a constraint on $\Gamma_3(\alpha)$ such that the corresponding coefficients of the system are reduced to a homogeneous form with respect to the function $\Delta(\alpha)$. Conditions of group (24) are called energy ones, because, among other things, the forces become, in a sense, "potential" with respect to the functions $\Delta^2(\alpha)/2$ and $\Delta(\alpha)$, so that the corresponding coefficients of the system are reduced to a homogeneous form (again with respect to $\Delta(\alpha)$). Moreover, it is the function $\Delta(\alpha)$ that introduces dissipation of variable sign into the system.

Theorem 3. *Suppose that conditions* (23) *and* (24) *are satisfied. Then system* (20)–(22) *has four independent, generally speaking, transcendental* [7, 8] *first integrals.*

In the general case, the first integrals have a cumbersome form (since the Abel equation has to be integrated [9]). Specifically, if $\kappa = -1$, then an explicit expression for the key first integral is given by

$$\Theta_1(z_3, z; \alpha) = G_1\left(\frac{z_3}{\Delta(\alpha)}, \frac{z}{\Delta(\alpha)}\right) = \frac{f_3^2(\alpha)(z_3^2+z^2) + (b-\lambda^1)z_3\delta(\alpha)f_3(\alpha) - \lambda_3^0\delta^2(\alpha)}{z\delta(\alpha)f_3(\alpha)} = C_1 = \text{const}. \tag{25}$$

An additional first integral of system (20) has the following structural form:

$$\Theta_2(z_3, z; \alpha) = G_2\left(\Delta(\alpha), \frac{z_3}{\Delta(\alpha)}, \frac{z}{\Delta(\alpha)}\right) = C_2 = \text{const}. \tag{26}$$

The first integral for system (21) is given by

$$\Theta_3(z_*; \beta_1) = \frac{\sqrt{1+z_*^2}}{\Phi(\beta_1)} = C_3 = \text{const}, \tag{27}$$

for the function $\Phi(\beta_1)$, see (15). The additional first integral coupling Eq. (22) is found by analogy with (16):

$$\Theta_4(z_*; \beta) = \beta_2 \pm \int_{\beta_{10}}^{\beta_1}\frac{C_3 g(b)}{\sqrt{C_2^2\Phi^2(b) - C_3^2}}db$$
$$= C_4 = \text{const}, \tag{28}$$

where, after the integral in (28) has been evaluated, the constants C_2 and C_3 can be replaced by the left-hand sides of (13) and (15), respectively.

The expressions for the first integrals (25)–(28) as finite combinations of elementary functions depend not only on computed quadratures, but also on the explicit form of the function $\Delta(\alpha)$. Indeed, for $\kappa = -1$, the additional first integral of system (20) is determined by the relation

$$d\ln|\Delta(\alpha)| = \frac{(b-u_3)du_3}{2W(u_3) - C_1\left\{C_1 \pm \sqrt{C_1^2 - 4W(u_3)}\right\}/2},$$

$$W(u_3) = u_3^2 + (b-\lambda^1)u_3 - \lambda_3^0, \quad u_3 = \frac{z_3}{\Delta(\alpha)}.$$

Moreover, after the integration, C_1 can be replaced by the left-hand side of (25). The right-hand side of this equality is expressed in terms of a finite combination of elementary functions, while the left-hand side is expressed depending on the function $\Delta(\alpha)$.

The following result holds, which is, in a sense, converse to Theorem 3.

Theorem 4. *Conditions* (23) *and* (24) (*e.g., for* $\kappa = -1$) *are necessary conditions for the existence of first integral* (25) *for system* (20)–(22).

4. STRUCTURE OF FIRST INTEGRALS FOR SYSTEMS WITH DISSIPATION

If α is a periodic coordinate of period 2π, then, under the conditions of Theorem 3, system (20)–(22)

becomes a dynamical system having variable dissipation with zero mean [2, 10]. For $b = -\lambda^1$, it turns into a conservative system with two smooth first integrals:

$$\Phi_1(-b; z_3, z; \alpha) = z^2 + z_3^2 + 2bz_3\Delta(\alpha) - \lambda_3^0\Delta^2(\alpha) = \text{const}, \quad (29)$$

$$\Phi_2(z_1; \alpha) = z\Delta(\alpha) = \text{const}. \quad (30)$$

Obviously, the ratio of two first integrals (29) and (30) is also a first integral of system (20)–(22) for $b = -\lambda^1$. However, for $b \neq -\lambda^1$, each of the functions

$$\Phi_1(\lambda^1; z_3, z; \alpha) = z^2 + z_3^2 + (b - \lambda^1)z_3\Delta(\alpha) - \lambda_3^0\Delta^2(\alpha) \quad (31)$$

and (30) taken separately is not a first integral of system (20)–(22). Nevertheless, the ratio of functions (31) and (30) is a first integral of system (20)–(22) (for $\kappa = -1$) for any b.

In general, for systems of any order with dissipation, transcendence of functions (in the sense of the presence of essential singularities) as first integrals is inherited from the existence of attracting or repelling limit sets in the system [11, 12].

5. SYSTEMS ON THE BUNDLE OF THREE-DIMENSIONAL SPHERE AND APPLICATIONS

In the above presentation, we identified, as an example, two classes of manifolds (surfaces of revolution and Lobachevsky spaces) for which the proposed technique of integration of systems with dissipation is applicable. Now we will note one-parameter family of functions $f(\alpha)$ and $f_3(\alpha)$ defining a metric on the three-dimensional sphere:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha\sqrt{1 + \mu_1\sin^2\alpha}}, \quad \mu_1 \in \mathbf{R}, \quad f_3(\alpha) \equiv -1.$$

Moreover, we distinguish between two important subcases:

$$\mu_1 = 0, \quad (32)$$

$$\mu_1 = -1. \quad (33)$$

Case (32) forms a class of systems corresponding to a dynamically symmetric four-dimensional rigid body moving at zero levels of cyclic integrals in a generally nonconservative force field in the case when the field depends additionally on (a second-rank tensor of) angular velocity [2, 10]. Case (33) forms a class of systems corresponding to the motion of a point on a three-dimensional sphere with a natural metric induced by the metric of the ambient four-dimensional Euclidean space. In particular, for $\delta(\alpha) = F_3(\alpha) \equiv 0$, the considered system describes a geodesic flow on a three-dimensional sphere. In the case of (32), if $\delta(\alpha) = \dfrac{F_3(\alpha)}{\cos\alpha}$, then the system describes the motion of a four-dimensional rigid body in the force field $F_3(\alpha)$ under the action of a follower force [2, 3]. Specifically, if $F_3(\alpha) = \sin\alpha\cos\alpha$ and $\delta(\alpha) = \sin\alpha$, then the system describes a generalized spherical pendulum placed in a material flow in four-dimensional space and has a complete set of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions [2, 3, 10, 11].

If the function $\delta(\alpha)$ is not periodic, then the considered dissipative system has variable dissipation with a nonzero mean (i.e., it is actually dissipative). Nevertheless, due to Theorems 3 and 4, closed-form expressions for transcendental first integrals in terms of a finite combination of elementary functions can also be obtained in this case. This result also determines new nontrivial cases of integrability (in explicit form) of dynamical systems with dissipation on the tangent bundle of a smooth three-dimensional manifold.

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Translated by I. Ruzanova