ISSN 1064-5624, Doklady Mathematics, 2020, Vol. 102, No. 3, pp. 487–492. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Doklady Rossiiskoi Akademii Nauk. Matematika, Informatika, Protsessy Upravleniya, 2020, Vol. 495, pp. 48–54.

MATHEMATICS

On the Finiteness of the Number of Expansions into a Continued Fraction of $\sqrt{f}$ for Cubic Polynomials over Algebraic Number Fields

Academician of the RAS V. P. Platonov[a,b,*] and M. M. Petrunin[a,]**

Received September 15, 2020; revised September 15, 2020; accepted September 21, 2020

Abstract—We obtain a complete description of cubic polynomials f over algebraic number fields $\mathbb{K}$ of degree 3 over $\mathbb{Q}$ for which the continued fraction expansion of $\sqrt{f}$ in the field of formal power series $\mathbb{K}((x))$ is periodic. We also prove a finiteness theorem for cubic polynomials $f \in K[x]$ with a periodic expansion of $\sqrt{f}$ for extensions of $\mathbb{Q}$ of degree at most 6. Additionally, we give a complete description of such polynomials f over an arbitrary field corresponding to elliptic fields with a torsion point of order $N \geq 30$.

Keywords: elliptic field, S-units, continued fractions, periodicity, modular curves, torsion point

DOI: 10.1134/S1064562420060137

Consider a square-free polynomial $f(x) \in \mathbb{K}[x]$ of degree $2g + 1$ over an algebraic number field $\mathbb{K}$. Assume that $f(x)$ is not divisible by x and its smallest coefficient is a complete square. Then the valuation ν_x corresponding to the linear polynomial x has two extensions ν_x^+ and ν_x^- to the field $\mathbb{K}(x)(\sqrt{f(x)})$. Therefore, $\sqrt{f(x)}$ (and, hence, $\mathbb{K}(x)(\sqrt{f(x)})$) can be embedded in the field of formal Laurent series $\mathbb{K}((x))$, which makes it possible to consider a continued fraction expansion of this element or any other element of $\mathbb{K}(x)(\sqrt{f(x)})$ (for more details, see [1]). Let $\mathscr{C}$ be a smooth compactification of the hyperelliptic curve $y^2 = f(x)$. Consider the embedding of the point $P = (0, \sqrt{f(0)})$ in the Jacobian of $\mathscr{C}$ that maps P into the class $P - \infty$. If $P - \infty$ has a finite order in the Jacobian, then there exist elements of $\mathbb{K}(x)(\sqrt{f(x)})$ for which the continued fraction expansion is periodic. These expansions have interesting properties, which are described in [1–3].

Note that, under the indicated assumptions on the pair $(\mathscr{C}, P)$, some elements are a fortiori periodic, for example, $\sqrt{f(x)}/x^g$ and $\sqrt{f(x)}/x^{g+1}$. In turn, the element $\sqrt{f(x)}$ is not always periodic, which is a substantial difference from the case of continued fraction expansions in $\mathbb{K}((1/x))$. In this context, the problem of describing all polynomials $f(x) \in \mathbb{K}[x]$ of degree $2g + 1$ for various classes of algebraic number fields $\mathbb{K}$ with a quasi-periodic continued fraction expansion of $\sqrt{f(x)}$ was raised in [3] (the quasi-periodicity of $\sqrt{f}$ is equivalent to periodicity, see [2]). This problem was completely solved in [3] for cubic polynomials over the rational number field by applying a theorem on the boundedness of torsion and using a rational parametrization of a pair consisting of an elliptic curve and a torsion point (see [4]). A similar result for the case of polynomials f of degree 4 was obtained in [5]. In [6] the case of algebraic number fields used as a coefficient field was studied and a generalization of the method of [3] was proposed. As a result, the problem of periodicity of $\sqrt{f}$ was completely solved for quadratic number fields and cubic polynomials f. Namely, a complete description of periodic expansions of pairs consisting of a quadratic number field and a periodic element $\sqrt{f}$ was obtained and a finiteness theorem for such polynomials f over third- and fourth-degree extensions of $\mathbb{Q}$ was proved. In [7] the periodicity of the expansion of $\sqrt{f(x)}$ was examined without using parametrizations, by assuming that its period is bounded, which was reached by bounding the order of a torsion point (which is equivalent to the assumption that the degree of a fundamental S-unit is bounded) and by finding solutions of a system of equations, for which the solvability condition is equivalent to the periodicity of $\sqrt{f(x)}$.

[a] *Scientific Research Institute for System Analysis, Russian Academy of Sciences, Moscow, Russia*
[b] *Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia*
**e-mail: platonov@mi-ras.ru*
***e-mail: petrushkin@yandex.ru*

Let E be an elliptic curve over an algebraic number field $\mathbb{K}$. By the Mordell–Weil theorem, the set of $\mathbb{K}$-points on E is a finitely generated Abelian group $E(\mathbb{K})$. In particular, its torsion subgroup $E(\mathbb{K})_{\text{tors}}$ is finite. Merel showed that $\#E(\mathbb{K})_{\text{tors}} \le B(d)$ for each elliptic curve E over a field $\mathbb{K}$ of degree d over $\mathbb{Q}$. However, the estimate for $B(d)$ that can be derived from Parent's result (see [8]) is too large for the current state of computational tools, and, in an attempt to generalize the theorem describing periodic $\sqrt{f}$ for cubic polynomials over $\mathbb{Q}$ to more general algebraic number fields $\mathbb{K}$, its use is inefficient. A complete description of orders of torsion points is available only for extensions of $\mathbb{Q}$ of degree at most 3 (see [9, 10]).

In this paper, the results of [6] and new results of other authors are used to obtain a complete solution of the problem of periodicity of $\sqrt{f}$ for cubic number fields. By optimizing algorithms and computer calculations, we prove a finiteness theorem for cubic polynomials f with a periodic expansion of $\sqrt{f}$ over algebraic number fields of degree at most 6 and give a complete description of such polynomials f over an arbitrary field corresponding to elliptic fields with a torsion point of order $N \le 30$.

Since the periodicity of the continued fraction expansion of $\sqrt{f(x)}$ is equivalent to the periodicity of $\sqrt{f^{\sigma}(x)}$, where $\sigma \in \operatorname{Aut}(\mathbb{K}/\mathbb{Q})$, and also to the periodicity of $\sqrt{a^2 f(bx)}$ for any a, $b \in \mathbb{K}^{\times}$, we consider polynomials up to this equivalence.

Our results are stated as the following theorems.

Theorem 1. *For square-free cubic polynomials $f \in \mathbb{K}[x]$ different from $cx^3 + 1$ over an algebraic number field $\mathbb{K}$ of degree $d = 3$ over $\mathbb{Q}$ that have a periodic continued fraction expansion of $\sqrt{f(x)}$ over $\mathbb{K}$, the number of equivalence classes is finite and is determined by the following representatives*:

$$12x^3 - 8x^2 + 4x + 1, \quad 12x^3 - 5x^2 + 2x + 1,$$
$$-120x^3 + 25x^2 + 2x + 1,$$

$$x^3 + (-6z^2 - 6)x^2 + (249z^2 + 105z + 360)x + \frac{2397}{2}z^2 + \frac{2055}{4}z + \frac{3495}{2},$$

where z is a root of the polynomial $t^3 - t^2 + \frac{1}{2}t - \frac{1}{12}$;

$$(x + 9z^2 - 6z + 2)(x^2 + (-48z^2 + 36z - 14)x + 216z^2 - 156z + 65),$$

where z is a root of the polynomial $t^3 - 3t^2 - 5$; and

$$x^3 + \frac{1}{84}(64z^2 - 40z - 123)x^2 + \frac{1}{49}(-32z^2 + 32z + 39)x + \frac{1}{343}(216z - 369),$$

where z is a root of the polynomial $t^3 + t^2 - 2t - \frac{9}{2}$.

This theorem is a corollary of the following more technical result, which provides a complete description of periodic $\sqrt{f}$ corresponding to an elliptic curve with torsion point $(0, \sqrt{f(0)})$ of order at most 30.

Theorem 2. *There exists only a finite number of equivalence classes of square-free cubic polynomials $f \in \mathbb{K}[x]$ over an arbitrary field $\mathbb{K}$ such that*

(i) *the point $P = (0, \sqrt{f(0)})$ of the corresponding elliptic curve has an order $5 \le N \le 30$;*

(ii) *the continued fraction expansion of $\sqrt{f(x)} \in \mathbb{K}((x))$ is periodic.*

Moreover, for each torsion order $5 \le N \le 30$, $N \ne 6$, there exists a unique polynomial f of this kind up to equivalence, while for $N = 6$ there is no such polynomial.

The degrees d of extensions of the coefficient field $\mathbb{K}$ and the period length Π of the continued fraction expansion of $\sqrt{f}$ for polynomials f from Theorem 2 are given in Table 1. Note that the quasi-period $\sqrt{f}$ in all cases given above is equal to a half of the period.

Relying on Theorem 2, we conclude that the number of periodic $\sqrt{f} \in \mathbb{K}[x]$ is finite if $\mathbb{K}$ is an algebraic number field of degree at most 6.

Theorem 3. *For square-free cubic polynomials $f \in \mathbb{K}[x]$ different from $cx^3 + 1$ over an algebraic number field $\mathbb{K}$ of degree $d \le 6$ over $\mathbb{Q}$ that have a periodic continued fraction expansion of $\sqrt{f(x)}$ over $\mathbb{K}$, the number of equivalence classes is finite.*

The proof sketch of Theorem 2 is similar to the one for the main theorems in [6]. For a fixed N, consider a modular curve $X_1(N)$ defined over $\mathbb{Q}$ such that its $\mathbb{K}$-points correspond to sets of pairs $(\mathcal{C}, P)$ consisting of an elliptic curve $\mathcal{C}$ over $\mathbb{K}$ and a $\mathbb{K}$-point of finite P order N on $\mathcal{C}$. The curves $X_1(N)$ can be described by equations $g_N(t,u) = 0$ in two variables, which were presented in [11]. Each pair (t,u), except for $(t,u) \in X_1(N)$ corresponding to cuspidal points, gives an elliptic curve in the Tate form:

$$y^2 + c(t,u)xy + b(t,u)y = x^3 + b(t,u)x^2. \qquad (1)$$

For such a curve, the point (0, 0) is a torsion point of order N if and only if $g_N(t,u) = 0$.

For all curves, the coefficients b and c are given in a unified manner by the formulas

$$\begin{aligned} c &= s - rs + 1, \\ b &= rs - r^2 s, \end{aligned} \tag{2}$$

where $r := r_N(t,u)$ and $s := s_N(t,u)$ depend on N. Replacing y by $y - \frac{cx+b}{2}$, we pass to the curve $y^2 = f(x)$ with the torsion point $\left(0, \frac{b}{2}\right)$, where

$$f = x^3 + \left(b + \frac{c^2}{4}\right)x^2 + \frac{bc}{2x} + \frac{b^2}{4}. \tag{3}$$

As was noted above, the continued fraction expansion of $\sqrt{f}/x^2$ is periodic. The nth step of this expansion is associated with a polynomial $L_n = (-1)^{n+1}(x^4 P_n^2 - fQ_n^2)$, where P_n/Q_n is the nth convergent of the continued fraction of $\sqrt{f}/x^2$. It was shown in [2] that the point $(0, \sqrt{f(0)})$ is a torsion point if and only if, for some n, the polynomial L_n is proportional to x^{2g+1} or x^{2g+2}. The degree of the S-unit equals the order of the torsion point and determines the parity of the degree of L_n for a minimal n such that L_n has the indicated property.

Thus, we have an equation $y^2 = f_N(x,t,u)$ in which the coefficients of x depend on the parameters (t, u), where t and u satisfy the relation $g_N(t,u) = 0$.

The element $\frac{\sqrt{f_N(x,t,u)}}{x^2}$ with (t, u) treated as formal variables is expanded into a continued fraction with respect to x^{-1} until the step at which L_n is proportional to x^3 or x^4. Next, according to the periodicity criterion for $\sqrt{f}$ given in [6], on the polynomials $P_n = p_0(t,u) + p_1(t,u)x^{-1} + \ldots$ and $Q_n = q_0(t, u) + q_1(t,u)x^{-1} + \ldots$, we impose the constraint $q_0(t,u) = 0$ or $p_1(t,u) = 0$, depending on the parity of the degree of L_n, which (under the condition $g_N(t,u) = 0$) implies the periodicity of $\sqrt{f}$. Finally, we solve the system consisting of $g_N(t,u) = 0$ and one of the following equations: $q_0(t,u) = 0$ or $p_1(t,u) = 0$.

Note that the free term in Q_n and the coefficient of x^{-1} in P_n depend only on the free terms and the coefficients of x^{-1} in P_m, Q_m, and A_m for $m < n$. Due to this circumstance, the number of arithmetic operations required for applying the periodicity criterion for $\sqrt{f}$ can be reduced significantly. The reduction in the number of arithmetic operations and the optimization of the algorithm and its software implementation made it possible to complete the computation for $N \le 30$, which, in turn, allowed us to complete the proof of the main results.

In the case of curves with a torsion point of order $N \ge 20$, we used the elimination of a variable based on Gröbner bases, which reduces the problem to a single equation in t.

Table 1. Orders N and degrees of the field $\mathbb{K}$ over $\mathbb{Q}$ for which $\sqrt{f}$ has a periodic expansion with period Π

N	Π	$\deg \mathbb{K}$
5	6	1
6	–	–
7	10	2
8	6	1
9	14	3
10	10	1
11	18	5
12	10	3
13	22	7
14	14	3
15	26	8
16	14	6
17	30	12
18	18	6
19	34	15
20	18	10
21	38	16
22	22	10
23	42	22
24	22	14
25	46	25
26	26	15
27	50	27
28	26	21
29	54	35
30	30	20

Proof sketch for Theorems 1 and 3. If $\sqrt{f}$ is periodic, then $\sqrt{f}/x^2$ is periodic as well and the point $(0, \sqrt{f(0)})$ on the curve $y^2 = f(x)$ has a finite order N (for more details, see [2]). We will use the following results on the finiteness of possible orders N. For the reader's convenience, they are stated in the form of a single theorem.

Theorem 4. *Let $\mathscr{C}$ be an elliptic curve over an algebraic number field $\mathbb{K}$ of degree $d \le 6$ over $\mathbb{Q}$. Then, for $\mathbb{K}$ and a torsion $\mathbb{K}$-point of order N on $\mathscr{C}$, the following assertions hold*:

(i) *If $\mathbb{K} = \mathbb{Q}$, then $N \le 12$ and $N \ne 11$ (see* [12]).

(ii) *If $d = 2$, then $N \le 18$ and $N \ne 17$ (see* [9]).

(iii) *If $d = 3$, then $N \le 21$ and $N \ne 17, 19$ (see* [10]).

(iv) *If* $d = 4$, *then the number of fields* $\mathbb{K}$ *and nonisomorphic elliptic curves* $\mathcal{E}_{\mathbb{K}}$ *with a torsion point of order distinct from* $N \le 24$, $N \ne 19, 23$, *is finite* (*see* [13]).

(v) *If* $d = 5$, *then the number of fields* $\mathbb{K}$ *and nonisomorphic elliptic curves* $\mathcal{E}_{\mathbb{K}}$ *with a torsion point of order distinct from* $N \le 25$, $N \ne 23$, *is finite* (*see* [14]).

(vi) *If* $d = 6$, *then the number of fields* $\mathbb{K}$ *and nonisomorphic elliptic curves* $\mathcal{E}_{\mathbb{K}}$ *with a torsion point of order distinct from* $N \le 30$, $N \ne 23, 25, 29$, *is finite* (*see* [14]).

In the formulation of Theorem 4 and in the proof of Theorem 1, we used the results of the recent preprint [10], which, according to one of its authors, is being prepared for publication. Earlier, it was shown in [15] that the number of cubic number fields $\mathbb{K}$ and nonisomorphic elliptic curves $\mathcal{E}_{\mathbb{K}}$ with a torsion point of order distinct from $N \le 20$, $N \ne 17, 19$, is finite.

According to Theorem 4, to prove that the number of classes of polynomials f over algebraic number fields of degree ≤ 6 is finite, it suffices to examine the periodicity of $\sqrt{f}$ only on curves with a torsion order $N \le 30$, $N \ne 23, 29$, which was done in Theorem 2.

Proof of Theorem 2. For reasons of space and due to the complexity of the numerical results, the complete proof is presented only for the case $N = 11$, which corresponds to expressions with moderate coefficients and gives a unique solution over an extension of degree 5, and for the case $N = 20$, which relies heavily on Gröbner bases. In all cases, except for $N = 6$, the system for (t, u) has exactly one solution (up to the choice of a root of an irreducible polynomial over $\mathbb{Q}$) for which the denominators of the coefficients of f_N and the free term of f_N do not vanish. Additionally, it should be noted that the cases $N \le 12$, $N \ne 11$, for which the parametrization $X_1(N)$ is rational were analyzed in [3]. The cases $N = 11, 13, 14, \ldots, 22, 24$ were announced in [6], where it was shown that, for $d = 2$, a nontrivial periodic root exists only for $N = 7$.

Theorem 4 and Table 1 show that nontrivial cases for $d = 3$ occur only for $N = 9, 12, 14$. Additionally, it has been shown that a periodic root occurs for $N = 11$ in the case $d = 5$ and for $N = 16, 18$ in the case $d = 6$.

Case $N = 11$. The curve $X_1(11)$ is defined by the relation $g_{11}(t,u) = u^2 - \frac{1}{4}t^4 - \frac{1}{2}t^2 + t - \frac{1}{4} = 0$, and the formulas

$$r(t,u) = tu - \frac{1}{2}t^3 - \frac{1}{2}t + 1, s(t,u) = -t + 1, \tag{4}$$

which are substituted into (2) and (3), determine the corresponding elliptic curve:

$$\begin{aligned} f_{11} = x^3 &+ \Big(\frac{1}{8}(t^8 + 2t^7 - t^6 + 2t^5 - 15t^4 + 14t^3 \\ &- 9t^2 + 6t + 2) - \frac{1}{4}(t-1)t(t^4 + 3t^3 + t^2 \\ &+ 3t - 6)u\Big)x^2 + \Big(\frac{1}{2}(t-1)t(t^7 - t^6 + 2t^5 - 4t^4 \\ &+ 2t^3 - 2t^2 + 1)u - \frac{1}{4}(t-1)^2 t(t^8 + 3t^6 - 4t^5 \\ &+ 2t^4 - 6t^3 - t - 1)\Big)x + \Big(\frac{1}{8}(t-1)^3 t^2 (t^9 + t^8 \\ &+ 5t^7 - t^6 + 5t^5 - 9t^4 + 4t^3 - 6t^2 + 3t - 1) \\ &- \frac{1}{4}(t-1)^3 t^2 (t^3 + t - 1)(t^4 + t^3 + 3t^2 + 1)u\Big). \end{aligned} \tag{5}$$

Consider a continued fraction expansion of the quadratic irrationality $\frac{\sqrt{f_{11}}}{x^2}$ in $\mathbb{K}(t,u)((x))$. In this case,

$$\begin{aligned} L_4 = t^{-2}\Big(&(-t^4 + 2t^3 - 2t + 1)u + \frac{1}{2}t^6 \\ &- t^5 + \frac{1}{2}t^4 - t^3 + \frac{5}{2}t^2 - 2t + \frac{1}{2}\Big)x^3, \end{aligned}$$

and L_n is not proportional to x^k for $0 \le n < 4$. The degree of an S-unit of the hyperelliptic field defined by the polynomial f_{11} coincides with the order of the torsion point with $x = 0$ and equals 11.

The quasi-period of the continued fraction expansion of $\sqrt{f_{11}}/x^2$ coincides with its period and equals 10. Applying the periodicity criterion for a square root (see [6]) in the case of an S-unit of odd degree, we conclude that $\sqrt{f_{11}}$ is periodic if and only if the coefficient of x^{-1} in the Laurent polynomial P_n vanishes:

$$p_1(t,u) = \frac{2(t^5 - 3t^3 + 4t^2 - 9t + 7)u - (t^7 - 2t^5 + 2t^4 - 12t^3 + 13t^2 + 5t - 7)}{4t} = 0.$$

Expressing u in terms of t by using this equation and substituting the result into $g_{11}(t,u) = 0$, we obtain

$$(3t^5 - 3t^4 - 12t^3 + 9t^2 - 35t + 63)(t-1)^3 t = 0. \tag{6}$$

Let us find irreducible factors of (6) associated with periodic expansions of $\sqrt{f_{11}}$. The roots $z = 0$, 1 do not correspond to f with a periodic expansion of $\sqrt{f}$, since the substitution of $t = z$ yields $f_{11}(0,z) = 0$, so $P = (0,0)$ is a second-order point.

A root z of the polynomial $t^5 - t^4 - 4t^3 + 3t^2 - \frac{35}{3}t + 21$, which is irreducible over $\mathbb{Q}$, is associated with

$u = \frac{6}{55}z^4 + \frac{3}{11}z^3 - \frac{53}{110}z^2 - \frac{6}{55}z - \frac{127}{110}$, and these values correspond to

$$f_{11}(x,z) = x^3 + \frac{1}{11}(-24z^4 + 72z^3 - 70z^2 + 112z - 76)x^2 + \frac{1}{11}(2877z^4 - 9984z^3 + 13080z^2 - 23436z + 24318)x + \frac{1}{4}(10224z^4 - 35451z^3 + 46509z^2 - 83811z + 87129). \quad (7)$$

The expansion of $\sqrt{f_{11}(x,z)}$ over an algebraic number field of degree 5 has a period of 18, a quasi-period of 9, and the quasi-periodicity coefficient

$$-\frac{56419}{33075}z^4 - \frac{77114}{33075}z^3 + \frac{43201}{33075}z^2 - \frac{3181}{1575}z + \frac{1501463}{99225}.$$

Case $N = 20$. The curve $X_1(20)$ is defined by the relation $g_{20}(u,t) = u^3 + (t^2+3)u^2 + (t^3+4)u + 2$ and, after substituting the expressions for $r(t,u)$ and $s(t,u)$ into (2) and (3), the corresponding elliptic curve is given by

$$f_{20} = x^3 + \frac{1}{4}(t-1)^{-6}(t^2-2t+2)^{-2}(t^2-t-1)^{-2} \times ((t-1)^{-1}t(t^{13}+\ldots)u + (t^{14}+\ldots))x^2 + (t-1)^{-10} \times t(t^2-2t+2)^{-2}(t^2-t-1)^{-3}((t^{15}+\ldots)u - \frac{1}{2}(t^{18}+\ldots))x + \frac{1}{4}(t-1)^{-11}t^2(t^2-2t+2)^{-2} \times (t^2-t-1)^{-4}((t^{19}+\ldots) - 3(t-1)^{-1}(t^{17}+\ldots)u). \quad (8)$$

Consider the continued fraction expansion of the quadratic irrationality $\frac{\sqrt{f_{20}}}{x^2}$ in $\mathbb{K}(t,u)((x))$. In this case, L_8 is proportional to x^4, while L_n is not proportional to x^k for $0 \le n < 8$. The degree of an S-unit of the hyperelliptic field defined by the polynomial f_{20} coincides with the order of the torsion point with $x = 0$ and equals 20. The expansion of $\frac{\sqrt{f_{20}}}{x^2}$ is quasi-periodic with a quasi-period of 9 and an expansion period of 18. Applying the periodicity criterion for $\sqrt{f}$ from [6] in the case of an S-unit of even degree, we conclude that $\sqrt{f_{20}}$ is periodic if and only if the free term of Q_n vanishes. This condition can be written as

$$q_0(t,u) = -(t^{10} - 8t^9 + 30t^8 - 68t^7 + 101t^6 - 100t^5 + 64t^4 - 24t^3 + 4t^2)^{-1}(-(3t^{12} - 33t^{11} + 177t^{10} - 606t^9 + 1453t^8 - 2555t^7 + 3362t^6 - 3340t^5 + 2505t^4 - 1413t^3 + 592t^2 - 174t + 28)u^2 - (3t^{13} - 30t^{12} + 141t^{11} - 398t^{10} + 679t^9 - 529t^8 - 533t^7 + 2257t^6 - 3545t^5 + 3488t^4 - 2349t^3 + 1100t^2 - 342t + 56)u + 2t^{11} - 19t^{10} + 111t^9 - 426t^8 + 1126t^7 - 2106t^6 + 2846t^5 - 2800t^4 + 1998t^3 - 1012t^2 + 336t - 56) = 0.$$

The Gröbner basis for the system of two conditions given above consists of three equations and has the form

$$u^2 + \frac{1}{2}(t^2+4)u + \frac{1788203968386774417}{1454626383087500}t^{31} + \ldots,$$

$$(t^5 - 4t^4 + 8t^3 - 8t^2 + 4t)u - \frac{1012274029378176552}{51950942253125}t^{31} + \ldots,$$

$$t\cdot(t-1)\cdot(t^2-2t+2)^2\cdot(t^4-2t^3+4t^2-3t+1)^4 \times (54t^{10} - 525t^9 + 2370t^8 - 6570t^7 + 12300t^6 - 16104t^5 + 14850t^4 - 9510t^3 + 4060t^2 - 1050t + 126).$$

For the last equation in the Gröbner basis, we find irreducible factors associated with periodic expansions of $\sqrt{f_{20}}$.

The roots $z = 0$ and $z = 1$ do not correspond to $f_{20}(x,z)$ with a periodic expansion of $\sqrt{f_{20}(x,z)}$, since the substitution of $x = 0$ and $t = z$ yields $f_{20}(0,z) = 0$, so $P = (0,0)$ is a second-order point. The case when z is a root of the polynomial $t^2 - 2t + 2$ or $t^4 - 2t^3 + 4t^2 - 3t + 1$ does not correspond to $f_{20}(x,z)$ with a periodic expansion of $\sqrt{f_{20}(x,z)}$, since z is also a root of the denominator of one of the coefficients of $f_{20}(x,t)$.

In turn, a root z of the polynomial $t^{10} - \frac{175}{18}t^9 + \frac{395}{9}t^8 - \frac{365}{3}t^7 + \frac{2050}{9}t^6 - \frac{2684}{9}t^5 + 275t^4 - \frac{1585}{9}t^3 + \frac{2030}{27}t^2 - \frac{175}{9}t + \frac{7}{3}$ is associated with

$$f_{20}(x,z) = x^3 + \frac{1}{360\,020}(121532184z^9 + \ldots)x^2 + \frac{1}{180010}(-3\,074\,046\,066z^9 + \ldots)x + \frac{1}{360020}(-40\,173\,695\,550z^9 + \ldots). \quad (9)$$

The expansion of $\sqrt{f_{20}}$ over a number field of degree 10 has a period of 9 and a quasi-period of 18.

FUNDING

This work was performed within the state assignment to basic scientific research, project no. 0065-2019-0011.

REFERENCES

1. V. P. Platonov, Russ. Math. Surv. **69** (1), 1–34 (2014).
2. V. P. Platonov and M. M. Petrunin, Proc. Steklov Inst. Math. **302**, 354–376 (2018).
3. V. P. Platonov and G. V. Fedorov, Sb. Math. **209** (4), 519–559 (2018).
4. D. S. Kubert, Proc. London Math. Soc. (3) **33** (2), 193–237 (1976).
5. V. P. Platonov and G. V. Fedorov, Russ. Math. Surv. **75** (4), 785–787 (2020).
6. V. P. Platonov, M. M. Petrunin and, V. S. Zhgoon, Dokl. Math. **102** (1), 288–292 (2020).
7. V. P. Platonov, M. M. Petrunin, and Yu. N. Shteinikov, Dokl. Math. **100** (2), 440–444 (2019).
8. P. Parent, J. Reine Angew. Math. **1999** (506), 85–116 (1999).
9. M. A. Kenku and F. Momose, Nagoya Math. J. **109**, 125–149 (1988).
10. M. Derickx, A. Etropolski, M. van Hoeij, J. S. Morrow, and D. Zureick-Brown, "Sporadic cubic torsion," arXiv:2007.13929 (2020).
11. A. Sutherland, Math. Comput. **81** (278), 1131–1147 (2012).
12. B. Mazur, "Rational points on modular curves," *Modular Functions of One Variable V*, Ed. by J. P. Serre and D. B. Zagier, *Lecture Notes in Mathematics* (Springer, Berlin, 1977), pp. 107–148.
13. D. Jeon, C. H. Kim, and E. Park, J. London Math. Soc. **74** (1), 1–12 (2006).
14. M. Derickx and A. Sutherland, Proc. Am. Math. Soc. **145** (10), 4233–4245 (2017).
15. D. Jeon, C. H. Kim, and A. Schweizer, Acta Arith. **113**, 291–301 (2004).

Translated by I. Ruzanova