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MATHEMATICS

On Stationary Nonequilibrium Measures for Wave Equations

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Abstract—In the paper, the Cauchy problem for wave equations with constant and variable coefficients is considered. We assume that the initial data are a random function with finite mean energy density and study the convergence of distributions of the solutions to a limiting Gaussian measure for large times. We derive the formulas for the limiting energy current density (in mean) and find a new class of stationary nonequilibrium states for the studied model.

Keywords: wave equations, random initial data, mixing condition, weak convergence of measures, Gaussian and Gibbs measures, energy current density, nonequilibrium states

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1. INTRODUCTION

In the paper, we consider the wave equations in $\mathbb{R}^d$ ($d \geq 3$ and odd) with constant or variable coefficients of the form

$$\ddot{u}(x,t) = \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j u(x,t)) - a_0(x) u(x,t), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \tag{1}$$

and with the initial data (as $t = 0$)

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad x \in \mathbb{R}^d. \tag{2}$$

Here $\partial_j \equiv \dfrac{\partial}{\partial x_j}$, $u(x,t) \in \mathbb{R}$. We assume that the coefficients of the equation are sufficiently smooth, for $|x| > R_0$ Eq. (1) has the form $\ddot{u}(x,t) = \Delta u(x,t)$; $a_0(x) \geq 0$, and the matrix $(a_{ij}(x))$ is positive definite for all $x \in \mathbb{R}^d$. In addition, we impose the so-called non-trapping condition (see condition D in [1]) which says that all rays of Eq. (1) go to infinity as $t \to \infty$.

The initial data $Y_0(x) = (u_0(x), v_0(x))$ are assumed to be a measurable random function with the distribution μ_0. We assume that the correlation functions of the initial measure μ_0,

$$Q_0^{ij}(x,y) = \int Y_0^i(x) Y_0^j(y) \mu_0(dY_0), \quad i,j = 0,1, \quad x,y \in \mathbb{R}^d,$$

have the form $Q_0^{ij}(x,y) = q_0^{ij}(\bar{x}, \bar{y}, \tilde{x} - \tilde{y})$, where $\bar{x} = (x_1, \dots, x_k)$, $\tilde{x} = (x_{k+1}, \dots, x_d)$, $x = (\bar{x}, \tilde{x})$, $y = (\bar{y}, \tilde{y}) \in \mathbb{R}^d$ with some $k \in \{1, \dots, d\}$. Moreover,

$$Q_0^{ij}(x,y) = q_{\mathbf{n}}^{ij}(x - y) \quad \text{for} \quad x,y \in D_{\mathbf{n}}, \tag{3}$$

where the regions $D_{\mathbf{n}}$ are defined as follows:

$$D_{\mathbf{n}} = \{x \in \mathbb{R}^d \colon (-1)^{n_1} x_1 > a, \dots, (-1)^{n_k} x_k > a\}, \quad \mathbf{n} = (n_1, \dots, n_k) \in \mathcal{N}^k. \tag{4}$$

Here $\mathcal{N}^k = \{\mathbf{n} = (n_1, \dots, n_k) \colon n_j \in \{1; 2\}, \forall j\}$, a is a fixed number, $a > 0$. Another words, condition (3) means that in the case when $(-1)^{n_j} x_j > a$ for all $j = 1, \dots, k$, the random function $Y_0(x)$ is equal to different, general speaking, translation invariant random processes $Y_{\mathbf{n}}(x)$ with distributions $\mu_{\mathbf{n}}$. Finally, we assume that the measure μ_0 has a finite mean energy density,

$$\int |Y_0(x)|^2 \mu_0(dY) = Q_0^{00}(x,x) + Q_0^{11}(x,x) \leq C < \infty. \tag{5}$$

Denote by μ_t, $t \in \mathbb{R}$, the distribution of the solutions

$$Y(t) \equiv Y(\cdot,t) = (u(\cdot,t), \dot{u}(\cdot,t)).$$

The main goal of the paper is to prove the weak convergence of the measures μ_t:

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$$\mu_t \to \mu_\infty \quad \text{as} \quad t \to \infty. \tag{6}$$

The similar convergence holds for $t \to -\infty$ since our system is time-reversible. In the paper, these results are applied in a particular case when the wave equations have the constant coefficients and the distributions $\mu_\mathbf{n}$ are Gibbs measures with temperatures $T_\mathbf{n} > 0$. However, Gibbs measures have the singular correlation functions and do not satisfy condition (5). Therefore, we introduce Gaussian random processes $Y_\mathbf{n}$ corresponding to the measures $\mu_\mathbf{n}$ and consider the "smoothened" measures $\mu_\mathbf{n}^\theta$ as the distributions of the convolutions $Y_\mathbf{n} * \theta$, where $\theta \in C_0^\infty(\mathbb{R}^d)$. The measures $\mu_\mathbf{n}^\theta$ satisfy condition (5). Denote by μ_t^θ the distribution of the convolution $Y(t) * \theta$. Then, the weak convergence of the measures $\mu_t^\theta \to \mu_\infty^\theta$ as $t \to \infty$ follows from convergence (6). This implies the convergence $\mu_t \to \mu_\infty$ as $t \to \infty$ because the function θ is arbitrary. In the case of the wave equations with constant coefficients, the explicit formulas for the covariance of the limiting measure μ_∞ are obtained. This allows us to calculate the coordinates of the limiting energy current density $\mathbf{J}_\infty = (J_\infty^1, \ldots, J_\infty^d)$ and to obtain that $J_\infty^l = 0$ for $l = k+1, \ldots, d$, and

$$J_\infty^l = -c_l \cdot 2^{-k} \Sigma(T_\mathbf{n}|_{n_l=2} - T_\mathbf{n}|_{n_l=1}) \quad \text{for} \quad l = 1, \ldots, k. \tag{7}$$

Here the summation is taken over all $n_j \in \{1,2\}$ with $j \neq l$, and $c_l = +\infty$. This infinity is connected with the "ultraviolet divergence." In the case of the smoothened measures μ_∞^θ, the energy current $\mathbf{J}_\infty$ has a finite value. Moreover, all numbers c_l are positive if the function $\theta(x)$ is axially symmetric with respect to all coordinate axes and not identically equal to zero.

At the present time, there is a large number of papers devoted to the study of the convergence to nonequilibrium states for various discrete and continuous systems, see review articles [2, 3]. For example, for an infinite one-dimensional chain of harmonic oscillators, the results similar to (6) were obtained in [4]. For many-dimensional harmonic crystals, convergence (6) and formula (7) were proved in [5, 6]. For continuous systems described by wave equations, results (6) and (7) were proved in [7] for a particular case when $k = 1$ (see condition (3)). Thus, in the given paper, we construct a more general (in comparison with [7]) class of nonequilibrium stationary states μ_∞, in which there is a nonzero heat flux in our model. Let us describe our results more precisely.

2. MAIN RESULTS

The following conditions (**A1**)–(**A3**) are imposed on Eq. (1).

(A1) $a_{ij}(x) = \delta_{ij} + b_{ij}(x)$,

where $b_{ij}(x) \in D \equiv C_0^\infty(\mathbb{R}^d)$, δ_{ij} is the Kronecker delta.

(A2) $a_0(x) \in D$, $a_0(x) \geq 0$, and the hyperbolicity condition holds, i.e., there exists $\alpha > 0$ such that

$$H(x,\xi) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \; x, \xi \in \mathbb{R}^d.$$

(A3) The nontrapping condition: for $(x(0), \xi(0)) \in \mathbb{R}^d \times \mathbb{R}^d$ with $\xi(0) \neq 0$, the convergence $|x| \to \infty$ as $t \to \infty$ holds, where $(x(t), \xi(t))$ is a solution to the Hamiltonian system $\dot{x}(t) = \nabla_\xi H(x(t), \xi(t))$, $\dot{\xi}(t) = -\nabla_x H(x(t), \xi(t))$.

In particular, condition (**A3**) is fulfilled in the case of constant coefficients, i.e., when $a_{ij}(x) \equiv \delta_{ij}$ and $a_0(x) \equiv 0$, because in this case $\dot{\xi}(t) \equiv 0$ and $x(t) = \xi(0)t + x(0)$.

The initial data $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0, v_0)$ of problem (1) belong to the phase space $\mathcal{H}$. By definition, $\mathcal{H} = H_{\text{loc}}^1(\mathbb{R}^d) \oplus H_{\text{loc}}^0(\mathbb{R}^d)$ is the Fréchet space of pairs $Y_0 = (u_0(x), v_0(x))$ of real functions $u_0(x)$ and $v_0(x)$ with local energy seminorms

$$\|Y\|_R^2 = \int_{|x|<R} (|u_0(x)|^2 + |\nabla u_0(x)|^2 + |v_0(x)|^2) dx < \infty, \quad \forall R > 0.$$

Proposition. *Let conditions* (**A1**)–(**A3**) *hold. Then for any initial data $Y_0 \in \mathcal{H}$ there exists a unique solution $Y(t) \in C(\mathbb{R}; \mathcal{H})$ to the Cauchy problem* (1), (2). *For any $t \in \mathbb{R}$, the operator $U(t)$: $Y_0 \to Y(t)$ is continuous in $\mathcal{H}$.*

Let us choose a function $\zeta(x) \in D$ with $\zeta(0) \neq 0$. Denote by $H_{\text{loc}}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e., the Fréchet spaces of distributions $u \in D'(\mathbb{R}^d)$ with finite local seminorms

$$\|u\|_{s,R} = \|\Lambda^s(\zeta(x/R)u)\|_{L^2(\mathbb{R}^d)},$$

where $\Lambda^s v = F_{\xi \to x}^{-1}(\langle\xi\rangle^s \hat{v}(\xi))$, $\langle\xi\rangle = \sqrt{|\xi|^2 + 1}$ and $\hat{v} = Fv$ is the Fourier transform of a tempered distribution v. For $\psi \in D$, write $F\psi = \int e^{i\xi \cdot x}\psi(x)dx$. By definition, $\mathcal{H}^s = H_{\text{loc}}^{1+s}(\mathbb{R}^d) \oplus H_{\text{loc}}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$.

Denote by μ_0 a probability Borel measure on $\mathcal{H}$ which is the distribution of the function Y_0, by $\mathbb{E}$ the integral with respect to the measure μ_0, by $Q_0(x, y) = (Q_0^{ij}(x,y))$ its correlation matrix. We assume that $\mathbb{E}(Y_0(x)) = 0$, $\mathbb{E}(|Y_0(x)|^2) \leq C < \infty$, the correlation functions $Q_0^{ij}(x,y)$ satisfy condition (3), where $q_\mathbf{n}(x-y) = (q_\mathbf{n}^{ij}(x-y))$ denote correlation matrices of some translation invariant measures $\mu_{\mathbf{n}}$ with zero mean value in the space $\mathcal{H}$. Finally, the measure μ_0 satisfies the strong uniform mixing condition of the Ibragimov type, see condition S4 in [7]. Roughly speaking, this condition means that $Y_0(x)$ and $Y_0(y)$ are asymptotically independent as $|x-y| \to \infty$.

Definition. μ_t, $t \in \mathbb{R}$, is a Borel probability measure on $\mathcal{H}$ which gives the distribution of the solution $Y(t)$, i.e., $\mu_t(B) = \mu_0(U(-t)B)$, $\forall B \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ stands for the Borel σ-algebra in $\mathcal{H}$.

Theorem. *Let conditions* **(A1)–(A3)** *and all assumptions imposed on the measure* μ_0 *be fulfilled. Then*,

(1) *the correlation functions of the measures* μ_t *converge to a limit as* $t \to \infty$;

(2) *for any* $\varepsilon > 0$, *the convergence* (6) *holds on the space* $\mathcal{H}^{-\varepsilon}$, *i.e., for any continuous bounded functional* $f(Y)$ *on the space* $\mathcal{H}^{-\varepsilon}$ *the convergence*

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY) \quad \text{as} \quad t \to \infty$$

is valid. Moreover, the limiting measure μ_∞ *is a Gaussian measure concentrated on* $\mathcal{H}$.

The proof of the theorem is based on the technique of [5], where the similar results were proved for discrete models (so-called harmonic crystals) and on the method of [7], where the theorem was proved in the particular case when $k = 1$.

Let now $u(x, t)$ be a random solution to problem (1) with constant coefficients, i.e., when $a_{ij}(x) \equiv \delta_{ij}$ and $a_0(x) \equiv 0$. Then, the mean energy current density is $\mathbf{J}(x, t) = -\mathbb{E}(\dot{u}(x,t)\nabla u(x,t))$. In the limit $t \to \infty$, we obtain $\mathbf{J}(x,t) \to \mathbf{J}_\infty = \nabla q_\infty^{10}(0)$, where $q_\infty(x) = (q_\infty^{ij}(x))$ is the correlation matrix of the measure μ_∞. We apply the obtained results to a particular case when $\mu_{\mathbf{n}}$ are Gibbs measures corresponding to the different temperatures $T_{\mathbf{n}} > 0$. For our model, the Gibbs measure g_T can be defined as the Gaussian measure with zero mean value and with the correlation matrix of a form

$$\begin{pmatrix} -T\mathcal{E}(x-y) & 0 \\ 0 & T\delta(x-y) \end{pmatrix},$$

where T stands for temperature, $T > 0$, $\mathcal{E}(x)$ is the fundamental solution of the Laplacian, i.e., $\Delta\mathcal{E}(x) = \delta(x)$ for $x \in \mathbb{R}^d$, $\delta(x)$ is the Dirac δ-function. In the case when $\mu_{\mathbf{n}} \equiv g_{T_{\mathbf{n}}}$ are Gibbs measures with temperatures $T_{\mathbf{n}}$, $\mathbf{n} \in \mathcal{N}^k$, formula (7) holds, where (formally) $c_l = (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{|\xi_l|}{|\xi|}d\xi$, $l = 1,\ldots,k$. Thus, we prove that there exist nonequilibrium states (or the probability limiting measures μ_∞), in which there is a nonzero heat flux in the studied model.

Let us consider a particular case of formula (7). Let $k = 1$ and $\mu_n = g_{T_n}$, $n = 1; 2$. Then, model (1) can be represented as a "system + two reservoirs," where reservoirs consist of the "points of the model" (i.e., of the solutions $Y(x, t)$) with coordinates $x \in D_1 = \{x \in \mathbb{R}^d: x_1 < -a\}$ and $x \in D_2 = \{x_1 > a\}$. Initially, the reservoirs have the Gibbs distributions with temperatures $T_{\mathbf{n}}$. It follows from formula (7) that in this case the limiting energy current density is equal to $\mathbf{J}_\infty = -(c_1(T_2 - T_1), 0, \ldots, 0)$, where $c_1 = +\infty$. In the case of the smoothened limiting measures μ_∞^θ, the number c_1 is finite and positive that corresponds to the Second Law of thermodynamics, i.e., the heat flows from the "hot reservoir" to the "cold" one.

In conclusion, we note that all results remain true for wave equations in $\mathbb{R}^d$ with even $d \geq 4$ which are an extension of [8] on a more general class of initial measures.

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