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MATHEMATICS

On Global Classical Solutions of Hyperbolic Differential-Difference Equations

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Abstract—A one-parameter family of global solutions of a two-dimensional hyperbolic differential-difference equation with an operator acting with respect to a space variable is constructed. A theorem is proved stating that the resulting solutions are classical for all parameter values if the symbol of the difference operator of the equation has a positive real part. Classes of equations for which this condition is satisfied are given.

Keywords: hyperbolic equation, differential-difference equation, classical solution, Fourier transform

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Interest in differential-difference (and, more generally, functional-differential) equations is motivated by problems arising in various applications for which classical models of mathematical physics based on only differential equations are insufficient (see, e.g., [1–4] and bibliography therein). For elliptic differential-difference equations, problems in bounded domains have been studied to date rather thoroughly (see the above references). In unbounded domains, problems for parabolic [5] and elliptic [6–10] differential-difference equations were investigated. Hyperbolic differential-difference equations were addressed in the case of shift operators acting with respect to the variable t (see [11, 12]).

In this paper, we study hyperbolic differential-difference equations with shift operators acting with respect to space variables.

In the half-plane $\mathbb{R}^1 \times (0,+\infty)$, we consider the equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = a_1 \frac{\partial^2 u}{\partial x^2}(x-h_1,t) + a_2 \frac{\partial^2 u}{\partial x^2}(x-h_2,t), \tag{1}$$

where a_j and h_j ($j = 1, 2$) are given real numbers, with h_1 and h_2 being not related by any commensurability conditions.

Since shift operators, like differential operators, are Fourier multipliers, solutions to Eq. (1) can be sought by applying the Gelfand–Shilov classical operation scheme (see, e.g., [13]). Generally speaking, this scheme leads to solutions in the sense of generalized functions, but, in the case under consideration, it can be shown that the found solutions are classical (smooth). In other words, they are functions all of whose derivatives (understood in the classical sense, i.e., as limits of corresponding ratios of finite differences) involved in the equation exist at each point of the half-plane $\mathbb{R}^1 \times (0,+\infty)$ and Eq. (1) holds for them at each point of this half-plane.

Applying the Gelfand–Shilov scheme, we conclude that smooth solutions of Eq. (1) have to be sought in the form

$$\begin{aligned} G(x,t;\xi) &= \sin\left(\rho(\xi)\xi t\cos\theta(\xi) + \theta(\xi) + \xi x\right) e^{\rho(\xi)\xi t\sin\theta(\xi)} \\ &\quad + \sin\left(\rho(\xi)\xi t\cos\theta(\xi) - \theta(\xi) - \xi x\right) e^{-\rho(\xi)\xi t\sin\theta(\xi)}, \end{aligned} \tag{2}$$

where

$$\rho(\xi) = (a_1^2 + a_2^2 + 2a_1a_2\cos(h_1-h_2)\xi)^{1/4}, \tag{3}$$

$$\theta(\xi) = \frac{1}{2}\arctan\frac{a_1\sin h_1\xi + a_2\sin h_2\xi}{a_1\cos h_1\xi + a_2\cos h_2\xi}. \tag{4}$$

Note that function (3) is well defined for all real values of the parameters a_j, h_j ($j = 1, 2$), and ξ.

The main result of this work is as follows.

Theorem. *The function* $G(x,t;\xi)$, *defined by formula* (2) *satisfies Eq.* (1) *for any real value of the parameter* ξ *such that*

$$a_1\cos h_1\xi + a_2\cos h_2\xi > 0. \tag{5}$$

If the assumption of the theorem is strengthen by requiring that the inequality involved be satisfied on the entire real line, then we obtain the following result.

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Corollary. *If inequality* (5) *holds for any real* ξ, *then the function* $G(x,t;\xi)$, *defined by formula* (2) *satisfies Eq.* (1) *for any real value of the parameter* ξ.

Note that, if inequality (5) holds, then the denominator in the argument of the arctangent function in formula (4) does not vanish; hence, under the conditions of the theorem and the corollary, any solution given by formula (2) is smooth.

The differential-difference operator on the right-hand side of Eq. (1) is the composition of the differential operator D_x^2 and the difference operator R defined as

$$Ru(x,t) = a_1 u(x - h_1, t) + a_2 u(x - h_2, t).$$

Its symbol is equal to $a_1 \cos h_1\xi + a_2 \cos h_2\xi - i(a_1 \sin h_1\xi + a_2 \sin h_2\xi)$, i.e., inequality (5) is equivalent to the fact that the real part of the symbol of the operator R (or, equivalently, of the symbol of the operator $R + R^*$) is positive at the point ξ. Thus, the condition of the corollary is equivalent to the condition that the real part of the symbol of the (only) difference operator involved in Eq. (1) is positive on the entire real line (see [1, Sections 8, 9; 5, Section 1.6]).

A natural question to ask is as follows: What are equations for which the indicated condition (that the real part of the symbol is positive) holds on the entire real line? An example of such equations is given by equations of the form (1) in which one of the shifts vanishes and the coefficient multiplying the (only) remaining nonlocal term does not exceed in absolute value the coefficient of the first term on the right-hand side, i.e., equations of the form

$$\frac{\partial^2 u}{\partial t^2}(x,t) = a_1 \frac{\partial^2 u}{\partial x^2}(x,t) + a_2 \frac{\partial^2 u}{\partial x^2}(x - h, t),$$

where $|a_2| < a_1$. For equations of this class, the symbol of the corresponding difference operator is equal to $a_1 + a_2 \cos h\xi - a_2 i \sin h\xi$; hence, the condition of the corollary is satisfied.

Another example of equations for which $G(x,t;\xi)$ is defined for all real values of the parameter ξ is equations of the form (1) in which vanishing is one of the coefficients on the right-hand side, rather than one of the shifts, i.e., equations with a single term on the right-hand side. This is a special case, in which the positivity condition for the real part of the symbol is not imposed and, for $G(x,t;\xi)$ to exist for any real parameter value, it is sufficient that the (only) coefficient on the right-hand side of the equation be positive. In this case, the equation becomes

$$\frac{\partial^2 u}{\partial t^2}(x,t) = a \frac{\partial^2 u}{\partial x^2}(x - h, t), \tag{6}$$

and the one-parameter family of its solutions has the form

$$\begin{aligned} G(x,t;\xi) = {} & \sin\left[\left(\sqrt{a}t \cos\frac{h\xi}{2} + x + \frac{h}{2}\right)\xi\right] e^{\sqrt{a}\xi t \sin\frac{h\xi}{2}} \\ & + \sin\left[\left(\sqrt{a}t \cos\frac{h\xi}{2} - x - \frac{h}{2}\right)\xi\right] e^{-\sqrt{a}\xi t \sin\frac{h\xi}{2}}. \end{aligned} \tag{7}$$

If $a > 0$, then, irrespective of the real value of h, function (7) is a smooth (classical) solution of Eq. (6) for any real value of ξ, which can be checked by directly substituting function (7) into Eq. (6).

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