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MATHEMATICS

Probabilistic Approximation of the Evolution Operator e^{-itH} where $H = \frac{(-1)^m}{(2m)!}\frac{d^{2m}}{dx^{2m}}$

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Abstract—Two approaches are suggested for constructing a probabilistic approximation of the evolution operator e^{-itH}, where $H = \frac{(-1)^m}{(2m)!}\frac{d^{2m}}{dx^{2m}}$, in the strong operator topology. In the first approach, the approximating operators have the form of expectations of functionals of a certain Poisson point field, while, in the second approach, the approximating operators have the form of expectations of functionals of sums of independent identically distributed random variables with finite moments of order $2m + 2$.

Keywords: Schrödinger equation, Poisson random measures, limit theorems

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Consider the Cauchy problem for the heat equation ($\sigma \in \mathbf{R}$)

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad u(0, x) = \varphi(x), \tag{1}$$

where the initial function φ is assumed to be continuous and bounded. It is well known that the solution of problem (1) has a probabilistic representation

$$u(t, x) = \mathbf{E}\varphi(x + \sigma w(t)), \tag{2}$$

where $w(t)$ is a standard Wiener process.

If the real number σ in Eq. (1) is replaced by the complex number $\sigma = e^{\frac{i\pi}{4}}$, then heat equation (1) turns into the Schrödinger equation (see [8]). In this case, representation (2) makes no sense, since the complex value $x + \sigma w(t)$ would be substituted into a function of a real argument. Formally, the solution of the Cauchy problem for the Schrödinger equation can be written using an integral (known as Feynman's path integral) with respect to the Feynman measure. The Feynman measure is a complex-valued finite additive function of a set defined on the algebra of cylinder sets and cannot be extended to a measure on the corresponding σ-algebra (see [1]).

A purely probabilistic method for approximating the solution of the Cauchy problem for the Schrödinger equation by mean values of functionals of stochastic processes was proposed in [2–4]. Two types of approximation of the solution to the Cauchy problem were constructed. In the first case, the solution was approximated by mean values of functionals of a Poisson point field, while, in the second case, it was approximated by mean values of functionals of normed sums of independent random variables with an identical symmetric distribution and a finite fourth moment.

The goal of this paper is to construct a probabilistic approximation of the solution to the Cauchy problem for the high-order Schrödinger equation

$$i\frac{\partial u}{\partial t} = \frac{(-1)^m}{(2m)!}\frac{\partial^{2m} u}{\partial x^{2m}}, \quad u(0, x) = \varphi(x), \tag{3}$$

where m is an arbitrary positive integer. As in [3], two approaches are proposed for approximating the solution of the Cauchy problem by mean values of functionals of stochastic processes. More specifically, the stochastic process is defined as an integral over a Poisson point field in the first approach and as normed sums of independent identically distributed random variables with a finite moment of order $2m + 2$ in the second approach. Note that approximations of the solution to the Cauchy problem for a Schrödinger-

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type equation with a right-hand side involving a fractional derivative of order $\alpha \in \bigcup_{m=2}^{\infty}(m-1,m)$ were constructed in [5–7].

APPROXIMATION OF THE SOLUTION TO THE CAUCHY PROBLEM BY MEAN VALUES OF FUNCTIONALS OF A POISSON POINT FIELD

Let ν be a Poisson random measure on $(0,\infty)\times(0,\infty)$ with intensity $\mathbf{E}\nu(dt,dx) = dt\mu(dx)$, where the measure μ has the form

$$d\mu(x) = \frac{dx}{x^{1+2m}}.$$

Given $\varepsilon > 0$, the compound Poisson process $\xi_\varepsilon(t)$ is defined by setting

$$\xi_\varepsilon(t) = \iint_{[0,t]\times[\varepsilon,e\varepsilon]} x\nu(ds,dx),$$

where e is the base of the natural logarithm.

Let $\hat{\varphi}(p)$ denote the direct Fourier transform of the function φ:

$$\hat{\varphi}(p) = \int_{-\infty}^{\infty}\varphi(x)e^{ipx}dx.$$

The initial function φ is represented in the form of a sum

$$\varphi(x) = P_+\varphi(x) + P_-\varphi(x) = \varphi_+(x) + \varphi_-(x),$$

where P_+ and P_- are Riesz projectors defined on $L_2(\mathbf{R})\cap L_1(\mathbf{R})$ as

$$\begin{aligned} P_+\varphi(x) &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ipx}\hat{\varphi}(p)dp,\\ P_-\varphi(x) &= \frac{1}{2\pi}\int_{0}^{\infty} e^{-ipx}\hat{\varphi}(p)dp. \end{aligned} \tag{4}$$

Note that the function φ_+ analytically extends to the upper half-plane, while the function φ_- analytically extends to the lower half-plane.

Let $\sigma = e^{\frac{\pi i}{2}\left(1-\frac{1}{2m}\right)}$. The complex number σ thus defined belongs to the upper half-plane $\mathbf{C}_+$ ($\mathrm{Re}\sigma > 0$) and satisfies the relation $(i\sigma)^{2m} = -i$.

For a fixed $\varepsilon > 0$, the operator semigroup P_ε^t acting on $\varphi \in L_2(\mathbf{R})$ is defined as

$$\begin{aligned} P_\varepsilon^t\varphi(x) = \mathbf{E}[&(\varphi_- * h_\varepsilon)(x - \sigma\xi_\varepsilon(t))\\ &+ (\varphi_+ * h_\varepsilon)(x + \sigma\xi_\varepsilon(t))], \end{aligned}$$

where the function $h_\varepsilon(x)$ is defined by its Fourier transform

$$\begin{aligned} \hat{h}_\varepsilon(p) = \exp\Bigg(-t\int_\varepsilon^{e\varepsilon}\Bigg(&i|p|\sigma x + \frac{(i|p|\sigma x)^2}{2} + \ldots\\ &+ \frac{(i|p|\sigma x)^{2m-1}}{(2m-1)!}\Bigg)d\mu(x)\Bigg)\exp\left(-t\int_\varepsilon^{e\varepsilon}\frac{(i|p|\sigma x)^{2m+1}}{(2m+1)!}d\mu(x)\right). \end{aligned}$$

Note that $\hat{h}_\varepsilon(p) \in L_q(\mathbf{R})$ for all $1 \le q \le \infty$, since $\mathrm{Re}(i\sigma)^{2m+1} > 0$.

Theorem 1. (i) *The operator P_ε^t is a pseudodifferential operator with the symbol*

$$r_{\varepsilon,t}(p) = \exp\left(-\frac{itp^{2m}}{(2m)!}\right)H(t,\varepsilon,p),$$

where

$$\begin{aligned} H(t,\varepsilon,p) = \exp\Bigg(t\int_\varepsilon^{e\varepsilon}\Bigg(&e^{i|p|\sigma x} - 1 - i|p|\sigma x\\ &-\frac{(i|p|\sigma x)^2}{2} - \ldots - \frac{(i|p|\sigma x)^{2m}}{(2m)!} - \frac{(i|p|\sigma x)^{2m+1}}{(2m+1)!}\Bigg)d\mu(x)\Bigg). \end{aligned}$$

(ii) *For any t,ε,p, it is true that $|H(t,\varepsilon,p)| \le 1$.*

Let $W_2^k(\mathbf{R})$ denote the Sobolev space of functions defined on $\mathbf{R}$ that have square integrable generalized derivatives up to the order k inclusive (see [9, p. 146]). The standard norm in $W_2^k(\mathbf{R})$ is defined by the formula

$$\|\psi\|_k^2 = \sum_{l=0}^{k}\int_{\mathbf{R}}|\psi^{(l)}(x)|^2dx.$$

It is convenient to equip $W_2^k(\mathbf{R})$ with a different norm equivalent to the standard one (see [9, p. 190]), namely,

$$\|\psi\|_{W_2^k(\mathbf{R})}^2 = \int_{\mathbf{R}}(1 + |p|^{2k})|\hat{\psi}(p)|^2dp.$$

Let P^t denote the semigroup $P^t = \exp\left(\frac{it(-1)^{m+1}}{(2m)!}\frac{d^{2m}}{dx^{2m}}\right)$. By definition, P^t maps the initial function φ to a solution of Cauchy problem (3) (see, e.g., [8, 10]).

Theorem 2. *There exists a number $C > 0$ such that, for any function $\varphi \in W_2^{2m+2}(\mathbf{R})$ and all $t \ge 0$,*

$$\left\|P^t\varphi - P_\varepsilon^t\varphi\right\|_{L_2(\mathbf{R})} \le Ct\varepsilon^2\|\varphi\|_{W_2^{2m+2}(\mathbf{R})}.$$

Corollary 1. *For any function $\varphi \in L_2(\mathbf{R})$,*

$$\|P_\varepsilon^t\varphi - P^t\varphi\|_{L_2(\mathbf{R})} \xrightarrow[\varepsilon\to 0]{} 0.$$

APPROXIMATION OF THE SOLUTION TO THE CAUCHY PROBLEM BY MEAN VALUES OF FUNCTIONALS OF SUMS OF INDEPENDENT RANDOM VARIABLES

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of independent identically distributed nonnegative random variables. Denote by $\mathcal{P}$ the distribution of the random variable ξ_1. Assume that ξ_1 has a finite moment of order $2m + 2$ and $\mathbf{E}\xi_1^{2m} = 1$.

Let $\eta(t)$, $t \in [0, \infty)$, be a Poisson process of intensity 1 independent of the sequence $\{\xi_j\}$. Define $\varkappa_1 = \mathbf{E}\xi_1^1$, $\varkappa_2 = \mathbf{E}\xi_1^2, \ldots, \varkappa_{2m+2} = \mathbf{E}\xi_1^{2m+2}$. For each positive integer n, the stochastic process $\zeta_n(t)$, $t \in [0, T]$, is defined as

$$\zeta_n(t) = \frac{1}{n^{1/2m}} \sum_{j=1}^{\eta(nt)} \xi_j.$$

For each positive integer n and $\varphi \in L_2(\mathbf{R})$, the operator semigroup P_n^t is defined by setting

$$P_n^t\varphi(x) = \mathbf{E}[(\varphi_- * R_n^t)(x - \sigma\zeta_n(t)) + (\varphi_+ * R_n^t)(x + \sigma\zeta_n(t))],$$

where, as before, the functions $\varphi_\pm$ are given by formula (4) and $\sigma = e^{\frac{\pi i}{2}\left(1 - \frac{1}{2m}\right)}$. The function $R_n^t(x)$ is defined by its Fourier transform

$$\hat{R}_n^t(p) = \exp\left(-ti|p|\sigma n^{1-\frac{1}{2m}}\varkappa_1 - \frac{t(i|p|\sigma)^2 n^{1-\frac{2}{2m}}\varkappa_2}{2!} - \ldots - \frac{t(i|p|\sigma)^{2m-1} n^{1-\frac{2n-1}{2m}}\varkappa_{2m-1}}{(2m-1)!}\right) \times \exp\left(-\frac{t(i|p|\sigma)^{2m+1} n^{1-\frac{2m+1}{m}}\varkappa_{2m+1}}{(2m+1)!}\right).$$

Theorem 3. (i) *The operator P_n^t is a pseudodifferential operator with symbol $r_{n,t}(p) = \exp\left(-\frac{itp^{2m}}{(2m)!}\right)H_n(t, p)$, where*

$$H_n(t, p) = \exp\left(nt\int_0^{\infty}\left(e^{\frac{i|p|\sigma y}{n^{1/2m}}} - 1 - \frac{i|p|\sigma y}{n^{1/2m}} - \ldots - \frac{(i|p|\sigma y)^{2m}}{(2m)!n} - \frac{(i|p|\sigma y)^{2m+1}}{(2m+1)!n^{(2m+1)/2m}}\right)d\mathcal{P}(y)\right).$$

(ii) *For any n, t, p, it is true that $|H_n(t, p)| \le 1$.*

Theorem 4. *There exists a number $C > 0$ such that, for any function $\varphi \in W_2^{2m+2}(\mathbf{R})$ and all $t \ge 0$,*

$$\|P^t\varphi - P_n^t\varphi\|_{L_2(\mathbf{R})} \le \frac{Ct}{n^{1/m}}\|\varphi\|_{W_2^{2m+2}(\mathbf{R})}.$$

Corollary 2. *For any function $\varphi \in L_2(\mathbf{R})$,*

$$\|P_n^t\varphi - P^t\varphi\|_{L_2(\mathbf{R})} \xrightarrow[n\to\infty]{} 0.$$

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