ISSN 1064-5624, Doklady Mathematics, 2019, Vol. 100, No. 1, pp. 329–331. © Pleiades Publishing, Ltd., 2019.
Russian Text © The Author(s), 2019, published in Doklady Akademii Nauk, 2019, Vol. 487, No. 1.

MATHEMATICS

# A Generalization of the Logarithmic Gross–Sobolev Inequality

**Sh. M. Nasibov**

Presented by RAS Academician V.P. Maslov January 13, 2019

Received January 28, 2019

**Abstract**—A sharp integral inequality is proved that is used to derive a Sobolev interpolation inequality. A generalization of the logarithmic Sobolev inequality is proposed based on the Sobolev interpolation inequality.

**DOI:** 10.1134/S1064562419040033

## 1. SOBOLEV INTERPOLATION INEQUALITY

In this section, we prove a sharp integral inequality implying, due to the Hausdorff–Young inequality, a Sobolev interpolation inequality.

### 1.1. Integral Inequality

For convenience, we use the following notation:

$$\|U\|_p = \left\{\int_{R^n} |U(x)|^p\,dx\right\}^{1/p}, \qquad p \geq 1,$$

is the norm in $L_p(R^n)$; the index $p$ in $\|\cdot\|_p$ with $p = 2$ is omitted, that is, we write $\|\cdot\|$ in this case. Let $k$ be any positive number. Let $\rho$ be a given positive number that is arbitrary if $n - k \leq 0$ and satisfies the inequality $\rho < \dfrac{2k}{n-k}$ if $n - k > 0$. We set $\alpha = \dfrac{n\rho}{k(\rho+2)}$; for a given $\alpha$, we introduce the quantity $\chi = \sqrt{\alpha^\alpha(1-\alpha)^{1-\alpha}}$.

For any $\theta > 0$, we define the Euler gamma function $\Gamma(\theta) = \int_0^{+\infty} e^{-t}t^{\theta-1}dt$; $B(\beta,\gamma) = \int_0^1 t^{\beta-1}(1-t)^{\gamma-1}\,dt$ for all $\beta > 0$ and $\gamma > 0$ is the Euler beta function; $\sigma_n = \dfrac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$;

$$K_g(\alpha) = \frac{1}{\chi}\left[\frac{\sigma_n}{k}B\left(\frac{n}{k}, \frac{n(1-\alpha)}{k\alpha}\right)\right]^{\alpha k/2n} = \frac{1}{\chi}\left[\frac{\frac{\sigma_n}{k}\Gamma\frac{n}{k}\Gamma\frac{n(1-\alpha)}{k\alpha}}{\Gamma(n/(k\alpha))}\right]^{\frac{\alpha k}{2n}}. \tag{1}$$

**Lemma 1.** *Let $k, \rho$, and $\alpha$ be the numbers defined above, $V(x) \in L_2(R^n)$, $r^{k/2}V(x) \in L_2(R^n)$, $r = |x|$. Then the following integral inequality holds:*

$$\|V\|_{(\rho+2)/(\rho+1)} \leq K_g(\alpha)\|r^{k/2}V\|^\alpha\|V\|^{1-\alpha}, \tag{2}$$

*where $K_g(\alpha)$ is the constant defined by (1). The constant is sharp: inequality (2) turns into equality with*

$$V(x) = V_0(r) = \frac{\omega_1}{(\omega_2 + \omega_2 r^k)^{1+1/\rho}},$$

*where $\omega_1, \omega_2$, and $\omega_3$ are arbitrary positive numbers.*

### 1.2. Hausdorff–Young Inequality

**Lemma 2.** *Let*

$$\hat{U}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{-i(x,\xi)}U(x)dx, \quad \xi \in R^n,$$

*be the Fourier transform of a function $U(x)$, $\hat{U} \in L_p(R^n)$, $1 \leq p \leq 2$. Then the Hausdorff–Young inequality*

$$\|U\|_{p'} \leq K_B(p)\|\hat{U}\|_p,$$

$$K_B(p) = \left[\left(\frac{p}{2\pi}\right)^{1/p}\left(\frac{p'}{2\pi}\right)^{-1/p'}\right]^{n/2},$$

*Institute of Applied Mathematics, Baku State University, Baku, AZ1148 Azerbaijan*
*e-mail: nasibov_sharif@mail.ru*

$1 \le p \le 2$, $\frac{1}{p}+\frac{1}{p'}=1$, *holds with the best Beckner–Babenko constant* [1–4].

### *1.3. Results*

**Theorem 1.** *Let* $k, \rho$, *and* $\alpha$ *be the numbers defined above,* $U(x) \in L_2(R^n)$, $r^{k/2}\hat{U}(\xi) \in L_2(R^n)$, $r = |\xi|$. *Then the following multiplicative Sobolev inequality holds*:

$$\|U\|_{\rho+2} \le \bar{K}_0 \|r^{k/2}\hat{U}(\xi)\|^{\alpha}\|U\|^{1-\alpha}. \quad (3)$$

*Here*, $\bar{K}_0 = K_g(\alpha)K_B\left(\frac{\rho+2}{\rho+1}\right)$, *where* $K_g(\alpha)$ *is defined by* (1).

We give a scheme for proving (3).

In view of inequality (2), we conclude that

$$\|\hat{U}\|_{\frac{\rho+2}{\rho+1}} \le K_g(\alpha)\||\xi|^{k/2}\hat{U}\|^{\alpha}\|\hat{U}\|^{1-\alpha}. \quad (4)$$

Due to the Plancherel–Parseval theorem, we have

$$\|\hat{U}\| = \|U\|. \quad (5)$$

Therefore, under the assumptions of Theorem 1, we deduce that $\hat{U} \in L_{\frac{\rho+2}{\rho+1}}(R^n)$. Then the Hausdorff–Young inequality implies

$$\|U\|_{\rho+2} \le K_B\left(\frac{\rho+2}{\rho+1}\right)\|\hat{U}\|_{\frac{\rho+2}{\rho+1}}. \quad (6)$$

Inequality (3) follows from (4)–(6).

Assume that $k = 2$ in Theorem 1. Owing to the relation $\||\xi|\hat{U}\| = \|\nabla U|$, Theorem 1 implies the following corollary.

**Corollary 1.** *Let* $\rho \in (0,\infty)$ *with* $n = 1,2$ *and* $\rho \in \left(0, \frac{4}{n-2}\right)$ *with* $n \ge 3$, *and let* $\propto = \rho n/[2(\rho+2)]$. *Let* $U(x) \in H^1(R^n)$. *Then the following Gagliardo–Nirenberg–Sobolev interpolation inequality holds*:

$$\|U\|_{\rho+2} \le \bar{K}_0\|\nabla U\|^{\alpha}\|U\|^{1-\alpha}. \quad (7)$$

*Here*, $\bar{K}_0 = K_g(\alpha)K_B\left(\frac{\rho+2}{\rho+1}\right)$, *where*

$$K_g(\alpha) = \frac{1}{\chi}\left[0.5\sigma_n B\left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha}\right)\right]^{\alpha/n} = \frac{1}{\chi}\pi^{\alpha/2}\left[\frac{\Gamma[(n-n\alpha)/2\alpha]}{\Gamma(n/2\alpha)}\right]^{\alpha/n}.$$

Assume that $k = 4$ in Theorem 1. Owing to the relation $\||\xi|^2\hat{U}\| = \|\Delta U\|$ [5], Theorem 1 implies the following corollary.

**Corollary 2.** *Let* $\rho \in (0,\infty)$ *with* $n \le 4$ *and* $\rho \in \left(0, \frac{8}{n-4}\right)$ *with* $n > 4$, *and let* $\propto = n\rho/[4(\rho+2)]$. *Let* $U(x) \in L_2(R^n)$ *and* $\Delta U \in L_2(R^n)$. *Then the following Sobolev interpolation inequality holds*:

$$\|U\|_{\rho+2} \le \bar{K}_0\|\Delta U\|^{\alpha}\|U\|^{1-\alpha}. \quad (8)$$

*Here*,

$$\bar{K}_0 = K_g(\alpha)K_B\left(\frac{\rho+2}{\rho+1}\right),$$

*where*

$$K_g(\alpha) = \frac{1}{\chi}\left[\frac{\sigma_n}{4}B\left(\frac{n}{4}, \frac{n(1-\alpha)}{4\alpha}\right)\right]^{2\alpha/n}.$$

## 2. LOGARITHMIC GROSS–SOBOLEV INEQUALITY

**Theorem 2.** *Let* $k$ *be an arbitrary positive number*, $U(x) \in L_2(R^n)$, *and* $|\xi|^{k/2}\hat{U}(\xi) \in L_2(R^n)$. *Then the following logarithmic Gross–Sobolev inequality holds*:

$$\int_{R^n}\frac{|U|^2}{\|U\|^2}\ln\left(\frac{|U|^2}{\|U\|^2}\right)dx \le \frac{n}{k}\ln\left[\frac{k\left(\frac{\sigma_n}{k}\Gamma\left(\frac{n}{k}\right)\right)^{k/n}\||\xi|^{k/2}\hat{U}\|^2}{n\pi^k e^{k-1}\|U\|^2}\right]. \quad (9)$$

Theorem 1 implies the following propositions.

**Proposition 1.** *Let* $U(x) \in H^1(R^n)$. *Then the following logarithmic Gross–Sobolev inequality holds*:

$$\int_{R^n}\frac{|U|^2}{\|U\|^2}\ln\left(\frac{|U|^2}{\|U\|^2}\right)dx \le \frac{n}{2}\ln\left(\frac{2\|\nabla U\|^2}{\pi e n\|U\|^2}\right). \quad (10)$$

*Inequality* (10) *is sharp*: *it turns into equality with*

$$U(x) = a\exp(-b|x|^2),$$

*where a and b are arbitrary positive constants.*

We put $k = 4$ in (9).

**Proposition 2.** *Let* $U(x) \in H^2(R^n)$. *Then the following logarithmic Gross–Sobolev inequality holds*:

$$\int_{R^n}\frac{|U|^2}{\|U\|^2}\ln\left(\frac{|U|^2}{\|U\|^2}\right)dx \le \frac{n}{4}\ln\left[\frac{4\left(\frac{\sigma_n}{4}\Gamma\left(\frac{n}{4}\right)\right)^{4/n}\|\Delta U\|^2}{n\pi^4 e^3\|U\|^2}\right]. \quad (11)$$

Inequality (10) was first proved by Gross [6]. Beckner [7] notes that after Gross found the logarithmic Sobolev inequality, it became folklore. Other inequalities of Gross–Sobolev type were proved in [4, 8–10] and etc.

We give a scheme for proving (9). We rewrite (3) in the form

$$\|U\|_{\frac{2n}{n-\alpha k}} \le K_B(\alpha) K_g(\alpha) \||\xi|^{k/2}\hat{U}\|^{\alpha} \|U\|^{1-\alpha}, \quad (12)$$

where

$$K_B(\alpha) = \left(\frac{n}{\pi}\right)^{\frac{\alpha k}{2}} \frac{(n-\alpha k)^{\frac{n-\alpha k}{4}}}{(n+\alpha k)^{\frac{n+\alpha k}{4}}},$$

$$K_g(\alpha) = \frac{1}{\sqrt{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}} \left[\frac{\sigma_n}{k} B\left(\frac{n}{k}, \frac{n(1-\alpha)}{k\alpha}\right)\right]^{\frac{\alpha k}{2n}}.$$

It is straightforward to show that $\lim_{\alpha\to 0+0} K_g(\alpha) = 1$ and $K_B(0) = 1$. Thus, inequality (12) remains valid even when $\alpha = 0$.

We consider the function

$$f(\alpha) = \|U\|_{2n/n-\alpha k} - K_0 \||\xi|^{k/2}\hat{U}\|^{\alpha} \|U\|^{1-\alpha},$$

where $K_0 = K_B(\alpha) K_g(\alpha)$. Since $f(\alpha) \le 0$ for $\alpha \in [0,1)$, we have $f'(0) \le 0$. When calculating $f'(0)$, we should take into account that $K_B'(0) = -\ln(\pi e)$ and $K_g'(0) = \frac{1}{2}\ln\left[\frac{ek\left(\frac{\sigma_n}{k}\Gamma\left(\frac{n}{k}\right)\right)^{k/n}}{n}\right]$.

Inequality (2) is also used to prove the generalized entropy inequality

$$-\int_{R^n} \frac{|U(x)|^2}{\|U\|^2} \ln\left(\frac{|U(x)|^2}{\|U\|^2}\right) dx \le \frac{n}{k}\ln\left[\frac{ek\left(\frac{\sigma_n}{k}\Gamma\left(\frac{n}{k}\right)\right)^{k/n} \|r^{k/n}U\|^2}{n\|U\|^2}\right]. \quad (13)$$

Under the condition $U(x) \in L_2(R^n)$, we have $r^{k/2}U(x) \in L_2(R^n)$ for any $k > 0$. Inequality (13) is sharp: it turns into equality with

$$U(x) = U_0(r) = a\exp(-b|x|^k),$$

where $a$ and $b$ are arbitrary positive constants [11].

**Remark 1.** Interpolation inequality (7) was also proved in [12] with another constant. This inequality is used to analyze the global solvability of the Cauchy problem for a nonlinear evolution Schrödinger equation [13], as well as in the spectral theory for Schrödinger operators [12].

**Remark 2.** Inequality (13) with $k = 2$ was announced in [13] and was proved in [14].


## ACKNOWLEDGMENTS

The author is grateful to Academician of the RAS V.P. Maslov for useful advice and support.


## REFERENCES


1. K. I. Babenko, Izv. Akad. Nauk SSSR, Ser. Mat. **25**, 531–542 (1961).
2. W. Beckner, Ann. Math. **102**, 159–182 (1975).
3. M. C. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 2: *Fourier Analysis and Self-Adjointness* (Academic, New York, 1975).
4. E. H. Lieb and M. Loss, *Analysis*, 2nd ed. (Am. Math. Soc., Providence, R.I., 2001).
5. Sh. Nasibov, Izv. Math. **73** (3), 555–577 (2009).
6. L. Cross, Am. J. Math. **97**, 1061–1683 (1975).
7. W. Beckner and M. Pearson, Bull. London Math. Soc. **30**, 80–84 (1998).
8. F. B. Weissler, Trans. Am. Math. Soc. **237**, 255–259 (1978).
9. W. Beckner, Forum Math. **11**, 105–137 (1999).
10. E. A. Carlen, J. Funct. Anal. **101**, 194–211 (1991).
11. Sh. M. Nasibov, Math. Notes **99** (2), 304–307 (2016).
12. E. J. Veling, J. Inequal. Pure Appl. Math. **3** (4), article 63 (2002).
13. Sh. M. Nasibov, Dokl. Akad. Nauk SSSR **307** (3), 538–542 (1989).
14. Sh. M. Nasibov, Math. Notes **84** (2), 218–223 (2008).


*Translated by N. Berestova*