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MATHEMATICS

On Infinite-Dimensional Integer Hankel Matrices

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Abstract—We construct a parametric family of infinite-dimensional integer Hankel matrices, each having the following property: its principal submatrices are nonsingular, and the set of prime divisors of the determinants of the principal submatrices coincides with the set of all primes.

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Investigated in [1], the deep relation between Hankel matrices and elliptic fields was used in [2] to construct by induction an infinite-dimensional Hankel matrix H_∞ with principal submatrices H_r of degree $r = 1, 2, 3, \ldots$ that has the following properties:

(i) all elements of the matrices have the form $m/2^l$, where m and l are integers;

(ii) $\det H_r \neq 0$ for all $r = 1, 2, \ldots$;

(iii) the set of prime divisors of the numerators of the determinants $\det H_r$, $r = 1, 2, 3, \ldots$, coincides with the set of all primes, except for 2.

Obviously, 2 can be included in the divisors of the numerators of the principal minors (i.e., of the determinants of the principal submatrices) of the matrix $2^t H_\infty$ by multiplying H_∞ by a suitable power 2^t. However, the existence of H_∞ (constructed as in [2] for monic polynomials) with a similar property remained an open question. It was answered in the affirmative in [3].

Next, a natural question arises as to whether there exists an integer infinite-dimensional Hankel matrix such that the set of prime divisors of its principal minors coincides with the set of all primes and the principal minors of this Hankel matrix do not vanish. This question is answered below; moreover, we construct a parametric family of such matrices.

Theorem 1. *There exists an infinite set of infinite-dimensional integer Hankel matrices such that any matrix H_∞ from this set with principal submatrices H_r of degree $r = 1, 2, 3, \ldots$ has the following properties*:

(i) *all elements of the matrices are integers*;

(ii) $\det H_r \neq 0$ *for all* $r = 1, 2, \ldots$;

(iii) *the set of prime divisors of the determinants* $\det H_r$, $r = 1, 2, 3, \ldots$, *coincides with the set of all primes*.

Proof. Consider the family of square-free polynomials of the form $f = a^2x^4 + bx + c$, where $a, b, c \in \mathbb{Q}$, $a \neq 0, b \neq 0, c \neq 0$, and $b^4 \neq 8a^2c^3$. It was shown in [1, Theorem 2.2] that the ring $D_f = \alpha + \beta\sqrt{f}, \alpha, \beta \in k[x]$ does not contain fundamental units. Moreover, arguments similar to those used in [2] show that, in this case, the set of prime divisors of the numerators of the principal minors of H_∞ contains all primes, except for possibly the prime divisors of the numerator of the discriminant of f and prime divisors of the denominators of the coefficients of f, and the finite-dimensional principal minors of H_∞ do not vanish, where

$$H_\infty = \begin{pmatrix} d_{-1} & d_{-2} & \cdots & d_{-r} & \cdots \\ d_{-2} & d_{-3} & \cdots & d_{-r-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ d_{-r} & d_{-r-1} & \cdots & d_{-2r+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix},$$

and d_i are the coefficients of the expansion of $\sqrt{f} = d_2x^2 + d_1x + d_0 + d_{-1}x^{-1} + \ldots$ into a formal Laurent power series in the field of formal power series $K((x^{-1}))$.

Let $f_4 = a^2, f_3 = f_2 = 0, f_1 = b, f_0 = c$, and $f_i = 0$ for $i < 0$. Then, without loss of generality, fixing the embedding of $K(x)(\sqrt{f})$ in $K((x^{-1}))$, we can calculate the coefficients d_s using the recurrence formulas

$$d_2 = \sqrt{f_4};$$
$$d_{2-k} = \frac{1}{2d_2}\left(f_{4-k} - \sum_{i=1}^{k-1} d_{2-i}d_{2-k+i}\right). \quad (1)$$

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Thus, assuming that $a = \frac{1}{2l}$ and $l, b, c \in \mathbb{Z}\setminus 0$, we conclude that all $d_i \in \mathbb{Z}$ for $i < 0$ and, hence, the matrix H_∞ are integer, and all its minors are integers as well.

Multiplying H_∞ by the prime divisors $p_1, p_2, \ldots$ of the numerator of the discriminant $\Delta(f)$ and the denominators of the coefficients of f raised to suitable powers, we can include the primes p_i in the set of divisors of the matrix $\prod p_i^{l_i} H_\infty$, which completes the proof of the theorem.

Example. Let $a = \frac{1}{2}, b = 2$, and $c = 3$. The discriminant $\Delta(f)$ of the polynomial $f = \frac{1}{4}x^4 + 2x + 3$ is equal to 3^4. From this, following the line of reasoning in Theorem 1, we see that, for each prime $p \neq 2, 3$, among the principal minors of the corresponding matrix H_∞ constructed using a Laurent series expansion of $\sqrt{f}$, there are minors that are divided by p.

The expansion of $\sqrt{f}$ into a Laurent series has the form

$$\sqrt{f} = \frac{1}{2}x^2 + 2x^{-1} + 3x^{-2} - 4x^{-4} - 12x^{-5} - 9x^{-6} + \ldots. \tag{2}$$

Consider the Hankel matrix H_2 associated with expansion (2):

$$H_2 = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}.$$

Its determinant is equal to -3^2, while the determinant of the matrix H_1 is 2. Thus, we conclude that the set of divisors of the principal minors of the infinite integer Hankel matrix H_∞ coincides with the set of all primes.

Theorem 1 presents a set of matrices with the indicated properties; however, this set is not representable in explicit form. As a result, the indicated matrices are obtained by calculating the discriminants of corresponding polynomials and depend substantially on the prime factorization of the discriminants.

The following theorem proves the existence of an explicitly constructed one-parameter family of infinite-dimensional integer Hankel matrices with the properties as in Theorem 1.

Theorem 2. *There exists a one-parameter family of infinite-dimensional integer Hankel matrices such that any matrix $H_\infty(t)$ from this family with principal submatrices $H_r(t)$ of degree $r = 1, 2, 3, \ldots$ has the following properties for every $t \in \mathbb{Z}\setminus 0$:*

(i) *all elements of the matrices are integers*;

(ii) $\det H_r(t) \neq 0$ *for all* $r = 1, 2, \ldots$;

(iii) *the set of prime divisors of the determinants* $\det H_r(t)$, $r = 1, 2, 3, \ldots$, *coincides with the set of all primes*.

Proof. Let $f = a^2x^4 + bx + c$. Then the discriminant of this polynomial is $\Delta(f) = -(27b^4 - 256c^3a^2)a^4$, and the Laurent series expansion of $\sqrt{f}$ can be calculated using formula (1). It is given by

$$\sqrt{f} = ax^2 + \frac{b}{2ax} + \frac{c}{2ax^2} - \frac{b^2}{(2a)^3x^4} - \frac{2bc}{(2a)^3x^5} - \frac{c^2}{(2a)^3x^6} + \frac{2b^3}{(2a)^5x^7} + \frac{6b^2c}{(2a)^5x^8} + \frac{6bc^2}{(2a)^5x^9} + \ldots. \tag{3}$$

Applying the argument used in the proof of Theorem 1, we construct an infinite-dimensional Hankel matrix from expansion (3):

$$H_\infty = \begin{pmatrix} \frac{b}{2a} & \frac{c}{2a} & 0 & -\frac{b^2}{8a^3} & -\frac{bc}{4a^3} & \ldots \\ \frac{c}{2a} & 0 & -\frac{b^2}{8a^3} & -\frac{bc}{4a^3} & -\frac{c^2}{8a^3} & \ldots \\ 0 & -\frac{b^2}{8a^3} & -\frac{bc}{4a^3} & -\frac{c^2}{8a^3} & \frac{b^3}{16a^5} & \ldots \\ -\frac{b^2}{8a^3} & -\frac{bc}{4a^3} & -\frac{c^2}{8a^3} & \frac{b^3}{16a^5} & \frac{3b^2c}{16a^5} & \ldots \\ -\frac{bc}{4a^3} & -\frac{c^2}{8a^3} & \frac{b^3}{16a^5} & \frac{3b^2c}{16a^5} & \frac{3bc^2}{16a^5} & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}. \tag{4}$$

The first few principal minors of H_∞ are given by

$$\det H_1 = \frac{b}{2a};$$
$$\det H_2 = -\frac{c^2}{(2a)^2};$$
$$\det H_3 = -\frac{(b^4 - 8c^3a^2)b}{(2a)^7};$$
$$\det H_4 = -\frac{b^8 - 8b^4c^3a^2 - 16c^6a^4}{(2a)^{12}};$$
$$\det H_5 = \frac{(b^8 - 12b^4c^3a^2 + 24c^6a^4)bc^2}{(2a)^{17}};$$
$$\det H_6 = \frac{b^{16} - 24b^{12}c^3a^2 + 208b^8c^6a^4 - 640b^4c^9a^6 - 256c^{12}a^8}{(2a)^{26}}.$$

If $a = \frac{1}{2l}$ and $l \in \mathbb{Z}\setminus 0$, then the infinite-dimensional Hankel matrix H_∞ is integer. If, additionally, $b \neq 0, c \neq 0$, and $b^4 \neq 8a^2c^3$, then the corresponding ring D_f has no nontrivial units. From this, following the argument used in the proof of Theorem 1, it remains to provide that the prime factors of the discriminant and the prime factors of the number $2l$ divide the principal minors.

Let $a = \frac{1}{4}$, $b = (2t)^3$, $c = (2t)^4$, and $t \in \mathbb{Z}\backslash 0$. Then $b^4 \neq 8a^2c^3$ and $\Delta(f) = -2^4 \cdot 11 \cdot t^{12}$. The prime divisors of $\Delta(f)$ are the prime divisors of t and the prime numbers 2 and 11. For the above-specified parameters a, b, c, the first principal minor is $\det H_1 = 2^4 t^3$, i.e., it is divided by all prime divisors of t and by 2, while the fifth principal minor is $\det H_5 = 2^{48} \cdot 11 \cdot t^{35}$.

Thus, the family of infinite-dimensional integer matrices $H_\infty(t)$ has the properties indicated in Theorem 2, where the matrix $H_\infty(t)$ is obtained from (4) by substituting the above-specified parameters a, b, c and $t \in \mathbb{Z}\backslash 0$:

$$H_\infty(t) = \begin{pmatrix} 16t^3 & 32t^4 & 0 & -512t^6 & -2048t^7 & -2048t^8 & \dots \\ 32t^4 & 0 & -512t^6 & -2048t^7 & -2048t^8 & 32\,768t^9 & \dots \\ 0 & -512t^6 & -2048t^7 & -2048t^8 & 32\,768t^9 & 196\,608t^{10} & \dots \\ -512t^6 & -2048t^7 & -2048t^8 & 32\,768t^9 & 196\,608t^{10} & 393\,216t^{11} & \dots \\ -2048t^7 & -2048t^8 & 32\,768t^9 & 196\,608t^{10} & 393\,216t^{11} & -2\,359\,296t^{12} & \dots \\ -2048t^8 & 32\,768t^9 & 196\,608t^{10} & 393\,216t^{11} & -2\,359\,296t^{12} & -20\,971\,520t^{13} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

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