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## CONTROL THEORY

# Class of Trajectories in $\mathbb{R}^3$ Most Remote from Observers

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**Abstract**—The set of extremal trajectories is completely described. Their construction is reduced to finding the best routes on a directed graph whose vertices are subsets (boxes) of $Y\setminus\bigcup_S K(S)$ and whose edges are segments $\mathcal{T}(S)$ of the trajectory $\mathcal{T}$ that intersect the cones $K(S)$ in the "best way." The edge length is the deviation of $S$ from $\mathcal{T}(S)$. The best routes are ones for which the length of the shortest edge is maximal.

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**1. Formulation of the problem.** Let $Y$ be a given neighborhood (corridor) of a trajectory without self-intersections that joins fixed points $t_* \neq t^*$ in the space $\mathbb{R}^3$. The boundary $\partial Y$ of the corridor is homeomorphic to a sphere. The collection of trajectories

$$\mathcal{T} = \{t(\tau) : 0 \le \tau \le 1,\ t(0) = t^*,\ t(1) = t^*\},$$

contained in $Y$ is denoted by $\mathbb{T}$. Given a set $\mathbb{S} = \{S\}$ of stationary observers $S \notin Y$, each having a fixed convex open visibility cone $K = K(S)$ with a vertex $S$ such that

$$K \cap \mathcal{T} \neq \phi \quad \forall \mathcal{T} \in \mathbb{T}.$$

Let $K_Y$ denote the maximal (with respect to inclusion) connected subset of $K \cap Y$ that is the nearest to $S$ and has the above-indicated property. Define the deviation

$$d(S,t) = \begin{cases} \|S - t\| & \text{if} \quad t \in K, \\ +\infty & \text{if} \quad t \notin K. \end{cases}$$

Assuming that the observers are hostile toward an object moving in $Y$, we consider the problem of finding the quantity

$$\mathbb{M} = \mathbb{M}(\mathbb{S}) = \max_{\mathcal{T}\in\mathbb{T}} \min\{d(S,t): t \in \mathcal{T},\ S \in \mathbb{S}\} \qquad (1)$$

and characterizing optimal trajectories in this problem. Below, the set of optimal trajectories is completely described. Their construction is reduced to finding the best routes on a digraph whose vertices are closed disjoint subsets of $Y\setminus\bigcup_S K(S)$ and the edges are trajectory fragments that "optimally" intersect $K(S)$.

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**2. Definitions.** In what follows, for any $S \in \mathbb{S}$, the truncated cone $K_Y(S)$ has a boundary with a left $\mathcal{L}K = \mathcal{L}K(S)$ and a right $\mathcal{R}K = \mathcal{R}K(S)$ side, i.e., the object moving from $t_*$ to $t^*$ along any trajectory $\mathcal{T} \in \mathbb{T}$ first crosses $\mathcal{L}K$ and escapes from $K_Y$, intersecting the side $\mathcal{R}K$ and $\mathrm{conv}\{S,\mathcal{L}K\} \cap \mathrm{conv}\{S,\mathcal{R}K\} = \phi$. A point $t \in Y$ is said to be to the left of $\mathcal{L}K$ (to the right of $\mathcal{R}K$) if $t$ lies on the segment of a trajectory $\mathcal{T}$ starting at the point $t_*$ and ending at the first intersection point with $\mathcal{L}K$ (if it lies on the trajectory segment between the last intersection point with $\mathcal{R}K$ and $t^*$).

Let a pair $\{S_1, S_2\}$ be such that $\mathcal{K} = K_Y(S_1) \cap K_Y(S_2) \neq \phi$. This pair is called even [1] if sets $Y \cap \mathrm{conv}(S_i \cup \mathcal{K})$ $(i = 1, 2)$ both lie to the left of the boundaries $\mathcal{R}K_i$ $(i = 1, 2)$ or to the right of the boundaries $\mathcal{L}K_i$ $(i = 1, 2)$. Here, conv denotes the convex hull of a set.

For a given set $\mathbb{S} = \{S_i\}$, the difference $Y\setminus\bigcup_i K_Y(S_i)$ is the union of closed connected disjoint sets—boxes. A box is formed by several surfaces $\mathcal{L}_i = \mathcal{L}K(S_i)$ and $\mathcal{R}_j = \mathcal{R}K(S_j)$, whose fragments make up its boundary. Thus, a box consists of the points of $Y$ lying to the left of the surfaces $\mathcal{L}_i, \ldots$ and to the right of the surfaces $\mathcal{R}_j, \ldots$. Additionally, a fragment of the surface $\partial Y$ can be a part of the box boundary. For such a box, we will use the notation $[\mathcal{L}_i, \ldots, \mathcal{R}_j, \ldots]$, omitting the boundary $\partial Y$. Since the box points are not visible to the observers, the trajectory segment lying within the box can be arbitrary.

**3. Observers' interests.** The side observing the moving object has a limited energy resource and seeks to reduce $\mathbb{M}(\mathbb{S})$ by choosing

—a suitable number of observers and a shape of the cones $K(S)$, in particular, their "thickness"

$$\min_H \max_x \{\rho(x,H): x \in K_Y\},$$

where the minimum is taken over the planes $H$ containing the point $S$;

—a constraint on the multiplicity of covering $Y$ by truncated cones $K_Y$;

—a distribution of the observers around $Y$ such that there are no extended segments of $Y$ that are not controlled by the observers. Note that it is reasonable to use even pairs of "neighboring" observers (see [1]). For technical reasons, pairs $S$, $S'$ of observers for which $[S,S'] \subset K(S) \cap K(S')$ are not admissible.

**4. Auxiliary result.** The lemma presented below is the basic tool for constructing optimal trajectories. Let $S \in \mathbb{S}$, $K = K(S)$, and $D$ be a connected subset of $K_Y$ that is the closure of an open set such that

$$D \cap \mathcal{L}K \neq \phi, \quad D \cap \mathcal{R}K \neq \phi,$$

in particular, it is possible that $D = K_Y$. Let $\mathbb{T}_D$ be a class of trajectories $\mathcal{T} \subset D$ joining the sets $D \cap \mathcal{L}K$, $D \cap \mathcal{R}K$. Consider the problem

$$\mathbb{M}(\mathbb{S})_D = \max_{\mathcal{T}\in\mathbb{T}_D} \min\{d(S,t):\ t \in \mathcal{T}\}. \tag{2}$$

It is convenient to monitor the motion of the object by inspecting intersection points of its trajectory with planes of a specially chosen one-parameter family of planes $\mathcal{H} = \{H\}$. Given two points $x_l \in \mathcal{L}K$ and $x_r \in \mathcal{R}K$, $x_l \neq x_r$, consider the supporting planes $\mathcal{H}_l$ and $\mathcal{H}_r$ of the cone $K$ at these points, and let

$$\mathcal{H} = \{H^\lambda = \lambda H_l + (1-\lambda) H_r: 0 \le \lambda \le 1\}.$$

To find optimal trajectories $\mathcal{T}$ of problem (2), we need to find the quantity

$$M(S)_D = \min_{H\in\mathcal{H}} \max\{d(S,x): x \in H \cap D\}, \tag{3}$$

the set $\mathcal{H}^*$ of planes providing the minimum in (3), and the set of points $\mathcal{P}_H = \{x\}$ providing the maximum in (3) for $H \in \mathcal{H}^*$. Define $\mathcal{P}(S)_D = \bigcup_{H\in\mathcal{H}^*} \mathcal{P}_H$ and $\mathring{V}_M(S) = \{x: \|x - S\| < M\}$.

**Lemma.** *For the observer $S$, it is true that $\mathbb{M}(S)_D = M(S)_D$. A trajectory $\mathcal{T} = \mathcal{T}_D \subset D$ is optimal in problem* (2) *if and only if*

$$\mathcal{T} \cap \mathcal{P}_H \neq \phi \ \ \forall H \in \mathcal{H}^*, \quad \mathcal{T} \subset D\backslash(K(S) \cap \mathring{V}_M(S)),$$

*where $M = M(S)_D$.*

**5. Main result.** Before studying problem (1) for an arbitrary number of observers, we note two special cases. For a single observer, the solution follows from the lemma at $D = Y$. Let $\mathbb{S} = \{S_1, S_2\}$ be an even pair and the sets $Y \cap \operatorname{conv}(S_i \cup \mathcal{H})$ $(i = 1, 2)$ lie to the left of the boundaries $\mathcal{R}K_i$ $(i = 1, 2)$. Suppose that the object moves from the point $t_*$ in the box $[\mathcal{L}_1, \mathcal{L}_2]$ and first intersects the cone $K_1 = K(S_1)$. To find an optimal intersection, we calculate $M(S_1) = M(S_1)_{D_1}$ (see (3)) and find the set $\mathcal{P}(S_1)_{D_1}$, where $D_1$ is the part of $K_1$ lying to the left of $\mathcal{L}K_2$. Since the cones $K_i$ are convex, the set $\mathcal{P}(S_1)_{D_1}$ lies on the arc $\mathcal{L}K_2 \cap \partial Y$. Intersecting this set, the object passes over this arc into the box $[\mathcal{R}K_1, \mathcal{L}K_2]$. A trajectory that passes from $[\mathcal{L}_1, \mathcal{L}_2]$ to $[\mathcal{R}K_1, \mathcal{L}K_2]$ and does not intersect $\mathcal{P}(S_1)_{D_1}$ is not optimal. This is easy to see by projecting it onto $\mathcal{L}K_2$ from the point $S_1$. For the transition from $[\mathcal{R}K_1, \mathcal{L}K_2]$ to $[\mathcal{R}K_1, \mathcal{R}K_2]$ to be optimal, we need to find $M(S_2) = M(S_2)_{D_2}$ and the set $\mathcal{P}_2(S_2)_{D_2}$ lying on $K_2 \cap \partial Y$, where $D_2 = K_2$. Thus, assuming that the object first intersects $K_i$ for $i = 1$, we find the quantities $M(S_1)^1 \overset{\text{def}}{=} M_1(S_1)$ and $M(S_2)^1 \overset{\text{def}}{=} M(S_2)$. If the object first intersects $K_2$, then we find $M(S_2)^2$ and $M(S_1)^2$ in a similar manner. Then

$$\mathbb{M}(\mathbb{S}) = \max\{\min\{M(S_1)^1, M(S_2)^1\},$$
$$\min\{M(S_2)^2, M(S_1)^2\}\}.$$

Let $\mathbb{S}$ be a set with an arbitrary number of observers constructed taking into account the observer's interests. This set defines a digraph $\mathbb{G}$ whose vertices are boxes. Suppose that the initial box containing the point $t_*$ has the form

$$B_0 = [\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{j_0}].$$

Let us describe a possible route $T$ of the object (Fig. 1). At the first step, the object must intersect one of the cones, say, $K_1$ in the direction from $\mathcal{L}K_1$ to $\mathcal{R}K_1$ and must pass into the box $B_1$ with the surfaces $\mathcal{R}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{j_0}$ being in the first places. For this purpose, we need to solve problem (3) (with piecewise convex constraints) in the case $S = S_1$ when the set $D_1$ is the part of the cone $K_1$ lying to the left of the surfaces $\mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_{j_0}$: find the quantity $M = M(S_1)_{D_1}$ and the set $\mathcal{P}(S_1)_{D_1}$. This intersection is performed in an optimal way, i.e., at the maximum possible distance from $S_1$ along a trajectory that does not intersect the cones $K_2, K_3, \ldots, K_{j_0}$ and is not seen by the observers $S_2, S_3, \ldots, S_{j_0}$. In the box $B_1$, additional boundaries $\mathcal{L}_i$ may appear on the right. Then $B_1$ becomes

$$B_1 = [\mathcal{R}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_{j_0}, \mathcal{L}_{j_0+1}, \ldots, \mathcal{L}_{j_1}].$$

At the second step, the object must intersect one of the cones $K_2, K_3, \ldots, K_{j_1}$, say, $K_2$ in the direction from

$\mathscr{L}K_2$ to $\mathscr{R}K_2$ and must pass into the box $B_2$ with the surfaces $\mathscr{R}_1, \mathscr{R}_2, \mathscr{L}_3, \mathscr{L}_4, \ldots, \mathscr{L}_{j_1}$ being in the first places. For this purpose, we need to solve problem (3) in the case $S = S_2$ when the set $D_2$ is the part of the cone $K_2$ lying to the right of $\mathscr{R}_1$ and to the left of $\mathscr{L}_3, \mathscr{L}_4, \ldots, \mathscr{L}_{j_1}$ and to find the quantity $M = M(S_2)_{D_2}$ and the set $\mathscr{P}(S_2)_{D_2}$. The cone $K_2$ is intersected at the maximum distance from $S_2$ along a trajectory that does not crosses $K_3, K_4, \ldots, K_{j_1}$. In the box $B_2$, additional surfaces $\mathscr{L}_i$ may appear on the right, while the surface $\mathscr{R}_1$ may disappear on the left if $\mathscr{R}_1$ lies to the left of $\mathscr{R}_2$. In this case, $B_2$ becomes

$$B_2 = \left[\mathscr{R}_2, \mathscr{L}_3, \mathscr{L}_4, \ldots, \mathscr{L}_{j_1}, \mathscr{L}_{j_1+1}, \ldots, \mathscr{L}_{j_2}\right].$$

After a finite number of steps, the object completes the chosen route $T$ and passes into the terminal box $B_N = \left[\mathscr{R}_{i_1}, \mathscr{R}_{i_2}, \ldots, \mathscr{R}_{i_N}\right]$, which contains the point $t^*$. In implementing the route $T$, we calculate all the quantities $M = M(S_i)_{D_i}$ (edge lengths) and construct the sets $\mathscr{P}(S_i)_{D_i}$. Find the minimum of these quantities

$$M_T = \min_i M(S_i)_{D_i}. \tag{4}$$

In constructing the route, each cone $K(S)$ is intersected by a trajectory at most once in the direction from $\mathscr{L}K$ to $\mathscr{R}K$. Each edge corresponds to a pencil of partial trajectories of the optimal intersection of $K(S)$, and these trajectories intersect the set $\mathscr{P}(S)_D$.

Let $\mathbf{T}$ denote the set of all possible routes $T$ joining the initial and terminal boxes. Define

$$\mathbb{M}(\mathbb{G}) = \max\left\{M_T \colon T \in \mathbf{T}\right\}. \tag{5}$$

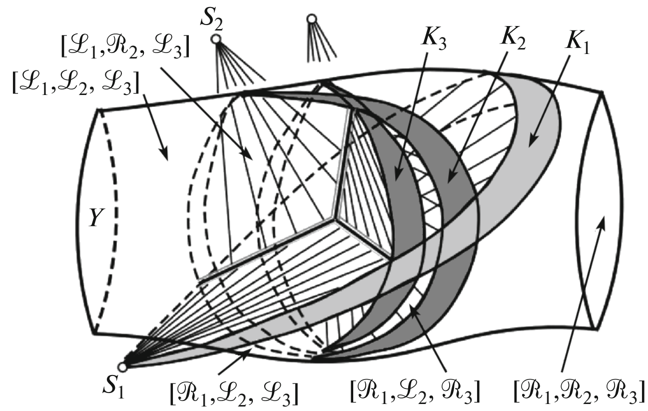


**Fig. 1.** Case $\mathbb{S} = \{S_i\}_{i=1}^3$, $\bigcap_{i=1}^{3} K(S_i) \neq \phi$.

**Theorem.** *It is true that*

$$\mathbb{M}(\mathbb{S}) = \mathbb{M}(\mathbb{G}).$$

*A trajectory* $\mathscr{T}$ *is optimal if and only if*

$$\mathscr{T} \subset Y \Big\backslash \left( \bigcup_{S \in \mathbb{S}} (K(S) \cap \mathring{V}_{\mathbb{M}}(S)) \right).$$

*The optimal trajectory* $\mathscr{T}$ *intersects the sets* $\mathscr{P}(S_i)_{D_i}$ *with indices* $i$ *that give the minimum in* (4) *for routes* $T$ *providing the maximum in* (5).

## REFERENCES


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