ISSN 1064-5624, Doklady Mathematics, 2018, Vol. 98, No. 2, pp. 515–517. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © A.M. Raigorodskii, T.V. Trukhan, 2018, published in Doklady Akademii Nauk, 2018, Vol. 482, No. 6.

MATHEMATICS

On the Chromatic Numbers of Some Distance Graphs

A. M. Raigorodskii[a,b,c,d,*] and T. V. Trukhan[b]

Presented by Academician of the RAS V.V. Kozlov June 13, 2018

Received June 15, 2018

Abstract—New estimates are obtained for the chromatic numbers of graphs from various classes of distance graphs with vertices in $\{-1, 0, 1\}^n$.

DOI: 10.1134/S1064562418060297

In the context of the classical Hadwiger–Nelson problem concerning the chromatic number of a space (see [1]), Raigorodskii began to study the chromatic numbers of distance graphs with vertices in $\{-1, 0, 1\}^n$ (see [2]). Specifically, the following problem was posed in [3, 4].

Let n be a positive integer and l_{-1}, l_0, l_1 be positive integers summing to n. Additionally, let $b \in \mathbb{N}$. Define

$$V_n(l_{-1}, l_0, l_1) = \{\mathbf{x} = (x_1, \ldots, x_n): x_i \in \{-1, 0, 1\},$$

$$|\{i: x_i = -1\}| = l_{-1},$$
$$|\{i: x_i = 0\}| = l_0, \quad |\{i: x_i = 1\}| = l_1\},$$

$$E_n(l_{-1}, l_0, l_1, b) = \{\{\mathbf{x}, \mathbf{y}\}: |\mathbf{x} - \mathbf{y}| = b\},$$

$$G_n(l_{-1}, l_0, l_1, b) = (V_n(l_{-1}, l_0, l_1), E_n(l_{-1}, l_0, l_1, b)).$$

The task is to find or estimate the quantity

$$\chi_1 = \chi(\{-1, 0, 1\}^n; l_{-1}, l_0, l_1) = \max_b \chi(G_n(l_{-1}, l_0, l_1, b)),$$

where $\chi(G)$ is the usual chromatic number of the graph G.

Two main theorems were proved in [3, 4]. Before formulating them, we introduce some notation. First, for $i_1, i_2 \in \{-1, 0, 1\}$, let $l(i_1, i_2) = l_{i_1} + l_{i_2}$,

$$V(l_{i_1}, l_{i_2}) = \{\mathbf{x} = (x_1, \ldots, x_{l(i_1, i_2)}): x_i \in \{i_1, i_2\},$$
$$|\{i: x_i = i_1\}| = l_{i_1}, |\{i: x_i = i_2\}| = l_{i_2}\}$$

$$\overline{s}_{i_1, i_2} = \max_{\mathbf{x}, \mathbf{y} \in V(l_{i_1}, l_{i_2})} (\mathbf{x}, \mathbf{y}), \quad \underline{s}_{i_1, i_2} = \min_{\mathbf{x}, \mathbf{y} \in V(l_{i_1}, l_{i_2})} (\mathbf{x}, \mathbf{y}),$$

$$\overline{s}_{-1,0,1} = \max_{\mathbf{x}, \mathbf{y} \in V_n(l_{-1}, l_0, l_1)} (\mathbf{x}, \mathbf{y}), \quad \underline{s}_{-1,0,1} = \min_{\mathbf{x}, \mathbf{y} \in V_n(l_{-1}, l_0, l_1)} (\mathbf{x}, \mathbf{y}).$$

Second, let $p_{0,1}$, $p_{-1,0}$, $p_{-1,1}$, and $p^m_{-1,0,1}$ be minimum odd primes ($m \geq 2$ is a positive integer that is not related to the dimension) such that

$$\overline{s}_{0,1} - 2p_{0,1} < \underline{s}_{0,1}, \quad \overline{s}_{-1,0} - 2p_{-1,0} < \underline{s}_{-1,0},$$

$$\overline{s}_{-1,1} - 8p_{-1,1} < \underline{s}_{-1,1}, \quad \overline{s}_{-1,0,1} - mp^m_{-1,0,1} < \underline{s}_{-1,0,1}.$$

Finally, let

$$P(l_{-1}, l_0, l_1) = |V_n(l_{-1}, l_0, l_1)| = \frac{n!}{l_{-1}! l_0! l_1!}.$$

In this notation, the following results are true.

Theorem A. *Let* $i_1 < i_2; i_1, i_2 \in \{-1, 0, 1\}$; *and* $i_3 \in \{-1, 0, 1\}$ *be the number remaining in the set* $\{-1, 0, 1\}$ *after* i_1, i_2 *have been removed from it. Define*

$$D_{i_1, i_2} = C_n^{l_{i_3}} \sum_{k_1 = 0}^{p_{i_1, i_2} - 1} C^{k_1}_{l_{i_1} + l_{i_2}}.$$

Then

$$\chi_1 \geq \max_{i_1, i_2} \frac{P(l_{-1}, l_0, l_1)}{D_{i_1, i_2}}.$$

Theorem B. *Let* $m \geq 2$ *be a positive integer independent of the dimension. Define*

$$D^m_{-1,0,1} = \sum_{(i,j) \in \mathcal{A}} C_n^i C_{n-i}^j,$$

where

$$\mathcal{A} = \{(i, j): i + j \leq n, i + 2j \leq p^m_{-1,0,1} - 1, i, j \in \mathbb{N} \cup \{0\}\}.$$

[a] *Moscow Institute of Physics and Technology (State University), Dolgoprudnyi, Moscow oblast, 141700 Russia*
[b] *Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119992 Russia*
[c] *Institute of Mathematics and Computer Science, Buryat State University, Ulan-Ude, 670000 Buryat Republic, Russia*
[d] *Caucasus Mathematical Center, Adyghe State University, Maikop, 385000 Republic of Adygea, Russia*
**e-mail: mraigor@yandex.ru*

Then

$$\chi_1 \geq \max_m \left(\frac{P(l_{-1}, l_0, l_1)}{D^m_{-1,0,1}} \right)^{\frac{1}{m-1}}.$$

FORMULATION OF THE RESULT

We managed to strengthen both results due to considerably wider choice of parameters $p_{0,1}$, $p_{-1,0}$, $p_{-1,1}$, and $p^m_{-1,0,1}$. It became fundamentally possible due to recent works [5–7]. The following new results are valid.

Theorem 1. *Let* $i_1 < i_2; i_1, i_2 \in \{-1, 0, 1\}$; *and* $i_3 \in \{-1, 0, 1\}$ *be the number remaining in the set* $\{-1, 0, 1\}$ *after* i_1, i_2 *have been removed from it. If, for the chosen* i_1, i_2, *we have one of the inequalities*

$$\overline{s}_{0,1} - 2p_{0,1} < \underline{s}_{0,1},$$

$$\overline{s}_{-1,0} - 2p_{-1,0} < \underline{s}_{-1,0}, \quad \overline{s}_{-1,1} - 8p_{-1,1} < \underline{s}_{-1,1},$$

where p_{i_1,i_2} *is an odd prime, then we define*

$$D_{i_1,i_2} = C_n^{l_{i_3}} \sum_{i=0}^{p_{i_1,i_2}-1} C^i_{l_{i_1}+l_{i_2}}.$$

If $i_3 \neq 0$ *and, on the contrary, for the chosen* i_1, i_2, *we have one of the inequalities*

$$\overline{s}_{0,1} - 2p_{0,1} \geq \underline{s}_{0,1}, \quad \overline{s}_{-1,0} - 2p_{-1,0} \geq \underline{s}_{-1,0},$$

where p_{i_1,i_2} *is an odd prime, then* t *is defined as* $t = \overline{s}_{i_1,i_2} - p_{i_1,i_2}$, *we set* $d = 2t - \overline{s}_{i_1,i_2} + 1$ *and* $k = \overline{s}_{i_1,i_2}$, *consider* $d_1, d_2 \in \mathbb{N} \cup \{0\}: d_1 + d_2 = d$, *set* $n_1 = n - l_{i_3} - d_1$ *and* $k_1 = \overline{s}_{i_1,i_2} - d_1$, *define* $r \in \mathbb{N}$ *by the relation*

$$(k_1 - d_2 + 1)\left(2 + \frac{d_2 - 1}{r + 1}\right)$$

$$\leq n_1 < (k_1 - d_2 + 1)\left(2 + \frac{d_2 - 1}{r}\right),$$

and set

$$D_{i_1,i_2} = C_n^{l_{i_3}} \frac{C_{n_1}^{d_2+2r} C^{d_1}_{n-l_{i_3}}}{C_{k_1}^{d_2+r} C^r_{n_1-k_1} C^{d_1}_k} \sum_{i=0}^{p_{i_1,i_2}-1} C^i_{n_1}.$$

Finally,

$$\chi_1 \geq \max_{i_1,i_2} \frac{P(l_{-1}, l_0, l_1)}{D_{i_1,i_2}}.$$

Theorem 2. *Let* $m \geq 2$ *be a positive integer independent of the dimension. If* $\overline{s}_{-1,0,1} - mp^m_{-1,0,1} < \underline{s}_{-1,0,1}$ *holds with some odd prime* $p^m_{-1,0,1}$, *then define*

$$D^m_{-1,0,1} = \sum_{(i,j)\in\mathcal{A}} C^i_n C^j_{n-i},$$

where

$$\mathcal{A} = \{(i, j): i + j \leq n, i + 2j \leq p^m_{-1,0,1} - 1, i, j \in \mathbb{N} \cup \{0\}\}.$$

If $\overline{s}_{-1,0,1} - mp^m_{-1,0,1} \geq \underline{s}_{-1,0,1}$, *then define* $d = \overline{s}_{-1,0,1} - mp^m_{-1,0,1} + 1$. *Let* $\mathfrak{m} = (m_1, m_2, m_3)$,

$$\mathfrak{M} = \begin{pmatrix} m_{-1,1} & m_{0,1} & m_{1,1} \\ m_{-1,2} & m_{0,2} & m_{1,2} \\ m_{-1,3} & m_{0,3} & m_{1,3} \end{pmatrix},$$

and all elements of the vector and the matrix are positive integers such that

$$m_1 + m_2 + m_3 = n,$$

$$m_{-1,1} + m_{0,1} + m_{1,1} = m_1,$$

$$m_{-1,2} + m_{0,2} + m_{1,2} = m_2, \quad m_{-1,3} + m_{0,3} + m_{1,3} = m_3,$$

$$m_{-1,1} + m_{-1,2} + m_{-1,3} = l_{-1},$$

$$m_{0,1} + m_{0,2} + m_{0,3} = l_0, \quad m_{1,1} + m_{1,2} + m_{1,3} = l_1$$

and another condition be satisfied. Suppose that the set $\{1, 2, \ldots, n\}$ *is represented as a union of three disjoint parts* M_1, M_2, M_3 *of respective cardinalities* m_1, m_2, m_3. *A vector* $\mathbf{x} = (x_1, \ldots, x_n) \in V_n(l_{-1}, l_0, l_1)$ *is said to satisfy the partition* $T(M_1, M_2, M_3)$ *if*

$$|\{i \in M_1: x_i = -1\}| = m_{-1,1},$$

$$|\{i \in M_1: x_i = 0\}| = m_{0,1}, \quad |\{i \in M_1: x_i = 1\}| = m_{1,1},$$

$$|\{i \in M_2: x_i = -1\}| = m_{-1,2},$$

$$|\{i \in M_2: x_i = 0\}| = m_{0,2}, \quad |\{i \in M_2: x_i = 1\}| = m_{1,2},$$

$$|\{i \in M_3: x_i = -1\}| = m_{-1,3},$$

$$|\{i \in M_3: x_i = 0\}| = m_{0,3}, \quad |\{i \in M_3: x_i = 1\}| = m_{1,3}.$$

Another condition on the parameters $\mathfrak{m}$ *and* $\mathfrak{M}$ *is that any two vectors* $\mathbf{x}, \mathbf{y}$ *satisfying the same partition have a scalar product equal to at least d. Define*

$$D^m_{-1,0,1} = \frac{n!}{m_1! m_2! m_3!} \cdot \frac{m_{-1,1}! m_{-1,2}! m_{-1,3}!}{l_{-1}!}$$

$$\cdot \frac{m_{0,1}! m_{0,2}! m_{0,3}!}{l_0!} \cdot \frac{m_{1,1}! m_{1,2}! m_{1,3}!}{l_1!} \cdot \sum_{(i,j)\in\mathcal{A}} C^i_n C^j_{n-i},$$

where

$$\mathcal{A} = \{(i, j): i + j \leq n, i + 2j \leq p^m_{-1,0,1} - 1\}.$$

Finally,

$$\chi_1 \geq \max_m \left(\frac{P(l_{-1}, l_0, l_1)}{D^m_{-1,0,1}} \right)^{\frac{1}{m-1}}.$$

Note that the improvements given by Theorems 1 and 2 as compared with Theorems A and B are the most significant in the cases $l_i \sim l'_l n$, $l'_l \in (0,1)$. In these cases, all estimates have the form $(c + o(1))^n$; moreover, for a large class of parameters l'_0, l'_1, l'_{-1}, the new values c are strictly greater than the previous ones, i.e., the well-known inequalities are strengthened exponentially.

Note also that similar problems for other parameter asymptotics were studied in recent works [8–15].

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 18-01-00355) and by the Russian Federation President Grant (project no. NSh-6760.2018.1).

REFERENCES

1. A. M. Raigorodskii, Fundam. Inf. **145**, 359–369 (2016).
2. A. M. Raigorodskii, Russ. Math. Surv. **55** (2), 351–352 (2000).
3. A. M. Raigorodskii, Dokl. Math. **68** (2), 216–220 (2003).
4. A. M. Raigorodskii, Sb. Math. **196** (1), 115–146 (2005).
5. E. I. Ponomarenko and A. M. Raigorodskii, Math. Notes **96** (1), 140–148 (2014).
6. E. I. Ponomarenko and A. M. Raigorodskii, Dokl. Math. **89** (1), 59–60 (2014).
7. A. M. Raigorodskii and A. A. Sagdeev, Dokl. Math. **97** (1), 47–48 (2018).
8. P. Frankl and A. Kupavskii, J. Combin. Theory Ser. A **155**, 157–179 (2018).
9. D. Cherkashin, A. Kulikov, and A. Raigorodskii, Discrete Appl. Math. **243**, 125–131 (2018).
10. D. D. Cherkashin and A. M. Raigorodskii, Dokl. Math. **95** (1), 5–6 (2017).
11. R. I. Prosanov, A. A. Sagdeev, and A. M. Raigorodskii, Dokl. Math. **96** (1), 336–338 (2017).
12. A. V. Bobu, A. E. Kupriyanov, and A. M. Raigorodskii, Dokl. Math. **96** (1), 354–357 (2017).
13. A. V. Bobu, A. E. Kupriyanov, and A. M. Raigorodskii, Sb. Math. **207** (5), 652–677 (2016).
14. D. Cherkashin, Electron. J. Combin. **25** (1), Paper P1.47 (2018).
15. A. Kupavskii and D. Zakharov, J. Combin. Theory Ser. A **155**, 180–189 (2018).

Translated by I. Ruzanova