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MATHEMATICS

Green's Function of Ordinary Differential Operators and an Integral Representation of Sums of Certain Power Series

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Abstract—The eigenvalues and eigenfunctions of certain operators generated by symmetric differential expressions with constant coefficients and self-adjoint boundary conditions in the space of Lebesgue square-integrable functions on an interval are explicitly calculated, while the resolvents of these operators are integral operators with kernels for which the theorem on an eigenfunction expansion holds. In addition, each of these kernels is the Green's function of a self-adjoint boundary value problem, and the procedure for its construction is well known. Thus, the Green's functions of these problems can be expanded in series in terms of eigenfunctions. In this study, identities obtained by this method are used to calculate the sums of convergent number series and to represent the sums of certain power series in an intergral form.

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1. Let $\alpha \in [0,1)$ and $L^2[0,2\pi]$ be the Hilbert space of Lebesgue square-integrable functions on the interval $[0,2\pi]$. Let S_α denote the extension of the minimal closed symmetric operator L_0 generated in $L^2[0,2\pi]$ by the expression

$$l_1[y] := \left(i\frac{d}{dx}\right)y$$

and the boundary condition

$$U_1(y) := y(0) - e^{2\pi i\alpha}y(2\pi) = 0.$$

It is well known that S_α is a self-adjoint operator with a discrete simple spectrum. The numbers $\lambda_{1k} := k + \alpha$ are the eigenvalues, while the functions

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}}e^{-i(k+\alpha)x}, \quad k = 0, \pm1, \pm2, \ldots$$

are the corresponding orthonormal eigenfunctions of the operator S_α. In addition, each self-adjoint extension of the operator L_0 is specified by a boundary condition of the form $U_1(y) = 0$, where $\alpha \in [0,1)$ is a fixed number (see, for example, [1, Chapter IV, Section 55]).

Let $P_n(x)$ be a polynomial of degree $n \geq 2$ with real coefficients. We consider the operator $P_n(S_\alpha)$. Evidently, the domain D_α of this operator is specified by the equality

$$D_\alpha = \{y | y^{(j-1)} \in AC[0,2\pi]; U_j(y) = 0, j = 1,2,\ldots,n\},$$

where

$$U_{j+1}(y) := y^{(j)}(0) - e^{2\pi i\alpha}y^{(j)}(2\pi), \quad j = 0,1,\ldots,n-1,$$

and, if $y \in D_\alpha$, then

$$P_n(S_\alpha)y = P_n\left(i\frac{d}{dx}\right)y =: l_n[y].$$

Thus, $P_n(S_\alpha)$ is a self-adjoint operator generated by a differential expression with constant coefficients $l_n[y]$ and the boundary conditions $U_j(y) = 0$ $(j = 1,2,\ldots,n)$. In addition, $P_n(S_\alpha)$ has a discrete spectrum of the form

$$\sigma = \{\lambda | \lambda = \lambda_{nk} := P_n(k+\alpha), k = 0; \pm1, \pm2; \ldots\},$$

and the eigenfunction φ_k corresponds to the eigenvalue λ_{nk}.

We now assume that the number $\lambda = 0$ is a regular point of the operator $P_n(S_\alpha)$ (that is, $0 \notin \sigma$) and consider its resolvent R. It is well known that R is an integral operator with the kernel $G_\alpha(x,t)$ being the Green's function of the problem

$$\begin{gathered} l_n[y] = f, \\ U_j(y) = 0, \quad j = 1,2,\ldots,n. \end{gathered} \tag{1}$$

A procedure for constructing this function is also well known (see, for example, [2, Part 2, Chapter I, Section 1.5]). To be precise, let $y_1(x), y_2(x), \ldots, y_n(x)$ be a

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fundanmental system of solutions to the equation $l_n[y] = 0$ and

$$g(x,t) = \frac{\operatorname{sgn}(x-t)}{2a_0 i^n W(t)} \begin{vmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \dots & \dots & \dots \\ y_1^{(n-2)}(t) & \dots & y_n^{(n-2)}(t) \\ y_1(x) & \dots & y_n(x) \end{vmatrix},$$

where a_0 is the leading coefficient of the polynomial $P_n(x)$ and $W(x)$ is the Wronskian of the functions $y_1(x), y_2(x), \dots, y_n(x)$. Then we have

$$G_\alpha(x,t) = \frac{1}{\det D} \begin{vmatrix} g(x,t) & y_1(x) \ \dots \ y_n(x) \\ U_1(g)(t) & \\ . & \\ . & D \\ . & \\ U_n(g)(t) & \end{vmatrix}, \quad (2)$$

where $D := (U_s(y_l))_{s,l=1}^n$ and values of the forms $U_j(g)(t)$ $(j = 1,2,\dots,n)$ can be obtained by the formulas

$$U_{j+1}(g)(t) = \left.\frac{\partial^{(j)} g(x,t)}{\partial x^j}\right|_{x=0} - e^{2\pi i\alpha} \left.\frac{\partial^{(j)} g(x,t)}{\partial x^j}\right|_{x=2\pi}, \quad j = 0,1,\dots,n-1.$$

Theorem 1. *Let $\alpha \in [0,1)$, and let $P_n(x)$ be a polynomial of degree n $(n \ge 2)$ with real coefficients such that $P_n(k+\alpha) \ne 0$ for $k = 0,\pm1,\pm2,\dots$*

Then the function $G_\alpha(x,t)$ defined by formula (2) is the Green's function of problem (1) and it holds that

$$G_\alpha(x,t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \frac{e^{-i(k+\alpha)(x-t)}}{P_n(k+\alpha)}.$$

2. Let $p_m(x)$ denote a polynomial of degree $m \ge 1$ with real coefficients. The following corollaries to Theorem 1 hold true.

Corollary 1. *Let a polynomial $p_m(x)$ be such that $p_m(k^2) \ne 0$ for $k = 0,1,\dots,$ and let $P_n(x) = p_m(x^2)$. Let $\alpha = 0$ in problem (1).*

Then the Green's function $G_0(x,t)$ of this problem can be represented in the form

$$G_0(x,t) = \frac{1}{2\pi p_m(0)} + \frac{1}{\pi} \sum_{k=1}^{+\infty} \frac{\cos k(x-t)}{p_m(k^2)}.$$

Corollary 2. *Let polynomials $p_m(x)$ and $P_n(x)$ and a number α be the same as those in Corollary 1, and let $f(x) \in L^2[0,2\pi]$. Then*

$$\int_0^{2\pi} G_0(x,t) f(t) dt = \frac{a_0}{2p_m(0)} + \sum_{k=1}^{+\infty} \frac{a_k \cos kx + b_k \sin kx}{p_m(k^2)},$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

and

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx, \quad n = 1,2,\dots,$$

are the Fourier coefficients of the function $f(x)$.

Corollary 3. *Let a polynomial $p_m(x)$ be such that $p_m\left(\left(k+\frac{1}{2}\right)^2\right) \ne 0$ for $k = 0,1,\dots,$ and let $P_n(x) = p_m(x^2)$. Let $\alpha = \frac{1}{2}$ in problem (1). Then the Green's function $G_{1/2}(x,t)$ of this problem can be represented in the form*

$$G_{1/2}(x,t) = \frac{1}{\pi} \sum_{k=0}^{+\infty} \frac{\cos\left(k+\frac{1}{2}\right)(x-t)}{p_m\left(\left(k+\frac{1}{2}\right)^2\right)}.$$

3. Evidently, Corollaries 1–3 deal with functions of the operator generated in the space $L^2[0,2\pi]$ by the expression $l_2[y] = -y''$ and by periodic boundary conditions ($y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$) or antiperiodic ones ($y(0) = -y(2\pi)$ and $y'(0) = -y'(2\pi)$). In addition, for the operators generated by the expression $l_2[y]$ and other appropriate boundary conditions, we can easily find explicit formulas for eigenvalues and the corresponding eigenfunctions (see, for example, [2, Part 3, Chapter II, problem 2.9]). A detailed analysis of self-adjoint operators obtained in such a way and the application of the above method to them will be addressed elsewhere. Here, we confine ourselves to the following particular case.

Let S denote the self-adjoint operator generated by the expression $l_2[y]$ and Dirichlet boundary conditions (that is, by the conditions $y(0) = y(\pi) = 0$) in $L^2[0,\pi]$. As before (see Section 2), let $p_m(x)$ be a polynomial of degree m with real coefficients. We consider the operator $p_m(S)$. The domain D of this operator is as follows:

$$D = \{y | y^{(j-1)} \in AC[0,\pi]; U_j(y) = 0, j = 1,2,\dots,2m\},$$

where the linear forms $U_j(y)$ are defined by the equalities

$$U_{j+1}(y) := y^{(2j)}(0), \quad U_{j+m+1}(y) := y^{(2j)}(\pi), \quad j = 0,1,\dots,m-1, \tag{3}$$

and, if $y \in D$, then

$$p_m(S)y = p_m\left(-\frac{d^2}{dx^2}\right)y =: l_{2m}[y]. \tag{4}$$

Obviously, the numbers k^2 and $p_m(k^2)$ are the eigenvalues of the operators S and $p_m(S)$, respectively, while the functions $\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin kx$ $(k = 1,2,\dots)$ are the corresponding orthonormal eigenfunctions.

Theorem 2. *Let a polynomial* $p_m(x)$ *be such that* $p_m(k^2) \neq 0$ *for* $k = 1,2,\dots$. *Then the function* $G(x,t)$ *defined by formula* (2) *is the Green's function of problem* (1), *where the expression* $l_n[y]$ $(=l_{2m}[y])$ *is given by formula* (4) *and the forms* $U_j(y)$ *are defined by formula* (3); *moreover,*

$$G(x,t) = \frac{2}{\pi}\sum_{k=1}^{+\infty}\frac{\sin kx \cdot \sin kt}{p_m(k^2)}. \tag{5}$$

4. Evidently, the results obtained in Sections 1–3 can be used to calculate the sums of certain convergent number series. We give several examples.

Example 1. In the notation of Theorem 1, the function $G_\alpha(x,x)$ obviously does not depend on x, that is, $G_\alpha(x,x) = G_\alpha(0,0)$ for any $x \in [0,2\pi]$, and it holds that

$$\sum_{k=-\infty}^{+\infty}\frac{1}{P_n(k+\alpha)} = 2\pi G_\alpha(0,0).$$

Example 2. In its turn, it follows from Corollary 1 that

$$\frac{1}{2p_m(0)} + \sum_{k=1}^{+\infty}\frac{1}{p_m(k^2)} = \pi G_0(0,0) \quad \text{and} \quad \frac{1}{2p_m(0)} + \sum_{k=1}^{+\infty}\frac{(-1)^k}{p_m(k^2)} = \pi G_0(0,\pi). \tag{6}$$

Example 3. Parseval's identity and Corollary 2 imply that

$$\frac{a_0^2}{2p_m^2(0)} + \sum_{k=1}^{+\infty}\frac{a_k^2 + b_k^2}{p_m^2(k^2)} = \frac{1}{\pi}\int_0^{2\pi}\left(\int_0^{2\pi}G_0(x,t)f(t)dt\right)^2 dx.$$

Example 4. It follows from Corollary 3 that

$$\sum_{k=0}^{+\infty}\frac{1}{p_m\left(\left(k+\frac{1}{2}\right)^2\right)} = \pi G_{1/2}(0,0).$$

Example 5. It follows from Theorem 2 that

$$\sum_{k=1}^{+\infty}\frac{1}{p_m(k^2)} = \int_0^{\pi}G(x,x)dx.$$

The list of these formulas can obviously be continued by choosing, for example, particular values of the variables x and t in the formulas of Corollaries 1 and 3 or Theorem 2.

Remark. There are well-known methods that make it possible to calculate the sums of convergent series with a general term of the form $\frac{q(k)}{p(k)}$, where $p(x)$ and $q(x)$ are polynomials, generally, in terms of special functions, in particular, of digamma functions (see [3, Chapter 1, Section 1.2; 4, Chapter 6, Section 6.8]). In some cases, these sums can be expressed in terms of values of elementary functions (see, for example, [5, Chapter 5, Section 5.1]). An advantage of the formulas given in Examples 1–5 lies in the fact that the sums in them are expressed directly in terms of values of an easy-to-construct quasi-polynomial.

Example 6. Let $a \in (0,1)$ and $P_2(x) = x^2 - a^2$. Let $\alpha = 0$ in problem (1). It follows from (2) that the Green's function $G_0(x,t)$ of the resulting problem is defined by the equality (see also [6, Chapter 3, problem 28])

$$G_0(x,t) = -\frac{\cos a(\pi - |x-t|)}{2a\sin(a\pi)},$$

while equalities (6) coincide with Euler's formulas for decomposition of the functions $\pi\cot(a\pi)$ and $\frac{\pi}{\sin(a\pi)}$ into simple fractions, that is,

$$\pi\cot(a\pi) = \frac{1}{a} + \sum_{k=1}^{+\infty}\left(\frac{1}{k+a} - \frac{1}{k-a}\right)$$

and

$$\frac{\pi}{\sin(a\pi)} = \frac{1}{a} + \sum_{k=1}^{+\infty}(-1)^k\left(\frac{1}{k+a} - \frac{1}{k-a}\right).$$

Thus, formulas (6) are, in a sense, generalizations of these decompositions.

5. Using Fourier series expansions of the functions

$$\frac{z\cos x - z^2}{1 - 2z\cos x + z^2}, \quad \ln(1 - 2z\cos x + z^2), \quad \ln 2\left|\sin\frac{x}{2}\right|, \quad \ln 2\left|\cos\frac{x}{2}\right|$$

where $z \in (-1,1)$ is a parameter (see [5, Chapter 5, Section 5.4.9, formulas (3) and (13), and Section 5.4.2, formulas (9) and (10)]), and taking into account equality (5), we can prove the following assertion.

Theorem 3. *Let* $z \in (-1,1]$, *and let* $p_m(x)$ *and* $G(x,t)$ *be the same functions as in Theorem* 2. *Then*

$$\sum_{k=1}^{+\infty}\frac{z^k}{p_m(k^2)} = 2\int_0^{\pi}G(x,x)\frac{z^2 - z\cos 2x}{1 - 2z\cos 2x + z^2}dx$$

and

$$\sum_{k=1}^{+\infty}\frac{z^k}{kp_m(k^2)} = \int_0^{\pi} G(x,x)\ln(1-2z\cos 2x + z^2)dx.$$

Example 7. Let $p_m(x) = x^m$. Comparing formula (5) at $t = x$ and the Fourier series expansion of the Bernoulli polynomial $B_n(x)$ (see, for example, [4, Section 23, formula (23.1.18); 5, Chapter 5, Section 5.4.2, formula (7)]) makes it possible to conclude that

$$G(x,x) = (-1)^m \frac{(2\pi)^{2m-1}}{(2m)!}\left(B_{2m}\left(\frac{x}{\pi}\right) - B_{2m}\right),$$

where B_{2m} are the Bernoulli numbers. Using this relation and the definition of the polylogarithmic function $\mathrm{Li}_j(z)$ of order j (see, for example, [7, Chapter 7, Section 7.1, formula (7.1)]), that is,

$$\mathrm{Li}_j(z) = \sum_{k=1}^{+\infty}\frac{z^k}{k^j},$$

we can prove that, for $z \in (-1,1]$, the identities in Theorem 3 can be written as

$$\mathrm{Li}_{2m}(z) = (-1)^{m-1}\frac{(2\pi)^{2m}}{(2m)!} \times \int_0^1 (B_{2m}(x) - B_{2m})\frac{z(\cos 2\pi x - z)}{1 - 2z\cos 2\pi x + z^2}dx \tag{7}$$

and

$$\mathrm{Li}_{2m+1}(z) = (-1)^{m+1}\frac{(2\pi)^{2m+1}}{(2m+1)!} \times \int_0^1 B_{2m+1}(x)\frac{z\sin 2\pi x}{1 - 2z\cos 2\pi x + z^2}dx. \tag{8}$$

Thus, the formulas in Theorem 3 are generalizations of (7) and (8). Note also that, for $z = 1$, formulas (7) and (8) are well known, since $\mathrm{Li}_{2m}(1) = \zeta(2m)$ and $\mathrm{Li}_{2m+1}(1) = \zeta(2m+1)$, where $\zeta(z)$ is the Riemann zeta function (see [4, Section 23; 5, Chapter 5, Section 5.1.2; 7, Chapter 7, Section 7.2.1]), while for $z \in (-1,1)$ these formulas were obtained in [8].

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