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MATHEMATICAL PHYSICS

Dynamics of a Delay Logistic Equation with Diffusion and Coefficients Rapidly Oscillating in Space Variable

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Abstract—The applicability of the averaging principle in the study of the dynamics of a practically important delay logistic equation with diffusion and coefficients rapidly oscillating with respect to the space variable is analyzed. A task of special interest is to address equations with rapid oscillations of the delay coefficient or a quantity characterizing the deviation of the space variable. Bifurcation problems arising in critical cases for the averaged equation are studied. Results concerning the existence, stability, and asymptotic behavior of periodic solutions to the original equation are formulated.

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1. Consider the scalar delay logistic equation with diffusion

$$\frac{\partial u}{\partial t} = d(\omega x)\frac{\partial^2 u}{\partial x^2} + r(\omega x)[1 - a(\omega x)u(t - T(\omega x), x)]u, \quad (1)$$

supplemented with the boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0. \quad (2)$$

As a phase space, it is convenient to use the Sobolev space $\overset{\circ}{W}{}_2^2([0,1]\times C_{[-T^0,0]}; T^0 = \max_s T(s))$ consisting of functions of the class W_2^2 satisfying the boundary conditions (2). All the coefficients in (1) are positive and have clear biological interpretations [1, 2]. Specifically, the coefficient d characterizes the mobility of the population, r is called the Malthusian coefficient, T characterizes the time delay, and the coefficient a describes the inhomogeneity of the habitat. Problem (1), (2) has been extensively studied (see, e.g., [1, 3]).

Assume that all coefficients in (1) are 2π-periodic functions of the space variable ωx. The basic assumption is that the parameter ω is sufficiently large, i.e., $\omega \gg 1$.

Boundary value problems of this type frequently arise in applications (see, e.g., [4–6]). In particular, numerous distributed problems describing fine-grained structures [6] are reduced to such problems.

The averaging principle in various problem formulations has been formulated and substantiated by numerous authors. Primarily, we note N.M. Krylov's and Bogolyubov's classical works [7, 8], where averaging over time was considered. First, we formulate a simple but fundamental result whose proof follows from the above-mentioned works. For this purpose, define

$$M_s(f(s)) = \frac{1}{2\pi}\int_0^{2\pi} f(s)ds.$$

Consider the averaged equation (1), namely,

$$\begin{aligned}\frac{\partial v}{\partial t} &= d_0\frac{\partial^2 v}{\partial x^2} + d_0(M_s(r(s)d(s)) \\ &- M_s(r(s)d^{-1}(s)a(s)v(t - T(s), x)))v,\end{aligned} \quad (3)$$

where $d_0 = (M_s(d^{-1}(s)))^{-1}$.

Given an arbitrary constant $L > 0$ and an arbitrary initial function $\varphi(x, s) \in \overset{\circ}{W}{}_2^2[0,1]\times C_{[-T^0,0]}$, let $u(t, x, \omega, \varphi)$ and $v(t, x, \varphi)$ denote the solutions (if any) of boundary value problems (1), (2) and (3), (2) for $t \in [0, L]$ and $x \in [0,1]$ with the same initial function $\varphi(x, s)$ at $t = 0$.

Theorem 1. *Suppose that the boundary value problem* (2), (3) h*as a solution* $v(t, x, \varphi)$ *for* $t \in [0, L]$. *Then*, *for* $t \in [0, L]$ *and sufficiently large* ω, *the boundary value problem* (1), (2) *also has a solution* $u(t, x, \omega, \varphi)$ *and*, *as* $\omega \to \infty$,

$$u(t, x, \omega, \varphi) = v(t, x, \varphi) + o(1). \quad (4)$$

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Let us examine the coincidence and differences in the dynamic properties of solutions to the original problem (1), (2) and averaged problem (3), (2). The following result deals with t-periodic solutions of these problems.

Theorem 2. *Assume that $v_0(t,x)$ is a periodic solution of the boundary value problem* (3), (2) *and the problem linearized about $v_0(t,x)$ has only one multiplier equal to* 1 *in absolute value. Then, for all sufficiently large* ω, *the boundary value problem* (1), (2) *has a t-periodic solution $u_0(t,x,\omega)$ such that, as $\omega\to\infty$,*

$$u_0(t,x,\omega) = v_0((1+o(1))t,x) + o(1).$$

For sufficiently large ω, *the periodic solutions $u_0(t,x,\omega)$ and $v_0(t,x)$ have identical stability properties.*

Below, we consider the most important cases. First, let

$$T(s)\equiv 1. \tag{5}$$

Note that, for constant coefficients, the stability condition for a positive equilibrium is $r\le\frac{\pi}{2}$, while, for $r>\frac{\pi}{2}$, this equilibrium is unstable.

Proposition 1. *Assume that condition* (5) *holds and only one of the coefficients in* (3) *is not identically constant. Then, for sufficiently large* ω, *the stability properties of the positive equilibrium for $r<\frac{\pi}{2}$ and $r>\frac{\pi}{2}$ are identical in problems* (1), (2) *and* (3) (2).

Proposition 2. *If constraint* (5) *holds and the coefficients $d(\omega x)$ and $a(\omega x)$ are not constant, then the stability condition for a positive equilibrium is the same as in the case of the equation with constant coefficients, but the equilibrium itself is different.*

Proposition 3. *If constraint* (5) *holds and the coefficients $d(\omega x)$ and $r(\omega x)$ are not constant, then the stability domain in the parameter space can be larger or smaller depending on the choice of these two coefficients. For example, let $d(s)=d_0(1+\alpha\cos s)$ and $r(s)=r_0(1+\beta\cos s)$. Then the stability domain with respect to the parameter r is larger for $\alpha=\beta$ and smaller for $\alpha=-\beta$.*

The most prominent role is played by all the coefficients in the critical case, i.e., for

$$r^0 = d_0M(r(s)d^{-1}(s)) = \frac{\pi}{2}.$$

Suppose that, in (1),

$$d(s)=d,\quad r(s)=\frac{\pi}{2}+\beta\cos s,\quad \left(r_0=\frac{\pi}{2}\right)\quad a(s)=a.$$

Then, in (2), (3), we have the critical case in the stability problem for the equilibrium $v_0\equiv a^{-1}$, i.e., an Andronov–Hopf bifurcation occurs.

It is well known (see, e.g., [11, 12]) that, in a sufficiently small neighborhood of the positive equilibrium of (2), (3), there exists a stable locally invariant integral two-dimensional manifold on which this boundary value problem becomes, up to $O|\xi|^5$,

$$\frac{d\xi}{dt} = \delta|\xi|^2\xi, \tag{6}$$

where $\delta = -\frac{\pi}{2}\left(3\pi-2+i(\pi+6)\left(10\left(1+\frac{4}{\pi^2}\right)\right)\right)^{-1}$ and the function $\xi(t)$ is related to solutions of (2), (3) by the equality

$$v(t,x) = a^{-1} + \xi(t)\exp\left(i\frac{\pi}{2}t\right) + \bar{\xi}(t)\exp\left(-i\frac{\pi}{2}t\right) + O(|\xi|^2).$$

Then, in a small neighborhood of the positive equilibrium, the original boundary value problem (1), (2) also has a stable invariant two-dimensional integral manifold on which this problem has the form

$$\begin{aligned}\frac{d\xi}{dt} &= (\omega^{-1}\alpha(\omega)+O(\omega^{-2}))\xi\\ &+ (\delta+O(\omega^{-1}))\xi|\xi|^2 + O(|\xi|^5).\end{aligned} \tag{7}$$

Here, δ is the same as in (6), while $\alpha(\omega)$ is given by the formula

$$\alpha(\omega) = -\left(1+i\frac{\pi}{2}\right)^{-1} i\frac{\pi}{2}\beta\sin\omega.$$

Note that $\operatorname{Re}\delta<0$, so, in the case under consideration, a stable cycle in problem (1), (2) is created and destroyed in an infinite process as $\omega\to\infty$.

Now, assume that constraint (5) is not satisfied. Let

$$T(s) = 1+\gamma\cos s,\quad |\gamma|<1, \tag{8}$$

and the other coefficients in (1) be constants. Then the averaged equation (3) can be represented in the form

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + r\left[1 - a\int_{-1}^{0}\frac{v(t-1+\gamma s,x)-v(t-1-\gamma s,x)}{\sqrt{1-s^2}}ds\right]v.$$

Accordingly, a criterion for the stability of the equilibrium $v_0\equiv a^{-1}$ is that all the roots of the characteristic equation

$$\lambda = -d\pi^2k^2 - 2\pi^{-1}r\exp(-\lambda)S(\gamma)\quad (k=0,1,2,\ldots),$$

where $S(\gamma)=\int_{-1}^{0}\frac{\cos(\pi\gamma/2)}{\sqrt{1-s^2}}ds$, have a negative real part. This condition holds for

$$4r < \pi^2 S^{-1}(\gamma).$$

Let us analyze how rapid oscillations of the medium (the coefficient $a(s)$) influence the stability of the equilibrium. Assume that $r(s) \equiv \text{const} = r_0 = \dfrac{\pi}{2}$.

We find the asymptotics of the positive equilibrium $u_0(x,\omega)$ in (1), (2) up to $O(\omega^{-3})$. Let

$$u_0(x,\omega) = a_0^{-1} + \omega^{-1}u_1(x,\omega) + \omega^{-2}(u_2(x,\xi,\omega) + u_3(x,\omega)) + \dots .$$

Then

$$u_1(x,\omega) = R(\omega) \equiv a_0^{-1}\int_0^{\omega}(1 - a(\xi)a_0^{-1}d\xi,$$

$$\frac{\partial^2 u_2}{\partial\xi^2} + r_0 a_0^{-1}(1 - a(\xi)a_0^{-1}) - r_0\omega^{-1}R(\omega) = 0,$$

$$\left.\frac{\partial u_2}{\partial\xi}\right|_{\xi=0} = \left.\frac{\partial u_2}{\partial\xi}\right|_{\xi=\omega} = 0,$$

$$u_3(x,\omega) = -3a_0R^2(\omega) + 2a_0^{-1}\lim_{T\to\infty}\Gamma(T),$$

where

$$\Gamma(\omega) = -\frac{1}{\omega}\int_0^{\omega} a(\xi)u_2(\xi,\omega)d\xi = \frac{a_0^2 r_0^{-1}}{\omega}\int_0^{\omega}\left(\frac{\partial u_2}{\partial\xi}\right)^2 d\xi > 0.$$

To analyze the stability of $u_0(x,\omega)$, we make the substitution $u = v + u_0(x,\omega)$ in (1), (2) and linearize the problem (with respect to v). The function $v(t,x)$ in the linearized problem is written as

$$v(t,x) = \exp\left[\left(i\frac{\pi}{2} + \frac{\lambda_1(\omega)}{\omega} + \frac{\lambda_2(\omega)}{\omega^2} + \dots\right)t\right] \times \left(\frac{v_1(x,\omega)}{\omega} + \frac{1}{\omega^2}(v_2(x,\xi,\omega) + v_3(x,\omega)) + \dots\right).$$

Corresponding computations yield $v_1(x,\omega) \equiv 0$,

$$\lambda_1(\omega) \equiv 0, \quad v_2(x,\xi,\omega) = (1-i)a_0u_2(\xi,\omega),$$

$$\mathrm{Re}\lambda_2(\omega) = -\left(1 + \frac{\pi^2}{4}\right)^{-1}\left(\frac{\pi}{2} - 1\right)[a_0^2R^2(\omega) + \Gamma(\omega)] < 0.$$

Thus, for all sufficiently large ω, the equilibrium $u_0(x,\omega)$ is exponentially stable, i.e., the rapid oscillations of $a(\omega x)$ stabilize the equilibrium.

2. Averaging of an equation with an oscillating deviation in the space variable. Equations of this type are frequently faced in applications [9, 10, 14]. Below, we study the boundary value problem

$$\frac{\partial u}{\partial t} = d(\omega x)\frac{\partial^2 u}{\partial x^2} + r(\omega x)[1 - a(\omega x)u(t, x + f(\omega x))]u \quad (9)$$

with periodic boundary conditions

$$u(t, x + 2\pi) \equiv u(t,x). \quad (10)$$

All the coefficients in (9) and $f(x)$ are 2π-periodic functions; moreover, $d(x)$, $r(x)$, and $a(x)$ are positive.

The equation

$$\frac{\partial v}{\partial t} = \delta\frac{\partial^2 v}{\partial x^2} + \frac{\delta u}{2\pi} \times \int_0^{2\pi} d^{-1}(s)r(s)[1 - a(s)u(t, x + f(s))]ds,$$

where $\delta = \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d^{-1}(s)ds\right)^{-1}$, with boundary conditions $v(t, x + 2\pi) \equiv v(t,x)$ is an averaged equation for (9).

Consider two simple cases that are of great interest in applications, namely, when the coefficients d, r, and a are independent of x and the function $f(x)$ has a zero mean:

$$\int_0^{2\pi} f(x)dx = 0$$

and one of the following two relations holds: either

$$f(x) = \begin{cases} A, & x \in [0,\alpha], \quad 0 < \alpha < 2\pi, \\ B, & x \in (\alpha, 2\pi], \quad B = \alpha A(\alpha - 2\pi)^{-1}, \end{cases}$$

or

$$f(x) = \alpha\sin x.$$

The averaged equations become

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + r\left[1 - \frac{a}{2\pi}(\alpha v(t, x + A) + (2\pi - \alpha)v(t, x + B))\right]v,$$

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + r\left[1 - \frac{a}{\pi}\int_{-1}^{0}\frac{v(t, x + \alpha s) + v(t, x - \alpha s)}{\sqrt{1 - s^2}}ds\right]v,$$

respectively.

Each of the considered boundary value problems has the equilibrium $v_0 \equiv a^{-1}$. The characteristic equations for the boundary value problems linearized about v_0 are

(i) $\lambda_k = -dk^2 - \dfrac{r}{2\pi}(\alpha\exp(ikA) + (2\pi - \alpha)\exp(ikB))$, $k = 0, \pm1, \pm2, \dots$;

(ii) $\lambda_k = -dk^2 - rJ_0(k\alpha), k = 0, \pm1, \pm2, \dots,$ where $J_0(\cdot)$ is a Bessel function of the first kind.

The stability of the equilibrium v_0 is determined by the sign of $\max_k(\mathrm{Re}\lambda_k)$. The above formulas imply that the equilibrium can be stabilized or destabilized by rapid oscillations in the space variable.

It is of interest to compare the present results with those obtained for delay equations with coefficients rapidly oscillating in time [14, 15].

3. Consider the delay logistic equation with diffusion

$$\frac{\partial u}{\partial t} = \lambda d \frac{\partial^2 u}{\partial x^2} + r[1 - au(t - T, x)]u \qquad (11)$$

with boundary conditions (2). All the coefficients d, r, a, and T are positive smooth functions of $x \in [0,1]$. The parameter λ is assumed to be sufficiently large: $\lambda \gg 1$. The averaging principle can be used in this case. The role of the averaged equation is played by the generalized logistic equation with a distributed delay:

$$\dot{v} = \left(\int_0^1 d^{-1}(x)dx\right)^{-1} \cdot \int_0^1 d^{-1}(x)r(x) \times [1 - a(x)v(t - T(x))]dx \cdot v.$$

Consider the following special case. Suppose that, in (7), $d(x) \equiv d_0$, $a(x) \equiv a_0$, $T(x) \equiv T_0$, and the function $r(x)$ is given by the formula

$$r(x) = r_0(1 + \alpha\cos\beta x), \qquad |\alpha| < 1.$$

Its equilibrium $v_0 = a_0^{-1}$ is asymptotically stable if

$$r_0T_0(1 + \alpha\beta^{-1}\sin\beta) < \frac{\pi}{2}.$$

Thus, the stability domain can be enlarged or reduced by varying the coefficient $r(x)$.

4. Conclusions. The above averaging principle also applies to other classes of time delay systems with deviations in space variables. It can be extended to systems of boundary value problems with space variables ranging in a two-dimensional domain. When the time delay or the spatial deviation is not constant, the corresponding averaged equation may be other than that with a single point time delay or a single spatial deviation. In other words, the averaged equation may contain several point time delays (deviations) or may contain a distributed time delay (deviation).

Criteria were obtained under which rapid oscillations lead to stabilization (i.e., enlarge the stability domain of certain solutions) or lead to destabilization (i.e., the corresponding stability domain is reduced).

Bifurcation problems were considered. For an equation with a rapidly oscillating time delay, it was shown that a stable cycle is created and destroyed in an infinite process (as the oscillation frequency is increased).

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