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COMPUTER SCIENCE

# Mathematical Models for Calculating the Development Dynamics in the Era of Digital Economy

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**Abstract**—Mathematical models for practical calculations of technological progress (total productivity of production factors) and economic growth in the era of widespread digitalization and robotization of national economies, where the main factor of production is technological information, are developed and verified. For this purpose, models using different modes of information production are proposed for the first time. It is shown that the economic effect of the digitalization of an economy will not come immediately, but with a lag of about eight years. For the US economy, forecast calculations show that this will happen in 2022–2026 with total productivity increasing by 1.1 percentage points up to 2.5% per year.

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Information communication technology (ICT) has become a major resource for the acceleration of technological progress and economic growth in the 21st century. M. Hirooka conclusively showed that ICT had become major innovation technologies and technological infrastructure connecting the fifth (1982–2018) and the sixth (2018–2050) Kondratieff long waves (KLWs) in the global economic development. ICT is based on computers and key Internet information technologies. Brynjolfsson and his colleagues were the first to answer the question as to why the economic effect of using ICT and, in particular, computers is observed with a considerable time lag rather than immediately [1]. They showed that this phenomenon agrees with the hypothesis viewing ICT as general-purpose technologies, which require a particular time lag for the formation of a technical-economic environment (paradigm), where their response is manifested as an increase in the labor productivity and the economic growth rate [2]. In the USA, which is the leader in the development and use of ICT, this moment occurred only in the mid-1990s [3], although the fraction of investments into ICT was high enough (about 6%) in the 1970–1980s.

In the late 1980s, the productivity paradox was formulated by the Nobel Prize winner R. Solow, who said that the era of computers can be seen everywhere around ourselves, but not in the growth of productivity [4]. Finally, in the late 1990s, the US economy exhibited the dynamic growth of the labor productivity, on average, by up to 2.8% per year, against 1.5% in 1990–1995, which was caused, to a large degree, by the widespread use of ICT [5]. This, in turn, led to the corresponding growth of the ICT fraction in the added value (GDP), which occurred in all developed countries in the late 1990s (see Fig. 1).

A similar effect will also be observed in the era of digital economy, since digital technologies are also of general-purpose type and will be widely used in all spheres of economy, management, and public life. The intensive development of ICT and NBIC technologies has led to the creation of intellectual computers and robots, thing Internet, and industrial Internet. The Internet facilitated the revolution of digital platforms as a new platform model of business and promoted networking forms of business organization and, through it, a new digital economy. Thus, the digital

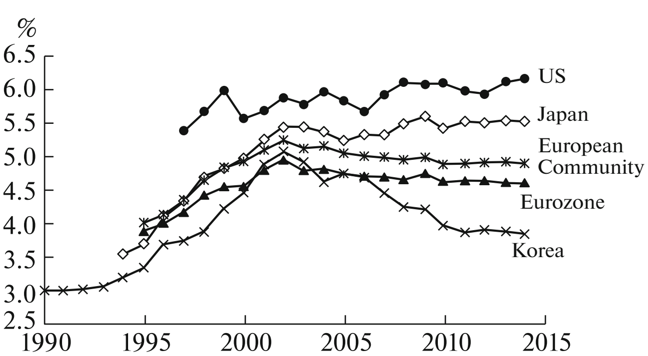


**Fig. 1.** ICT fraction of the added value in the national income (GDP).

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economy and Industry 4.0 are a natural development of information industry, but at a higher technology level. Accordingly, the nearest decade will be devoted to intensive formation of digital platform infrastructure in order to more quickly overcome the critical threshold barrier, behind which the digital economy will begin to facilitate an increase in the labor productivity and the economic growth rate, as was observed in ICT in the 1990s.

## 1. PRODUCTION FUNCTION IN THE ERA OF DIGITAL ECONOMY

To describe long-term economic dynamics in a technologically developed country, we can use the classical Cobb–Douglas production function with labor-saving technological progress [6, p. 142]:

$$Y = \gamma K^{\alpha} (AL)^{1-\alpha+\delta}, \tag{1}$$

where $Y(t)$ is the current amount of the national income (GDP), $K(t)$ is capital, $L(t)$ is the number of people occupied in the economy, $A(t)$ is technological progress, $\alpha$ is the surplus fraction in GDP, $\delta$ is a parameter characterizing the increasing return to the production scale ($\delta > 0$), and $\gamma$ is a constant normalizing coefficient.

This is explained by the fact that a digital economy is an actual economy in which a key role in a increase in the labor productivity is played by digital technologies and platforms. In production function (1), the capital $(K)$ and labor $(L)$ can be described by conventional methods and models [6]. However, the technological progress $A(t)$ cannot be described by conventional models [6, Chapter 3], since they do not contain the main factor of information and a fortiori digital economy, namely, the rate of production of technological information. Below, we propose mathematical models for calculating technological progress in an information digital economy based on the use of the rate of technological information production.

Logarithmically differentiating production function (1), we can write it in the rate form

$$q_Y = \alpha q_K + (1 - \alpha + \delta)(q_A + q_L), \tag{2}$$

where

$$q_Y = \frac{\dot{Y}}{Y}; \quad q_K = \frac{\dot{K}}{K}; \quad q_L = \frac{\dot{L}}{L}; \quad \text{and} \quad q_A = \frac{\dot{A}}{A}.$$

As was noted above, methods for computing the rates of capital accumulation $(q_K)$ and the variation in the number of workers $(q_L)$ are well known [6]. For this reason, we focus on the contribution of technological progress $(q_A)$ to the rate of economic growth $(q_Y)$.

If one of Eq. (1) or (2) is known, then the other is easy to obtain. For example, given $q_Y$ (2), we easily find

$$Y(t) = Y_0 \exp\left[\int_{T_0}^{t} q_Y(\tau)\, d\tau\right]$$

or

$$A(t) = A_0 \exp\left[\int_{T_0}^{t} q_A(\tau)\, d\tau\right].$$

Here and above, we mean the technological level $A(t)$ averaged over the entire economy, which is uniquely determined by the high technological level of newly formed innovation industries and sectors (in our case, this is the digital economy branch) $A_d(t)$ [7].

Thus, our problem is reduced to determining the function $A_d(t)$, which describes the trajectory of motion of the technological level of the digital economy, or the rate of its growth $q_{Ad}(t)$. As a baseline model for calculating the rate of technological progress in the information digital industry, we use the information model proposed by Yablonskii [8, p. 163]:

$$q_{Ad}(t) = \xi \frac{I_d}{K_d} \frac{\dot{S}_{Ad}}{S_{Ad}} = \xi \varepsilon_d(t) q_S(t), \tag{3}$$

where $I_d(t)$ are the current investments in the fixed capital $K_d(t)$ of the information digital industry; $S_{Ad}(t)$ is a function describing the dynamics of accumulation of industrial technological knowledge, which determines $A_d(t)$; $\varepsilon_d(t) = \frac{I_d}{K_d}$; $q_S = \frac{\dot{S}_{Ad}}{S_{Ad}}$; and $\xi$ is a constant tuning coefficient taking into account the R&D characteristics in the industry.

However, formula (3) is not valid, since it does not take into account the correspondence of the dimensions on the right- and left-hand sides of the equation. On the basis of the $\pi$ theorem in the theory of dimensions [9], formula (3) has to be written as

$$q_{Ad}(t) = \xi \sqrt{\varepsilon_d(t) q_S(t)}. \tag{4}$$

Thus, given the law of variation in industrial technological knowledge $S_{Ad}(t)$ in the information digital industry, the contribution of the industry to the rate of technological progress can be calculated using formula (4), since data characterizing the degree of renewal of the fixed capital $\varepsilon_d(t)$ are available in various databases, for example, at the OECD site [10]. The prognostic function of the relative investments $\varepsilon_d(t)$ can be obtained by approximating retrospective data with the subsequent extrapolation, as shown in Fig. 2 as applied to the US economy. Inspection of Fig. 2 shows that the sought function is well approximated by a linear function:

$$\varepsilon_d(t) = \varepsilon_0 + \varepsilon_1(t - T_0), \tag{5}$$

where $T_0 = 1982$, $\varepsilon_0 = 0.09$, and $\varepsilon_1 = 0.002$. The local outliers in the data in the years 2000, 2005, and 2009 are explained by investments nonmaterialized in capital because of the crisis phenomena.

According to Kurzweil [11, pp. 491–496], the accumulation of industrial technological knowledge in the information digital industry $S_{Ad}(t)$ obeys an exponential law:

$$S_d(t) = S_{d0}\exp[g(t)]. \quad (6)$$

Most studies deal with the simplest case, where $g(t) = g_0 t$ and $g_0 = \text{const}$:

$$S_d(t) = S_{d0}\exp(g_0 t).$$

We consider the general case. Therefore, for the calculation of the rate of technological information production (6), we have

$$q_{Sd}(t) = \frac{\dot{S}_d}{S_d} = \dot{g}(t). \quad (6a)$$

Finally, in view of (6a), formula (4) becomes

$$q_{Ad}(t) = \xi\sqrt{\varepsilon_d(t)\dot{g}(t)}. \quad (7)$$

## 2. ACTUAL MODES OF TECHNOLOGICAL INFORMATION PRODUCTION

Actual modes of information production in modern society have been addressed in numerous works. For example, they are tabulated in [12, Chapter 1, pp. 41–90]; additionally, [12] provides Lagrangian functions producing a number of practically important modes of information production, which will be used in the subsequent presentation. For our purposes, the following four modes of technological information production are of primary interest.

**2.1. Constant mode**. In this case, information is produced with a constant growth rate $\dot{g} = \nu_0 = \text{const}$. The Lagrangian has the form

$$L(\dot{g}, g, t) = \dot{g}^2.$$

The corresponding Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{g}} - \frac{\partial L}{\partial g} = \ddot{g} = 0$$

has the solution

$$g(t) = g_0 + \nu_0 t, \quad \dot{g}_0 = \nu_0. \quad (8)$$

In this case, $S_d(t) = S_{d0}\exp(\nu_0 t)$. The accumulation of industrial technological knowledge obeys the simplest exponential law characterizing a steady state within a single KLW.

**2.2. Blowup mode**. The rate of growth of information production increases exponentially with its accumulation [11, p. 492], i.e., $\dot{g} \sim e^g$. This mode is associated with the Lagrangian

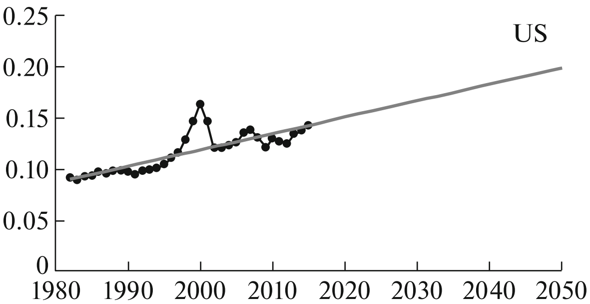


**Fig. 2.** Dynamics of the ratio of investments to the production capital in the information digital industry of the economy.

$$L(\dot{g}, g, t) = \dot{g}^2 e^{-2g}.$$

The corresponding Lagrange equation has the form $\ddot{g} = \dot{g}^2$. The solution of this equation with the initial conditions $g(t = 0) = g_0$ and $\dot{g} = (t = 0) = \nu_0$ leads to a hyperbolic increase in the rates of growth of information production:

$$\dot{g}(t) = \frac{1}{T_S - t}, \quad (9a)$$

$$g(t) = g_0 - \ln\left(1 - \frac{t}{T_S}\right), \quad T_S = \frac{1}{\nu_0}, \quad (9b)$$

where $T_S$ is a singularity point. Equation (9a) resembles the hyperbolic equation of demographic dynamics, which was first derived by von Foerster, Mora, and Amiot [13] with a singularity point at $T_S = 2026$. In reality, the explosive demographic growth was replaced by a stabilization mode—demographic transition. The same occurred for the mode of information production in the fifth KLW, when a smoothly varying logistic growth was observed instead of explosive growth (9). We assume that a blow mode with the subsequent stabilization will also occur in the sixth KLW.

**2.3. Mode with stabilization**. This mode is a combination of the blowup mode $\dot{g} \sim e^g$, which occurs at the initial stage development, and the constant mode $\dot{g} = \text{const}$ at the final stage: $\dot{g} \sim \frac{e^g}{1 + e^g}$. The Lagrangian is

$$L(\dot{g}, g, t) = \frac{\dot{g}^2 e^{-2g}}{1 - \dot{g}},$$

and the Lagrange equation is

$$\ddot{g} = \dot{g}^2(1 - \dot{g}).$$

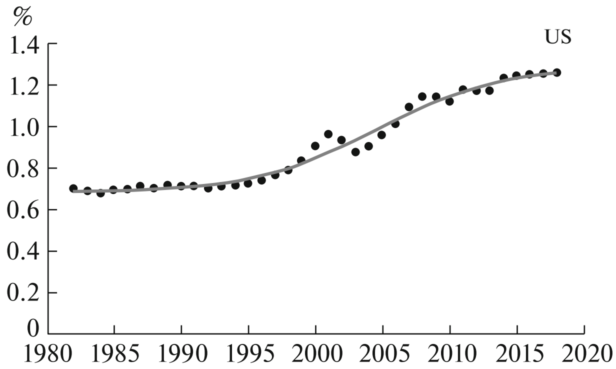


**Fig. 3.** Contribution of ICT to the total productivity of labor.

The solution is usually scaled to make it adequate for the considered problem. For this purpose, we introduce the new variables

$$g = \frac{g^1}{s_g}, \quad t = \frac{t^1}{s_t},$$

where $s_g$ and $s_t$ are constant factors. The most frequently used value is $s_t = 1$, which will also be used below.

The scaled solution of the last Lagrange equation has the form

$$\dot{g}(t) = \frac{1}{s_g}[1 + c_1 e^{-s_g g(t)}]^{-1}, \quad c_1 = e^{s_g g_0}\left(\frac{1}{\nu_0} - 1\right); \tag{10a}$$

$$t = s_g g(t) - c_1 e^{-s_g g(t)} + c_2, \quad c_2 = \frac{1}{\nu_0} - 1 - s_g g_0. \tag{10b}$$

It can be seen from Eq. (10a) that the production of technological information increases monotonically according to the logistic law at a variable rate, since, as follows from Eq. (10b), $g(t)$ is not a strictly linear function of *t*.

It this mode of technological information production that was observed in the fifth KLW (1982–2018). Indeed, Fig. 3 shows that the function $q_{Ad}(t)$ calculated using formulas (7) and (10) coincides, up to a high determination coefficient ($R^2 = 0.998$), with the curve of actual points characterizing the ICT contribution to the rate of technological progress [10]. To overlap the plot of $q_{Ad}(t)$ on the actual data, we needed to determine the initial data $g_0$ and $\nu_0$ and the scaling multiplier $s_g$, which are involved in the function $\dot{g}(t)$ (see (10)). first was required as. Since the actual data $\overline{q}_{Ad}(t)$ over the fifth KLW (1982–2018) are known [10], substituting the complete analytical expression for $\dot{g}(t)$ from (10) into (7), we obtain an equation for determining $g(t)$, i.e., the exponential growth rate for technological information in (6):

$$g(t) = g_0 + \frac{1}{s_g}\left(t - \frac{1}{\nu_0}\right) + \frac{\xi^2}{s_g^2}\frac{\varepsilon_d(t)}{q_{Ad}^2(t)}. \tag{11}$$

On the other hand, according to Varakin's empirical information economic law [15], the amount of technological information (6) is proportional to the added value generated by the information industry $\overline{Y}_{IE}$ (see Fig. 1):

$$S_d(t) = S_{d0}\exp[g(t)] = \omega\overline{Y}_{IE}(t),$$

where ω is a coefficient. It follows that

$$g(t) = \omega^1 + \ln\overline{Y}_{IE}(t), \quad \omega^1 = \ln\frac{\omega}{S_{d0}}. \tag{12}$$

First, using formula (9), we find $\nu_0 = \frac{1}{14}$. Indeed, $\nu_0 = \frac{1}{T_S}$, and the information industry of the US economy, which began to form in 1982 with the beginning of the economic boom in the fifth KLW, did not make an appreciable contribution to the productivity growth until 1996 [3], i.e., for 14 years, as can easily be seen in Fig. 1. Now we determine the coefficient $s_g$. Since the third term in Eq. (11) makes a small contribution, in the stabilization mode, it is described by the linear part of Eq. (11), which coincides with function (8). As a result, $s_g \cong \frac{1}{\nu_0} = 14$. Next, maximally fitting functions (11) and (12) to each other by applying the least squares method, we estimate the other two parameters: $g_0 = 2.77$ and $\xi = 0.07$.

**2.4. Blowup mode with return to a steady state.** In this development scenario, the process follows a blowup mode ($\dot{g} \sim e^g$) at the initial stage; by inertia, proceeds beyond the stationary level ($\dot{g} = \text{const}$); and, after reaching a maximum value $\dot{g}_m$, returns to a steady state. This mode can be described by the relation $\dot{g} \sim \frac{e^g}{1 + c(g)e^g}$, where $c(g)$ is the deceleration function, which, in the simplest case, has the form $c(g) = 1 - \frac{1}{1-\alpha}e^{-\alpha g}$, where $\alpha = \text{const}$ and $\alpha \neq 1$; moreover, in the limit as $\alpha \to 0$, we obtain the blowup mode, while, as $\alpha \to \infty$, the stabilization mode occurs. This mode is produced by the Lagrangian $L(\dot{g},g,t) = \frac{\dot{g}^2 e^{-2g}}{1-c(g)\dot{g}}$ and the corresponding Lagrange equation has the form

$$\ddot{g} = \dot{g}^2\left\{1 - \dot{g}\left[c(g) + \frac{dc}{dg}\right]\right\}.$$

The scaled solution of this equation is given by

$$\dot{g} = \frac{1}{s_g}\left(1 - \frac{e^{-\alpha s_g g}}{1-\alpha} + c_1 e^{-s_g g}\right); \quad c_1 = e^{s_g g_1}\left(\frac{1}{\nu_1} - 1 + \frac{e^{-s_g g_1}}{1-\alpha}\right); \qquad (12a)$$

$$t = s_g g + \frac{e^{-\alpha s_g g}}{\alpha(1-\alpha)} - c_1 e^{-s_g g} + c_2; \quad c_2 = \frac{1}{\nu_1} - 1 - s_g g_1 - \frac{1}{\alpha}e^{-\alpha s_g g_1}. \qquad (12b)$$

Here, $\nu_1 = \dot{g}\,(t = 2018)$ and $g_1 = g\,(t = 2018)$ are the initial values for the production of technological information in the sixth KLW and $s_g$ is a rescaling coefficient. According to this scenario of growth with return, the population of certain countries is stabilized in the 21st century.

It should be expected that, in the coming years, the production of technological knowledge will develop according to the blowup scenario under the influence of revolutionary digital technologies with artificial intelligence elements. However, then, the process will be affected by deceleration forces associated with resource and other limiters. As a result, the blowup mode with the subsequent stabilization will occur. To calculate the contribution of the digital economy to the rate of technological progress in the sixth KLW (2018–2050) by using formula (7), we first need to determine the initial values $g_1$ and $\nu_1$ and the parameter $\alpha$.

Note that the values of the coefficients $\xi$ and $s_g$ found for the fifth KLW are preserved in the sixth KLW, and the prognostic values of $\varepsilon_d\,(t)$ are calculated using formula (5) by its extrapolation. The initial value $g_1 = g \;\; (t = 2018)$ is calculated by formula (11): $g_1 = 5.3$. Since the logistic function (10a) is symmetric, we obtain $\nu_1 = \nu_0 = \frac{1}{14}$. Concerning the deceleration parameter $\alpha$, a theoretical basis is difficult to construct, so, following [12, pp. 56, 57], we empirically chose three values: $\alpha_1 = 0.005$, $\alpha_2 = 0.008$, and $\alpha_3 = 0.015$.

Finally, for each $t \in (2018–2050)$, we solve Eq. (13b) to find $g(t)$. Then formula (13a) is used to determine $\dot{g}(t)$, which is next substituted into formula (7). In this way, all values $q_{Ad}\,(t)$ are calculated and the curve characterizing the contribution of the digital economy is constructed. This curve is presented in Fig. 4 as an extension of the logistic curve (1982–2018) depicting the contribution of the information economy.

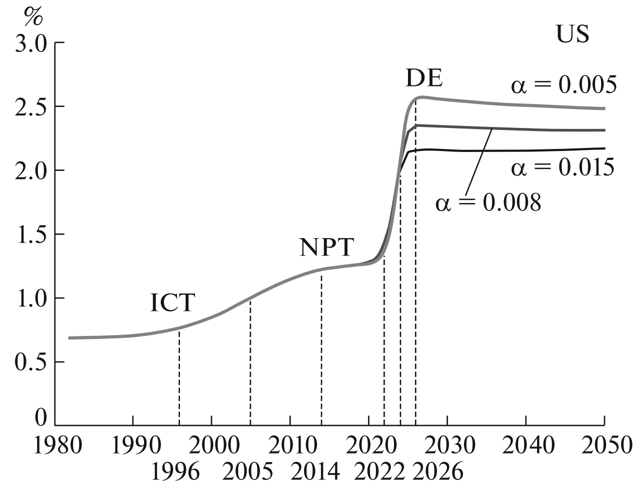


**Fig. 4.** Contribution of ICT and digital technologies in the total productivity of labor.

Inspection of Fig. 4 shows that a noticeable contribution to the US digital economy will begin to be exhibited only in 2022. Moreover, the productivity in 2022–2026 will grow in a jump by about 1.1 percentage points (for $\alpha = 0.008$), while the similar ascent in the preceding fifth KLW was smooth and extended over 12 years (1996–2008) roughly the same pattern will be observed in other developed countries. Since the digital industry in the US economy overcame the 5%-level necessary for establishing an innovation industry in 2014 and the economic effect will be observed in 2022–2026, the time lag is about eight years.

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## REFERENCES

1. E. Brynjolfsson and S. Adam, *Wired for Innovation: How Information Technology Is Reshaping the Economy* (MIT Press, Cambridge, Mass., 2010).
2. T. Bresnahan and M. Trajtenberg, J. Econ. **65** (1), 83–108 (1995).
3. E. Brynjolfsson and L. Hitt, Manage. Sci. **42** (4), 541–558 (1996).
4. R. Solow, New York Times Books Rev., July 12 (1987).
5. D. Jorgenson and K. Stiroh, *Brookings Papers on Economic Activity* (2000), Vol. 1.

6. D. Romer, *Advanced Macroeconomics* (McGraw-Hill Education, New York, 2011).
7. A. A. Akaev and V. A. Sadovnichii, Herald Russ. Acad. Sci. **86** (5), 371–383 (2016).
8. A. I. Yablonskii, *Mathematical Models in Science Studies* (Mysl', Moscow, 1986) [in Russian].
9. G. I. Barenblatt, *Scaling* (Cambridge Univ. Press, Cambridge, Mass., 2003).
10. OECD. http://stats.oecd.org/index.aspx?DatasetCode=KEI#.
11. R. Kurzweil, *The Singularity Is Near* (Viking Books, New York, 2005).
12. B. M. Dolgonosov, *Nonlinear Dynamics of Ecological and Hydrological Processes* (Librokom, Moscow, 2009) [in Russian].
13. H. von Foerster, P. Mora, and L. Amiot, Science **132**, 1291–1295 (1960).
14. S. P. Kapitsa, *Essay on the Theory of Human Population Growth: Demographic Revolution and Information Society* (Nikitskii Klub, Moscow, 2008) [in Russian].
15. L. E. Varakin, *Information-Economic Law* (MAS, Moscow, 2006) [in Russian].

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