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MATHEMATICS

On Two-Dimensional Polynomially Integrable Billiards on Surfaces of Constant Curvature

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Abstract—The algebraic version of the Birkhoff conjecture is solved completely for billiards with a piecewise C^2-smooth boundary on surfaces of constant curvature: Euclidean plane, sphere, and Lobachevsky plane. Namely, we obtain a complete classification of billiards for which the billiard geodesic flow has a nontrivial first integral depending polynomially on the velocity. According to this classification, every polynomially integrable convex bounded planar billiard with C^2-smooth boundary is an ellipse. This is a joint result of M. Bialy, A.E. Mironov, and the author. The proof consists of two parts. The first part was given by Bialy and Mironov in their two joint papers, where the result was reduced to an algebraic-geometric problem, which was partially studied there. The second part is the complete solution of the algebraic-geometric problem presented below.

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1. INTRODUCTION AND THE MAIN RESULTS

Let Σ be an arbitrary two-dimensional surface with a Riemannian metric and $\Omega \subset \Sigma$ be a connected domain with a piecewise smooth boundary. The geodesic flow of the billiard Ω on the tangent bundle $T\Sigma|_{\Omega}$ is defined as follows. A point $(Q, P) \in T\Sigma|_{\Omega}$, where $Q \in \Omega$ and $P \in T_Q\Sigma$, moves along the trajectory of the geodesic flow of Σ until Q hits the boundary $\partial\Omega$. In the collision, the point Q remains unchanged, while the velocity P reflects from the boundary (to become a vector P^* directed inside Ω) according to the usual reflection law: the angle of incidence is equal to the angle of reflection and $|P| = |P^*|$. Then the motion continues along the trajectory of the geodesic flow issued from the point (Q, P^*), etc.

The billiard geodesic flow thus defined has a trivial first integral: the squared magnitude of the velocity. A billiard in a convex domain with a smooth boundary is called integrable if its flow has an additional first integral independent of the squared velocity magnitude on a neighborhood in $T\Sigma|_{\bar{\Omega}}$ of the unit tangent bundle to the boundary of the billiard. It is well known that elliptic billiards in the plane are integrable. The famous Birkhoff conjecture states the converse: any integrable convex planar billiard with a smooth boundary is an ellipse. The particular case of the Birkhoff conjecture when the additional integral is assumed to be a polynomial in the velocity has led to the algebraic Birkhoff conjecture for billiards on surfaces of constant curvature, which was studied by Bolotin [7, 8] and recently by Bialy and Mironov [4, 5]. This paper presents a complete solution of the algebraic Birkhoff conjecture. Let us formulate the corresponding results.

Definition 1. A billiard $\Omega \subset \Sigma$ with a piecewise smooth boundary is *polynomially integrable* if its flow has a first integral that is a polynomial in the velocity and is not constant on the hypersurface $\{|P|^2 = 1\}$. Note that, according to Bolotin's result [7, 8], for a surface Σ of constant curvature and for a smooth connected boundary $\partial\Omega$, polynomial integrability is equivalent to the existence of the above-mentioned integral in a neighborhood in $T\Sigma|_{\Omega}$ of the unit tangent bundle of the boundary.

Let Σ be a two-dimensional surface with a metric of constant curvature and $\Omega \subset \Sigma$ be a connected domain with a piecewise smooth boundary. Assume without loss of generality that the curvature of the metric is either zero or ± 1: this can be achieved by multiplying the metric by a positive constant, which changes neither geodesics nor the polynomial integrability of the billiard in Ω. The surface Σ is represented by one of three standard models in the space $\mathbb{R}^3|_{(x_1,x_2,x_3)}$ equipped

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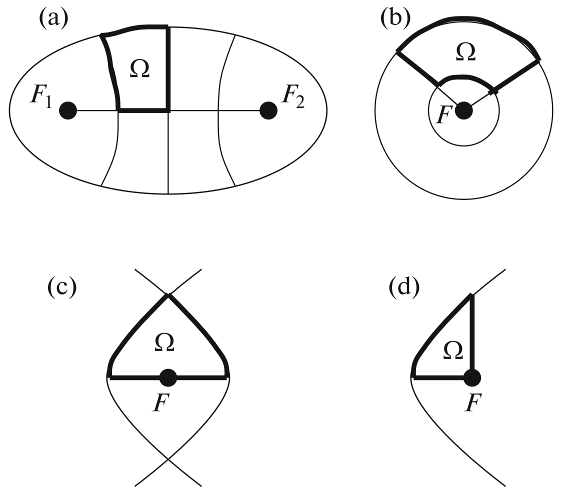

Fig. 1. Examples of confocal planar billiards: F_1, F_2, F are foci. Billiards (a–c) have integrals of degree 2, and billiard (d) has an integral of degree 4.

with a suitable (pseudo) Euclidean metric $\langle Ax, x\rangle$, where

$$A \in \{\mathrm{diag}(1,1,0), \mathrm{diag}(1,1,\pm 1)\},$$

$$x = (x_1, x_2, x_3), \quad \langle x, x\rangle := \sum_{j=1}^{3} x_j^2,$$

namely, Euclidean plane: $\Sigma = \{x_3 = 1\}$, $A = \mathrm{diag}(1, 1, 0)$; the unit sphere: $\Sigma = \{|x_1|^2 + |x_2|^2 + |x_3|^3 = 1\}$, $A = \mathrm{Id}$; and the Lobachevsky plane: $\Sigma = \{|x_1|^2 + |x_2|^2 - |x_3|^2 = -1\}$, $A = \mathrm{diag}(1,1,-1)$.

The quadratic form on $T\mathbb{R}^3$ defined by $\langle Ax, x\rangle$ induces a metric of constant curvature on the surface Σ. Geodesics on Σ are its intersections with two-dimensional vector subspaces of $\mathbb{R}^3$. Conics on Σ are its intersection with the quadrics $\{\langle Cx, x\rangle = 0\} \subset \mathbb{R}^3$, where C is a symmetric matrix. Recall the following generalization of the confocality notion.

Definition 2 [13, p. 84]. The pencil of confocal conics on the surface Σ is defined using a symmetric matrix B nonproportional to A and consists of the conics

$$\Gamma_\lambda = \Sigma \cap \{\langle B_\lambda x, x\rangle = 0\}, \qquad B_\lambda = (B - \lambda A)^{-1}, \quad \lambda \in \mathbb{R}. \tag{1}$$

In the Euclidean case, when $A = \mathrm{diag}(1, 1, 0)$, we suppose in addition that the x_3 axis does not lie in the kernel of the matrix B. For λ such that $\det(B - \lambda A) = 0$ and the kernel $K_\lambda = \mathrm{Ker}(B - \lambda A)$ is one-dimensional, let Γ_λ denote the geodesic

$$\Gamma_\lambda = \Sigma \cap K_\lambda^\perp. \tag{2}$$

If $\dim K_\lambda = 2$ (in the Euclidean case, this corresponds to a pencil of concentric circles), for each two-dimensional orthogonal vector subspace $H \perp K_\lambda$ we also set $\Gamma_\lambda = \Gamma_\lambda(H) = \Sigma \cap H$.

Definition 3. A billiard $\Omega \subset \Sigma$ with a piecewise smooth boundary is *confocal* if its boundary consists of arcs of confocal conics (and contains a nonlinear conical arc) and, possibly, of some segments of geodesics from the following list of admissible geodesics (here, B is the matrix determining the corresponding pencil of confocal conics):

(I) all the geodesics Γ_λ in (2) and (or) $\Gamma_\lambda(H)$ are admissible;

(II) for $B = Aa \otimes b + b \otimes Aa$ (modulo $\mathbb{R}$A), where $a, b \in \mathbb{R}^3$ and $\langle a, b\rangle = 0$;

(IIa) in the non-Euclidean subcase, the following geodesics are also admissible:

$$\{r \in \Sigma | \langle r, a\rangle = 0\}, \quad \{r \in \Sigma | \langle r, Ab\rangle = 0\}; \tag{3}$$

(IIb) in the subcase where $\Sigma = \{x_3 = 1\}$ is Euclidean plane and the vector b is not parallel to it, only Γ_λ and the first geodesic in (3) are admissible.

Note that, in case (II), the subcase where Σ is Euclidean plane and the vector b is parallel to it is impossible, since, in this subcase, the x_3 axis would lie in the kernel of the matrix B, which is forbidden by the assumptions.

Confocal billiards were introduced in [8], where their polynomial integrability with an integral of the first, second, or fourth degree was proved [8, Section 2, Proposition 1; the theorem in Section 4]. The case of a fourth-degree integral that is not reducible to an integral of degree no higher than two corresponds to a billiard whose boundary contains conics from a pencil of type (IIa) or (IIb) and also segments of some of the admissible geodesics from (3) mentioned in (IIa) and (IIb), respectively.

Example 1. A planar billiard bounded by arcs of confocal parabolas and by a segment of the line passing through a focus orthogonally to the axis of the parabolas (and, possibly, by segments of the axis, see Fig. 1d)) has type (IIb). This example of a billiard having an integral of degree 4 was first found in [10]. Similar billiards on surfaces of nonzero constant curvature were constructed in [2].

Below is the main result of this paper.

Theorem 1. *Let a billiard with a piecewise C^2-smooth boundary on a surface Σ of constant curvature be polynomially integrable, and let its boundary contain at least one nonlinear smooth arc. Then the billiard is confocal.*

Corollary 1. *Every bounded polynomially integrable billiard with a C^2-smooth boundary on Euclidean plane is an ellipse.*

Theorem 1 is a joint result of Bialy, Mironov, and the author of the present paper. Its proof consists of two parts.

(i) Bialy and Mironov's papers [4, 5], where a geometric construction was used to reduce Theorem 1 to an algebraic-geometric problem, which was partially studied there:

(ii) a complete solution of the above algebraic-geometric problem (Theorem 5, see below). For the complete proof, see https://arxiv.org/abs/1706.04030.

A detailed overview on the history of studying the Birkhoff conjecture can be found in [3, 14], where its local version is proved, namely, any integrable deformation of an ellipse is an ellipse. The complete polynomial integrability of billiards of arbitrary dimension on confocal quadrics in Euclidean space and on quadrics in spheres and in Lobachevsky spaces is proved in [12, 13]. The above-mentioned results of Bolotin, Bialy, and Mironov are presented below. The families of planar billiards with piecewise smooth boundaries depending continuously on a single parameter that have a common polynomial integral were classified by Abdrakhmanov in [1]. (In fact, it is sufficient to require that the union of the boundaries not lie in an algebraic curve, see [1, p. 30].) The solution of the algebraic Birkhoff conjecture for planar billiards with an integral of degree at most 4 was obtained by Bialy and Mironov in [6]. An analogue of the algebraic Birkhoff conjecture for outer planar billiards was formulated and partially studied in [11] and was completely proved in [15].

2. PROOF OF THEOREM 1

A point $r \in \Sigma$ is identified with its position vector in $\mathbb{R}^3$.

Theorem 2 (see [7; 8, p. 119; 9, Chapter 5, Section 3, Proposition 5]). *For every polynomially integrable billiard $\Omega \subset \Sigma$ with a piecewise C^2-smooth boundary, there exists a first integral that is not constant on the level hypersurface $\{|P|^2 = 1\}$ and is a homogeneous polynomial $\Psi(M)$ of even degree in the components of the moment vector*

$$M = [r, P] = (x_2P_3 - x_3P_2, -x_1P_3 + x_3P_1, x_1P_2 - x_2P_1), \quad (4)$$
$$r = (x_1, x_2, x_3) \in \Sigma,$$
$$P = (P_1, P_2, P_3) \text{ is the velocity vector.}$$

Every C^2-smooth arc in $\partial\Omega$ lies in an algebraic curve.

Theorem 3 [8, Section 4]. *Let a billiard with a piecewise C^2-smooth boundary on the surface Σ be polynomially integrable. Suppose that its boundary contains a nonlinear arc of a conic. Then the billiard is confocal.*

Definition 4. Let $\mathbb{I} \subset \mathbb{CP}^2$ be an arbitrary conic (either regular or a pair of distinct straight lines) and $B \in \mathbb{CP}^2 \setminus \mathbb{I}$. For every straight line passing through B, consider its projective involution with the fixed point B that permutes its two points of intersection with $\mathbb{I}$. The transformation thus constructed is a projective involution $\mathbb{CP}^2 \to \mathbb{CP}^2$, which is called the $\mathbb{I}$-angular symmetry centered at B.

Definition 5. Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic. An irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ different from $\mathbb{I}$ and from a straight line generates a rationally integrable $\mathbb{I}$-angular billiard if there exists a rational function $G \not\equiv$ const on $\mathbb{CP}^2$ (called an integral) with poles lying in $\mathbb{I}$ such that, for any point $B \in \gamma \setminus \mathbb{I}$, the restriction of the function G to the projective tangent line $T_B\gamma$ is invariant under the $\mathbb{I}$-angular symmetry centered at B.

Let $\pi: \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ be the tautological projection. The form $\langle x, x\rangle$ on $\mathbb{R}^3$ determines an orthogonal polarity that takes a vector subspace to its orthogonal complement and induces a projective duality $\mathbb{RP}^{2*} \to \mathbb{RP}^2$ sending projective lines to points.

Theorem 4. *Let $\mathbb{I} = \{\langle AM, M\rangle = 0\} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}$ and $\Omega \subset \Sigma$ be a polynomially integrable billiard with a homogeneous integral $\Psi(M)$ of even degree $2n$. Consider the corresponding function*

$$G(M) = \frac{\Psi(M)}{\langle AM, M\rangle^n}$$

regarded as a rational function on $\mathbb{CP}^2_{(M_1:M_2:M_3)}$. Let $\alpha \subset \partial\Omega$ be a nonlinear C^2-smooth arc of the boundary and $\alpha^* \subset \mathbb{RP}^2$ be the curve dual to its projection $\pi(\alpha)$: points of α^* are orthogonally polar-dual to the tangent lines to $\pi(\alpha)$. Every nonlinear irreducible component of the complex projective Zariski closure of α^* generates a rationally integrable $\mathbb{I}$-angular billiard with integral G.

Theorem 4 follows from [5, Theorem 1.3, p. 151] and the results of [4].

Theorem 1 is implied by Theorems 3 and 4 and the following theorem.

Theorem 5. *Given an arbitrary conic $\mathbb{I} \subset \mathbb{CP}^2$ that is either a regular conic or a pair of distinct lines, every irreducible algebraic curve generating a rationally integrable $\mathbb{I}$-angular billiard is a conic.*

Sketch proof of Theorem 5. Let $\gamma \subset \mathbb{CP}^2$ be an irreducible curve generating a rationally integrable $\mathbb{I}$-angular billiard with an integral G. Then, according to [4, Theorem 1; 5, Theorem 1.2],

(0) all the singular and inflection points of the curve γ are contained in $\mathbb{I}$.

We study local branches (irreducible components of a germ) of the curve γ at points of the intersection $C \in \gamma \cap \mathbb{I}$. Each local branch is nonlinear; thus, in a suitable local affine chart (z, w) centered at C, it can be parametrized holomorphically bijectively by a small complex parameter t:

$$t \mapsto (t^q, ct^p(1 + o(1))) \quad \text{as} \quad t \to 0;$$
$$q, p \in \mathbb{N}, \quad 1 \leq q < p, \quad c \neq 0.$$

Recall that a branch is quadratic if $\frac{p}{q} = 2$ and subquadratic if $\frac{p}{q} \leq 2$. We prove the following assertions and theorem.

(i) If $\mathbb{I}$ is a pair of straight lines intersecting at a point C, then any branch at C that is transversal to both lines is quadratic.

(ii) If C is a regular point of a conic $\mathbb{I}$, then

(iia) any branch at C that is tangent to $\mathbb{I}$ is quadratic;

(iib) any branch transversal to $\mathbb{I}$ is regular and quadratic.

Theorem 6. *Let* $\mathbb{I} \subset \mathbb{CP}^2$ *be a conic* (*either a regular conic or a pair of distinct lines*). *Suppose that* $\gamma \subset \mathbb{CP}^2$ *is an irreducible algebraic curve different from a straight line that satisfies assertion* (0). *Let each of its local branches of type* (i) (*if any*) *be subquadratic and all branches of types* (iia) *and* (iib) *at points of the intersection* $\gamma \cap \mathbb{I}$ *be quadratic* (*respectively, regular and quadratic*). *Then* γ *is a conic.*

The proof of Theorem 6 is based on Shustin's argument about invariants of plane curves in [15, Section 4], where a similar result was proved in the case where $\mathbb{I}$ is a straight line. Theorem 5 follows from assertions (0), (i), (iia), and (iib) and Theorem 6.

It is well known that $G|_\gamma \equiv \text{const}$ (analogously to [4, Theorem 3; 5, Theorem 1.3]). Let $\gamma \subset \Gamma = \{G = 0\}$. The proof of (i) and (iia) makes use of only the invariance of the intersections $T_B\gamma \cap \Gamma$ under $\mathbb{I}$-angular symmetry centered at B and consists in its asymptotic analysis as $B \to C \in \gamma \cap \mathbb{I}$. Namely, we study those points of the intersection $T_B\gamma \cap \Gamma$ for which a suitable coordinate (w or $z^{\pm 1}$) is asymptotically equivalent to the corresponding coordinate of B up to a nonzero constant asymptotic factor. It is proved that all the "nontrivial" asymptotic factors are exactly suitable powers of the roots of a finite set of polynomials of the form $c_i z^p - rz^q + r - 1$, $r = \frac{p}{q}$, $c_i \neq 0$, constructed using the germ (Γ, C). From this and the symmetry of the intersections $T_B\gamma \cap \Gamma$, it follows that the set of these powers of roots (except possibly for a small explicit exception) is invariant under a suitable involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$: either taking the inverse or the central symmetry with respect to 1. Then we deduce that $\frac{p}{q} = 2$. The proof of (iib) is the most technical part of the proof. It relies on the symmetry of the intersections $T_B\gamma \cap \Gamma$ and the Bialy–Mironov formula for the Hessian of the polynomial defining the curve γ (see [4, Theorem 6.1; 5, formulas (16) and (32)]).

Any nonlinear arc of the boundary of a polynomially integrable billiard $\Omega \subset \Sigma$ contains an arc of a conic (Theorems 4 and 5). Therefore, Ω is confocal by Theorem 3. Theorem 1 is proved.

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REFERENCES

1. A. M. Abdrakhmanov, Moscow Univ. Mech. Bull. **45** (6), 13–17 (1990).
2. A. M. Abrdrakhmanov, Moscow Univ. Mech. Bull. **45** (5), 14–16 (1990).
3. A. Avila, J. De Simoi, and V. Kaloshin, Ann. Math. **184**, 527–558 (2016).
4. M. Bialy and A. Mironov, Adv. Math. **313**, 102–126 (2017).
5. M. Bialy and A. E. Mironov, Geom. Phys. **115**, 150–156 (2017).
6. M. Bialy and A. E. Mironov, Proc. Steklov Inst. Math. **295**, 27–32 (2016).
7. S. V. Bolotin, Moscow Univ. Mech. Bull. **45** (2), 10–13 (1990).
8. S. V. Bolotin, Math. Notes **51** (2), 117–123 (1992).
9. V. V. Kozlov and D. V. Treshchev, *Billiards. A Genetic Introduction to the Dynamics of Systems with Impacts* (Mosk. Gos. Univ., Moscow, 1991); Transl. Math. Monographs **89**, Am. Math. Soc., Providence, RI, 1991.
10. A. Ramani, A. Kalliterakis, B. Grammaticos, and B. Dorizzi, Phys. Lett. A **115** (1–2), 13–17 (1986).
11. S. Tabachnikov, Pacific J. Math. **235** (1), 101–104 (2008).
12. A. P. Veselov, Funct. Anal. Appl. **22** (2), 83–93 (1988).
13. A. P. Veselov, J. Geom. Phys. **7** (1), 81–107 (1990).
14. V. Kaloshin and A. Sorrentino, Ann. Math. **188** (1), 315–380 (2018).
15. A. Glutsyuk and E. Shustin, Math. Ann. (to appear). https://doi.org/.10.1007/s00208-018-1726-4.

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