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MATHEMATICS

Approximation of Solution Components for Ill-Posed Problems by the Tikhonov Method with Total Variation

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Abstract—An ill-posed problem in the form of a linear operator equation given on a pair of Banach spaces is considered. Its solution is representable as a sum of a smooth and a discontinuous component. A stable approximation of the solution is obtained using a modified Tikhonov method in which the stabilizer is constructed as a sum of the Lebesgue norm and total variation. Each of the functionals involved in the stabilizer depends only on one component and takes into account its properties. Theorems on the componentwise convergence of the regularization method are stated, and a general scheme for the finite-difference approximation of the regularized family of approximate solutions is substantiated in the n-dimensional case.

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INTRODUCTION

Consider an ill-posed problem in the form of a linear equation

$$Au = f \tag{1}$$

with an operator $A: U \to F$ acting on a pair of Banach spaces. The ill-posedness of the problem means that the inverse operator A^{-1} is discontinuous, which leads to the instability of the solution with respect to perturbations of the right-hand side f of Eq. (1). The solution is assumed to have different smoothness properties on different segments of its domain. In the case of the conventional (one-component) regularization method, the task arises of choosing an adequate stabilizing functional that recovers the solution equally well on all segments, while capturing its fine structure. An approach to addressing this task is to represent the solution as a sum of several components and to construct a stabilizer in the form of a sum of functionals, each depending only on one of the components. As a result, each functional takes into account the smoothness property characteristic for the given component if there is a priori information on this property of the desired solution. This technique is used, for example, when noisy signals are processed by applying the Tikhonov method to recover the continuous and discontinuous components of the solution [1]. Theoretical substantiation of this two–three-component approach can be found in [2, 3], where the discontinuous component is approximated using the total variation, which leads to the necessity of solving a nonsmooth optimization problem.

In this paper, as a stable method for the approximate solution of Eq. (1), we propose a modified version of the Tikhonov method, which, assuming that the solution of Eq. (1) can be represented as the sum $u = u_1 + u_2$ of a smooth and a nonsmooth component, has the form

$$\min\{\|A(u_1 + u_2) - f_\delta\|_{L_2}{}^2 + \alpha[\|u_1\|^2_{L_q} + \Omega^\beta(u_2)]: \\ u_1 \in L_q, u_2 \in BV\} = \Phi^*. \tag{2}$$

Here, $\|f - f^\delta\| \le \delta, L_q(D), D \in R^d, 1 < q < \infty, d = 1, 2, 3$ the stabilizing functional $\Omega^\beta(u)$ for the second component is defined by the formulas

$$\Omega^\beta(u) = l^\beta(u) + J^\beta(u), \quad \beta > 0, \tag{3}$$

$$l^\beta(u) = \int_D \sqrt{|u(x)|^2 + \beta}dx, \\ J^\beta(u) = \sup_{v \in V}\left\{\int_D\int_D(-u\operatorname{div}v + \sqrt{\beta(1-|v|^2)})dx\right\}, \tag{4}$$

where $V = \{v: v \in C_0^1(D, R^d), |v(x)| \le 1\}, d = 1,2,3.$ Let $J(u)$ denotes $J^\beta(u)$ in (4) at $\beta = 0$, and let $\Phi(u_1, u_2)$ denote the objective functional in problem (2). Obviously, the functional $l^\beta(u)$ is a smooth approximation of the L_1 norm. The functional $J^\beta(u)$, which can be treated as a smooth approximation of the total varia-

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tion $J(u)$, was introduced in [4] and was used as a stabilizer in the traditional one-component version of the Tikhonov method ($u_1 = 0$, $l^\beta(u_2) = 0$).

In Section 1, we establish the existence of a normal solution of Eq. (1) that minimizes the stabilizing functional. Additionally, the convergence of regularized solutions is proved. In Section 2, for problem (2) a convergence theorem for discrete approximations formed of extremal elements of finite-dimensional minimization problems with differentiable convex objective functions is stated. As a result, these problems can be solved by applying gradient and Newton-type methods.

1. REGULARIZING PROPERTIES OF THE METHOD

First, we prove the existence of a pair (u_1, u_2) minimizing the functional

$$\Omega(u_1, u_2) = \|u_1\|^2_{L_q} + \Omega^\beta(u_2), \tag{5}$$

where Ω^β is given by formula (3). Note that, even if Eq. (1) has a unique solution, its representation in the form of the sum $u = u_1 + u_2$ is not unique. For example, along with (u_1, u_2), the pair $(u_1 + v, u_2 - v)$, when summed, also yields a solution.

Theorem 1. *Suppose that A is a linear continuous operator from $L_p(D)$ to $L_2(S)$, where D, $S \in R^d$, $1 < p \le \frac{d}{d-1}$, $p \le q$; the domain D satisfies the cone condition; and Eq.* (1) *has a unique solution $\hat{u} = A^{-1}f$, $\|f - f^\delta\| \le \delta$. Then the problem*

$$\begin{gathered}\min\{\|u_1\|^2_{L_q} + \Omega^\beta(u_1, u_2) \colon A(u_1 + u_2) = f, \\ u_1 \in L_q, u_2 \in L_1\} = \Psi^*,\end{gathered} \tag{6}$$

where the functional Ω is defined by formula (5), *has a solution (u_1, u_2), possibly nonunique.*

Proof. Let (u_1^k, u_2^k) be a minimizing sequence, i.e., $\Psi(u_1^k, u_2^k) \to \Psi^*$, where $\Psi(u_1^k, u_2^k)$ is the objective functional in problem (5). Therefore, in view of [4, Theorem 2.2], each of its terms is uniformly bounded with respect to k:

$$\begin{gathered}\|u_1^k\| \le c_1, \quad \|u_2^k\| \le l^\beta(u_2^k) \le c_2, \\ J(u_2^k) \le J^\beta(u_2^k) \le c_3.\end{gathered} \tag{7}$$

Therefore, in the uniformly convex space L_q, there is a weakly converging sequence

$$u_1^{k_i} \to \bar{u}_1 \quad \text{(weakly) in} \quad L_q, \tag{8}$$

and the embedding theorem $BV \to L_p$ (see [4, Theorem 2.5]) implies that

$$u_2^{k_i} \to \bar{u}_2 \quad \text{(weakly) in} \quad L_p, p < \frac{d}{d-1}, \tag{9}$$

$$u_2^{k_i} \to \bar{u}_2 \quad \text{(weakly) in} \quad L_p, \quad p = \frac{d}{d-1}, \quad d \ge 2.$$

Since (7) and (9) hold, we can assume that

$$u_2^{k_i} \to \bar{u}_2 \text{ (almost everywhere)},$$

$$\lim_{i\to\infty} \int_D \sqrt{|u_2^{k_i}|^2 + \beta}\, dx = \bar{d},$$

whence, by the Fatou theorem,

$$\int_D \sqrt{\bar{u}_2(x) + \beta}\, dx \le \lim_{i\to\infty} \int_D \sqrt{|u_2^{k_i}(x)|^2 + \beta}\, dx = \bar{d}. \tag{10}$$

Combining the conditions of the theorem with relations (8)–(10) and Theorem 2.3 from [4] on the weak lower semicontinuity of the functional J^β, we obtain the inequalities

$$\Psi^* \le \Psi(\bar{u}_1, \bar{u}_2) \le \lim_{i\to\infty} \Psi(u_1^{k_i}, u_2^{k_i}) = \Psi^*,$$

i.e., the pair $(\bar{u}_1, \bar{u}_2)$ is a minimizer in problem (6) and, for any such pair, $\bar{u}_1 + \bar{u}_2 = A^{-1}f = \hat{u}$.

Now we consider the Tikhonov regularization method (2).

Theorem 2. *Let the conditions of Theorem* 1 *hold. Then, for any $\alpha > 0$, problem* (2) *has a solution (u_1^α, u_2^α) such that, if the regularization parameter is related to the error δ as*

$$\alpha(\delta) \to 0, \quad \frac{\delta^2}{\alpha(\delta)} \to 0, \quad \delta \to 0,$$

then the following properties hold:

(i) $\{u_1^{\alpha(\delta)}\}$ *is relatively compact in the space L_q.*

(ii) $\{u_2^{\alpha(\delta)}\}$ *is relatively compact for $1 < p < \frac{d}{d-1}$ and relatively weakly compact for $1 \le p \le \frac{d}{d-1}$, $(d \ge 2)$ in the space L_p.*

(iii) *Any pair (u_1^α, u_2^α) defines a unique element $u^\alpha = u_1^\alpha + u_2^\alpha$.*

(iv) *If $(\bar{u}_1, \bar{u}_2)$ are limit points of the sequences $u_1^{\alpha(\delta_k)}, u_2^{\alpha(\delta_k)}$ $(\delta_k \to 0)$, then this pair solves problem* (6).

(v) $\lim_{k\to\infty} \Omega^\beta(u_2^{\alpha(\delta_k)}) = \Omega^\beta(\bar{u}_2)$.

Proof. Let us sketch the scheme for checking properties (i)–(v). The existence of the solution (u_1^α, u_2^α) to problem (2) can be proved by following the argument used in Theorem 1. The uniqueness of the function $u^\alpha = u_1^\alpha + u_2^\alpha$ follows from the strict convexity of L_2

and the existence of an inverse operator A^{-1}. The compactness of the families $\{u_1^{\alpha(\delta)}\}$ and $\{u_2^{\alpha(\delta)}\}$ follows from the fact that, under the conditions imposed on the parameter $\alpha(\delta)$, the functionals involved in the stabilizer Ω are bounded, i.e., we have inequalities (7), where u_1^k, u_2^k are replaced by $u_1^{\alpha(\delta_k)}$, $u_2^{\alpha(\delta_k)}$. Next, we use the theorem of embedding of the space BV in L_p, the E-property of L_q, and the (weak) lower semicontinuity of the functionals involved in $\Phi(u_1, u_2)$.

Remark 1. Theorems 1 and 2 remain valid if the functional $l^{\beta}(u_2)$ in problem (2) is replaced by $\|u_2\|_{L_r}$, $1 < r < \infty$. In this case, the solution $(\hat{u}_1, \hat{u}_2)$ is guaranteed to be unique and the substantiation of the scheme for the finite-dimensional approximation of regularized problem (2) is simplified.

2. DISCRETIZATION OF THE REGULARIZATION METHOD

Now we consider regularization method (2) with

$$\Omega^{\beta}(u) = \|u\|_{L_r}^2 + J(u), \qquad (11)$$

i.e., in (3) we make the substitution indicated in Remark 1 and set $\beta = 0$. Thus, we examine the regularization method (2) with $J^{\beta}(u_2)$ in functional (11) replaced by the nonsmooth total variation $J(u_2)$, which corresponds to the discontinuous component u_2. To construct numerical algorithms, we need to pass from the infinite-dimensional problem (2) to its finite-dimensional (discrete) analogue. For the Tikhonov method in the one-dimensional case ($d = 1$), a semidiscretization scheme based on a piecewise linear approximation of the function only in the solution space was proposed and studied in [5]. In this section, we substantiate the general scheme for the discrete approximation of problem (2) with a nonsmooth stabilizer by a sequence of finite-dimensional minimization problems with convex differentiable objective functions.

Let D be a d-dimensional rectangular domain, for example, a unit cube. A grid analogue of the space R^d is defined as

$$R_n^d = \{x \in R^d : x = (j_1 h, j_2 h, \ldots, j_d h),\; j_1, j_2, \ldots, j_d = 0, \pm 1, \ldots\}.$$

Consider grid functions $u_n : R_n^d \to R$, where R_n^d is an n^d-dimensional vector space and the index n means that $u_n(x)$ is defined on a grid with step size $h = \frac{1}{n}$ in each variable. Define the family of coupling operators

$$P = \left\{ p_n : p_n(u) = h^{-d} \int\limits_{\omega_d(x)} u(y)\, dy \right\},$$

where $\omega_d(x)$ is an elementary cell of volume h^d around the node $x = (x_1, x_2, \ldots, x_d)$, $\omega_d(x) = \{y \in R^d : x_j - h < y_j \le x_j\}$. The family P provides a discrete approximation of the space $L_p(D)$ $(1 < p < \infty)$ by the sequence of spaces l_p^n with the norm

$$\|u_n\|_{l_p^n} = \left(\sum_{x \in D_n} h^d |u_n(x)|^p \right)^{1/p}, \qquad D_n = D \cap R_n^d$$

and generates the discrete and discrete weak convergence of elements and operators. The basic properties and the necessary facts concerning the discrete convergence properties to be used to substantiate the general scheme for the discrete approximation of problem (2) can be found in [6–8].

Problem (2) is associated with the sequence of finite-dimensional problems

$$\min\{\|A_n(u_{1n} + u_{2n}) - f_n\|_{l_2}^2 + \alpha[\|u_{1n}\|_{l_q^n}^2 + \|u_{2n}\|_{l_r^n}^2 + J_n^{\beta_n}(u_{2n})] : u_{1n} \in l_q^n, u_{2n} \in l_r^n\} = \Phi_n^*, \qquad (12)$$

where

$$J_n^{\beta_n}(u_{2n}) = \sup\left\{ \sum_{x \in D_n} h^d \left(-u_{2n}(x) \sum_{j=1}^d \partial_j v_{jn}(x) + \sqrt{\beta_n (1 - |v_n(x)|)} \right) : |v_n(x)| \le 1, v_n \in C_0^1(D_n, R_n^d) \right\},$$

$$\partial_j v_n(x) = \frac{v_n(x) - v_n(x - h_j e_j)}{h}, \quad e_j = (0, \ldots, 1, 0, \ldots, 0),$$
$$v_n(x) = (v_{1n}(x), v_{2n}(x), \ldots, v_{dn}(x)).$$

Let $\Phi_n(u_{1n}, u_{2n})$ denote the objective functional in problem (12).

Lemma 1. *It is true that*

$$J_n^{\beta}(u_n) = \sup\left\{ \sum_{x \in D_n} h^d \left(-u_n(x) \sum_{j=1}^d \partial_j v_{jn}(x) + \sqrt{\beta (1 - |v_n(x)|^2)} \right) : v_n \in C_0^1(D_n, R_n^d) \right\} = \sum_{x \in D_n} h^d \sqrt{\sum_{j=1}^d |\partial_j u_n(x)|^2 + \beta}. \qquad (13)$$

Proof. Taking into account the property

$$\sqrt{|z|^2 + \beta} = \sup\{\langle z, y\rangle + \sqrt{\beta(1 - |y|^2)} : y \in R^d, |y| \le 1\}$$

of the Fenchel transformation [9, p. 289] and applying summation by parts, we obtain

$$\Delta_n(v_n) = \sum_{x \in D_n} h^d \left(-u_n(x) \sum_{j=1}^d \partial_j v_{jn}(x) + \sqrt{\beta (1 - |v_n(x)|^2)} \right)$$

$$= \sum_{x\in D_n} h^d \left(\sum_{j=1}^{d} \partial_j u_n(x) v_{jn}(x) + \sqrt{\beta(1-|v_n(x)|^2)} \right)$$

$$\leq \sum_{x\in D_n} h^d \sqrt{\sum_{j=1}^{d} |\partial_j u_n(x)|^2 + \beta},$$

therefore,

$$J_n^{\beta}(u_n) \leq \sum_{x\in D_n} h^d \sqrt{\sum_{j=1}^{d} |\partial_j u_n(x)|^2 + \beta}. \tag{14}$$

On the other hand, direct verification shows that, for $\overline{v}_n(x) = (\overline{v}_{1n}(x), \overline{v}_{2n}(x), \ldots, \overline{v}_{dn}(x))$, where

$$\overline{v}_{jn} = \frac{\partial_j u_n(x)}{\sqrt{\sum_{j=1}^{d} |\partial_j u_n(x)|^2 + \beta}},$$

we have

$$\Delta(\overline{v}_n) = \sum_{x\in D_n} h^d \sqrt{\sum_{j=1}^{d} |\partial_j u_n(x)|^2 + \beta}. \tag{15}$$

Representation (13) follows from (14) and (15).

Corollary 1. *It is true that*

$$J_n(u_n) \leq J_n^{\beta_n}(u_n) \leq J_n(u_n) + \sqrt{\beta_n}. \tag{16}$$

The notation "$-\to$" and "$-\to$ (weakly)" will be used for discrete and discrete weak convergence, respectively.

Lemma 2. *The pair of functionals* $J, (J_n^{\beta_n})$, *where* $\beta_n \to 0$, *is discretely weakly lower semicontinuous, i.e.*,

$$u_n - \to u \ (weakly) \Rightarrow J(u) \leq \liminf_{n\to\infty} J_n^{\beta_n}(u_n). \tag{17}$$

Proof. Relation (17) follows from the discrete weak lower semicontinuity of the pair J, (J_n) [10, Lemma 4.3] and from inequalities (16).

Retaining the notation (u_1^α, u_2^α) for the solution of problem (2) at $\beta = 0$, we denote by $(\hat{u}_{1n}, \hat{u}_{2n})$ the solution of problem (12) at $\beta = \beta_n$, where the functional $l_\beta(u_n)$ is replaced by $\|u_2\|^2_{L_r}, 1 < r$.

Theorem 3. *Suppose that problems* (2) *and* (12) *satisfy the discrete approximation conditions*

$$A_n - \to A, \quad A_n \to A \, (weakly), \quad f_n - \to f^\delta, \tag{18}$$

where A, A_n *are continuous operators from* L_r *to* L_2 *and from* l_r^n *to* l_2^n, *respectively, and let* $\beta_n \to 0$. *Then problem* (12) *has a unique solution* $(\hat{u}_{1n}, \hat{u}_{2n})$ *such that*

$$\hat{u}_{1n} - \to u_1^\alpha, \quad \hat{u}_{2n} - \to u_2^\alpha. \tag{19}$$

Moreover, the total variations and the optimal values satisfy the limit relations

$$\lim_{n\to\infty} J_n^{\beta_n}(\hat{u}_n) = J(u_2), \quad \lim_{n\to\infty} \Phi_n^* = \Phi^*. \tag{20}$$

Proof. In view of the conditions of the theorem and Lemmas 1 and 2, the solvability of problem (12), i.e., the existence of a pair $(\hat{u}_{1n}, \hat{u}_{2n})$, is proved using the scheme applied in the one-component case [10, Theorem 4.1]. Following this scheme in checking the relation

$$\limsup_{n\to\infty} \Phi_n^* = \limsup_{n\to\infty} \Phi_n(\hat{u}_{1n}, \hat{u}_{2n}) \leq \Phi^*, \tag{21}$$

we need to prove that, for any $\varepsilon > 0$, there exists a pair $u_{1\varepsilon}$, $u_{2\varepsilon} \in C^\infty$ such that

$$\Phi(u_{1\varepsilon}, u_{2\varepsilon}) \leq \Phi^* + \varepsilon, \tag{22}$$

$$\lim_{n\to\infty} J_n^{\beta_n}(\overline{p}_n u_{2\varepsilon}) = J(u_{2\varepsilon}) \tag{23}$$

for the family of grid projection operators $\overline{p}_n$. Relation (22) follows from Lemma 2.1 in [4], while the validity of (23) follows from (16) and the inequalities

$$|J_n^{\beta_n}(p_n u_{2\varepsilon}) - J(u_{2\varepsilon})|$$
$$\leq |J_n^{\beta_n}(p_n u_{2\varepsilon}) - J(p_n u_{2\varepsilon})| + |J(p_n u_{2\varepsilon}) - J(u_{2\varepsilon})|.$$

According to (21), the components $\hat{u}_{1n}, \hat{u}_{2n}$ are uniformly bounded and, hence, discretely weakly compact [8] (see also [10, Theorem 5.1]), i.e., for a subsequence of indices $\{n'\} \in \{n\}$,

$$\hat{u}_{1n'} - \to \overline{u}_1 \text{ (weakly)}, \quad \overline{u}_{2n'} - \to \overline{u}_2 \text{ (weakly)}. \tag{24}$$

Combining conditions (18) with Lemma 2 and relation (21), we obtain the chain of inequalities

$$\begin{aligned} \Phi^* &\leq \Phi(\overline{u}_1, \overline{u}_2) \leq \liminf_{n'\to\infty} \Phi_{n'}(\hat{u}_{1n'}, \hat{u}_{2n'}) \\ &\leq \limsup_{n'\to\infty} \Phi_{n'}^* \leq \Phi^*, \end{aligned} \tag{25}$$

which implies that the pair $(\overline{u}_1, \overline{u}_2)$ solves problem (2), where $\beta = 0$ and $l^\beta(u_2)$ is replaced by $\|u_2\|^2$, i.e., $\overline{u}_1 = u_1^\alpha$, $\overline{u}_2 = u_2^\alpha$. Inequalities (25) and the discrete weak semicontinuity of the functionals involved in $\Phi_n(u_{1n}, u_{2n})$ imply the norm convergence relations

$$\lim_{n\to\infty} \|\hat{u}_{1n}\| = \|\overline{u}_1\|, \quad \lim_{n\to\infty} \|\hat{u}_{2n}\| = \|\overline{u}_2\|,$$

which, when combined with (24) and the uniqueness of the solution, yield (19). Applying the same argument, we conclude that relations (20) hold as well.

Corollary 2. *Let* $\{r_n\}$ *be a family of piecewise constant extension operators. Then*

$$\lim_{n\to\infty}\|r_n\hat{u}_{1n} - u_1^{\alpha}\|_{L_q} = 0,$$
$$\lim_{n\to\infty}\|r_n\hat{u}_{2n} - u_2^{\alpha}\|_{L_r} = 0.$$

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